Cauchy sequences and Euclidean Geometry: A foundational method in computing

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1 Timeline

- Pythagoras (570-495 BC)
- Euclid (at Alexandria, 323-283 BC)
- Archimedes (287-212 BC)
- Oresme (1323-1382)
- Viète (1540-1603)
- Newton (1642-1727)
- Jakob Bernoulli (1646-1716)
- Johann Bernoulli (1654-1705)
- Taylor (1685-1731)
- Euler (1707-1783)
- Lagrange (1736-1813)
- Laplace (1749-1827)
- Legendre (1752-1853)
- Fourier (1768-1830)
- Gauss (1777-1855)
- Cauchy (1789-1857)
- George Green (1793-1841)
- Olinde Rodrigue (1795-1851)
- Abel (1802-1829)

- Dirichlet (1805-1899)
- Liouville (1809-1882)
- Ludwig Schläfli (1814-1895)
- Weierstrass (1815-1897)
- Riemann (1826-1866)
- Mittag-Leffler (1846-1927)
- Volterra (1860-1940)
- Baire (1874-1932)
- Lebesgue (1875-1941)
- Čech (1893-1960)

2 What is a Euclidean space?

A **Euclidean space** is a separable real Hilbert space with norm $||x|| = <x, x>^{\frac{1}{2}}$ and orthonormal basis e_i,

$$lim_{n\to\infty} ||x - \sum_{k=1}^{n} a_i\, e_i|| = 0$$

where $a_i = <x, e_i>$.

Note that $a_1 e_1 + \cdots + a_n e_n$ is the closest vector in $span\{e_1, \ldots, e_n\}$ to x (in the metric $||y - z||$). This is the (Euclidean!) theory of best least-square fit.

Pthagorean theorem = Parsival's equality
For orthonormal e_i,

$$||\sum a_i\, e_i||^2 \;=\; \sum\, a_i^2$$

If $x = \sum a_i\, e_i$ with e_i only orthogonal,

$$||x||^2 \;=\; \sum\,(||e_i||\, a_i)^2$$

3 Why π?

- $A = \pi r^2$ (Archimedes, 287–212 BC)
- $V = \frac{4}{3}\pi r^3$ (Archimedes)
- Ellipse, $A = \pi ab$

Green's theorem in the plane (George Green, 1828; Gauss, 1813; Lagrange, 1760)

Let C be a closed curve (not necessarily simple) bounding a (not necessarily simply-connected) region R. For force field $(u(x,y), v(x,y))$ with continuous first partial derivatives and defined on and inside C,

$$\int_C (u,v)\cdot(dx,dy) \;=\; \int\int_R (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\, dx\, dy$$

For $u = 0$, $v = x$,

$$\text{Area } R \;=\; \int_C x\, dy$$

Example $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Parametrize ∂E by $x = a\,cos\,\theta$, $y = b\,sin\,\theta$, $0 \le \theta \le 2\pi$. Note the use of the (Euclidean) Pthagorean theorem ($sin^2 + cos^2 = 1$). Then the area is

$$\int_{\partial E} x\,dy = \int_0^{2\pi} a\,cos\,\theta\,(b\,cos\,\theta\,d\theta) = ab\int_0^{2\pi} cos^2\theta\,d\theta$$

By Euler's formula, $cos^2\theta = \frac{1+cos\,2\theta}{2}$, so the area is

$$ab\,[\frac{1}{2}\theta + \frac{1}{4}sin\,2\theta]_0^{2\pi} = \pi ab$$

Because of the *definition* of π ($= 180^o$), more general ∂R are parametrized by $x = f(\theta)$, $y = g(\theta)$, $0 \le \theta \le 2\pi$, with area

$$\int_0^{2\pi} f(\theta)\,g'(\theta)\,d\theta = H(\theta)|_0^{2\pi}$$

so it's no surprise π appears.

4 What is π?

- Archimedes: 2 places (inscribed, circumscribed 96-gon)
- François Viète (1540–1603): 9 places
- Mathematica (1980s):
  NumberForm[N[π,500],500] produces

  3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342117067982148086513282306647093844609550582231725359408128481117450284102701938521105559644622948954930381964428810975665933446128475648233786783165271201909145648566923460348610454326648213393607260249141273724587006606315588174881520920962829254091715364367892590360011330530548820466521384146951941511609433057270365759591953092186117381932611793105118548074462379962749567351885752724891227938183011949129833673362

- $\frac{\pi}{4} = arc\ tan\ 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$. This converges slowly, *but* it is the basis of the Rabinowitz-Wagon spigot algorithm which is best implemented in a functional language such as Haskell –see J. Gibbons, **Monthly** 113, 2006.
- Bailey, Borwein and Plouffe 1997:

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right)$$

  Can generate *mth* base-16 digit without knowing earlier digits.

Newton's approximation of π

On the board we derive

$$\pi = 24\,(\frac{\sqrt{3}}{32} + \int_0^{\frac{1}{4}} x^{\frac{1}{2}}\,(1-x)^{\frac{1}{2}}\,dx)$$

Newton had proved the "binomial theorem" to get the Taylor expansion

$$(1-x)^{\frac{1}{2}} \;=\; 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \frac{7}{256}x^4 - \cdots - \frac{883631595}{2^{38}}x^{20} - \cdots$$

Multiplying through by $x^{\frac{1}{2}}$ and taking antiderivatives to integrate, all of which Newton had invented, note that

$$\left(\frac{1}{4}\right)^{n+\frac{1}{2}} = \frac{1}{2\cdot 4^n}$$

so, taking the approximation above up to the x^{20} term computes

$$\pi \;\sim\; 3.1415926535897935$$

which is accurate to 15 places.

5 Taylor's theorem and the irrationality of e

Taylor's theorem Let $z_0 \neq z_1, a_k$ be complex numbers such that $\Sigma_{k=0}^{\infty} a_k(z_1 - z_0)^k$ converges. Then $f(z) = \Sigma_{k=0}^{\infty} a_k(z - z_0)^k$ converges and is infinitely differentiable for all z with $|z - z_0| < |z_1 - z_0|$. Moreover, if $0 < s < |z_1 - z_0|$ then, for $|z - z_0| \leq s$,

$$f(z) - \sum_{k=0}^{n} a_k(z - z_0)^k \;=\; \frac{f^{(n+1)}(\xi(z))}{(n+1)!}\,(z - z_0)^{n+1}$$

for some $|z_0 - \xi(z)| \leq |z_0 - z|$.

The a_k are unique and satisfy

$$a_k \;=\; \frac{f^{(k)}(z_0)}{k!}$$

Now apply this to $f(z) = e^z$ at $z_0 = 0$ which converges everywhere with partial sums

$$S_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$$

Now suppose $e = \frac{a}{b}$ were rational and seek a contradiction. This proof is due to Euler.

By Taylor's Theorem, $e - S_n(1) = \frac{e^\xi}{(n+1)!} z^n$ for $|z| < 1$, so $e - S_n(1) \leq \frac{e}{(n+1)n!}$.

Now choose $n > b$ with $\frac{e}{n+1} < 1$. Then $e - S_n(1) < \frac{1}{n!}$. We have

$$0 \ < \ \frac{a}{b} - (1 + \frac{1}{2!} + \cdots + \frac{1}{n!}) \ < \ \frac{e^s}{(n+1)n!} \ < \ \frac{1}{n!}$$

Now multiply through by $n!$:

$$0 \ < \ \frac{an!}{b} - n!(1 + \frac{1}{2!} + \cdots + \frac{1}{n!}) \ < \ 1$$

Since no integer is strictly between 0 and 1, we have found a contradiction.

6 Continuity

Karl Weierstrass (1815–1897)

Choose

$$\frac{1+\frac{3}{2}\pi}{a} \; < \; b \; < \; 1, \quad a \text{ an odd integer}$$

and define $W(x) = \Sigma_{k=0}^{\infty}\, b^k\, cos(\pi a^k x)$.

- By the "Weierstrass M-test", the series converges absolutely and uniformly, so W is continuous.
- Differentiating term-by-term yields a divergent series, so W has a continuous derivative nowhere.
- A more delicate proof shows W is differentiable nowhere.

For $b = 0.8$, $a = 9$, the first three terms of $W(x)$ are

$$cos(\pi x) + 0.8\,cos(9\pi x) + 0.64\,cos(81\pi x)$$

Here is a plot of this fragment on the interval $[-\pi, \pi]$:

$$f(x) = \begin{cases} x^2\, sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable, but f' is not continuous at 0.

René Baire, thesis 1899

- In $\mathbb{R}$, every countable intersection of open dense sets is dense.
- If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $\{x : f' \text{ is continuous at } x\}$ is a countable intersection of open dense sets, hence is dense.

If f is differentiable on $[a, b]$ is it true that $\int_a^b f'(x)\, dx = f(b) - f(a)$?

- With Lebesgue integral, yes.
- (Volterra, 1860–1940). f' may fail to be Riemann integrable.

7 Baire spaces

A non-empty $T1$ topological space is a **Baire space** if every countable intersection of open dense sets is dense.

Well known theorems:

- A locally compact, Hausdorff space is Baire.
- A complete metric space is Baire.

But also, the irrationals are Baire. What is the general principle?

A **compactification** of X is a dense subspace embedding $X \to C$ with C compact Hausdorff.

Thus, a compactification exists $\Leftrightarrow$ X is completely regular, Hausdorff $\Leftrightarrow$ X is uniformizable (and separated).

In a space, a countable union of closed sets is said to be F_σ.

Theorem Given (completely regular, Hausdorff) X, there exists a compactification $X \to C$ with $C \backslash X$ F_σ $\Leftrightarrow$ for every compactification $X \to C$ with $C \backslash X$ F_σ.

We say X is **Čech complete** in that case.

Examples

- A locally compact, Hausdorff space is Čech complete.
- The irrationals are Čech complete.
- $\mathbb{R}$ is Čech complete.
- (harder) A complete metric space is Čech complete.

Theorem (Čech 1937) A Čech complete space is Baire.

Observation If X is a Baire space in which every non-empty open set has a least two elements, every countable intersection of open dense sets is uncountable. In particular, X is uncountable.

Corollary $\mathbb{R}$ is uncountable.

8 Classical PDEs and complex analysis

Laplacian of $u(x, y, z, t)$

$$\bigtriangledown u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Wave equation: $\bigtriangledown u = \frac{\partial^2 u}{\partial t^2}$

Fourier heat equation for $u(x, y, z, t)$ the temperature at (x, y, z) at time t:

$$\bigtriangledown u = \frac{\partial u}{\partial t}$$

$u(x, y, z)$ or $u(x, y)$ is **harmonic** if $\bigtriangledown u = 0$. This is the heat equation at thermal equilibrium, and is also the **Laplace equation**.

For U an open subset of the complex plane $\mathbb{C}$ and for $f : U \to \mathbb{C}$ the following conditions are equivalent and we say f is **holomorphic** on U if any, hence all, hold.

- $f'(z) = lim_{w \to 0} \frac{f(z+w)-f(z)}{w}$ exists
- Writing $f = u + iv$, the **Cauchy-Riemann equations** hold:
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
- For $z_0 \in U$ there exists $r > 0$ with $S_r(z_0) \subset U$ and
$$f(z) = \sum_{k=0}^{\infty} a_k \, (z - z_0)^k, \quad z \in S_r(z_0)$$

For holomorphic $f = u(x, y) + i\, v(x, y)$, $f'(z) = \frac{\partial u}{\partial x} + i \, \frac{\partial v}{\partial x}$.

First properties of holomorphic functions

- Every holomorphic function is infinitely differentiable inside any of its circles of convergence.
- As a real function $U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$, because of the Cauchy-Riemann equations, the Jacobian matrix is

$$\begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{pmatrix}$$

  where $tan\,\theta = \frac{\partial v}{\partial x} / \frac{\partial u}{\partial x}$ and $r^2 = (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2$.
- Two functions holomorphic on an open set U, if they agree on infinitely many points in a closed and bounded subset of U, must agree everywhere in U.
- If $u + i\,v$ is holomorphic, u and v are harmonic functions.

Using complex line integrals to expand a holomorphic function

Let $f(z)$ be holomorphic on and inside a closed curve C.

Cauchy's theorem.

$$\int_C f(z)\, dz = 0$$

Cauchy's integral formulas. For C a simple curve and z_0 inside C,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \\ f^{(n)}(z_0) &= \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \end{aligned}$$

Thus, Taylor coefficients can be found by

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}$$

9 From Legendre to wiggly polynomials

Our goal: To approximate a wiggly curve on an interval by a polynomial.

The story begins with Legendre's study of the gravational potential V between point mass at (a, b, c) and a solid with typical point (x, y, z) (see blackboard picture), $V(x, y, z) = \frac{K}{\delta}$, $\delta = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$.

Indeed the gradient is given by

$$\bigtriangledown V = (\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}) = -\frac{K}{\delta^2}(\frac{(x-a)}{\delta}, \frac{(y-b)}{\delta}, \frac{(z-c)}{\delta})$$

By the law of cosines (Euclidean!), $\delta^2 = \alpha^2 + \rho^2 - 2\alpha\rho\, cos\,\theta$. Define $\alpha = \tau\rho$ so that

$$\begin{aligned} \delta^2 &= \rho^2\tau^2 + \rho^2 - 2\tau\rho^2 cos\,\theta \\ &= \rho^2(1 - 2\tau cos\,\theta + \tau^2) \end{aligned}$$

Choose units with $K = \rho$. Then

$$V = \frac{\rho}{\delta} = \frac{1}{\sqrt{1 - 2\tau\, cos\,\theta\, + \tau^2}}$$

Write $x = cos\,\theta$, $\tau = z \in \mathbb{C}$ so that

$$V(x, z) = \frac{1}{\sqrt{1 - 2xz + z^2}}$$

(from previous slide)

$$V(x,z) = \frac{1}{\sqrt{1-2xz+z^2}}$$

Fix x. At this point, following the example of Newton and Euler, Legendre used the binomial theorem to expand $V(x,z)$. We do the same thing, in effect, but by virtue of regarding V as a holomorphic function in a radius about 0.

Thus $V(x)$ expands as follows:

$$\begin{aligned} \frac{1}{\sqrt{1-2xz+z^2}} &= P_0(x) + P_1(x)\,z + P_2(x)\,z^2 + \cdots \\ P_n(x) &= \frac{1}{2\pi\, i}\int_C \frac{1}{z^{n+1}\sqrt{1-2xz+z^2}}\,dz \end{aligned}$$

Where the heck is this going?

Schäfli observed that if $w = \sqrt{1 - 2\frac{x}{z} + \frac{1}{z^2}} + \frac{1}{z}$ then, on a large circle E centered at x,

$$P_n(x) \;=\; \frac{1}{2\pi i} \int_E \frac{(w^2-1)^n}{2^n(w-x)^{n+1}}\, dw$$

so, if $g(w) = (w^2-1)^n$, we have

$$g^{(n)}(x) = \frac{n!}{2\pi\, i} \int_E \frac{(w^2-1)^n}{(w-x)^{n+1}}\, dw \;=\; 2^n\, n!\, P_n(x)$$

and we arrive at **Rodrigue's formula**

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n}(x^2-1)^n$$

We thus learn that $P_n(x)$ is a polynomial of degree n. These polynomials are called the **Legendre polynomials**.

Summary: $V = \frac{1}{\delta}$ as a function of $cos\,\theta$ and $\frac{\rho}{\alpha}$ expands as

$$\sum_{n=0}^{\infty} P_n(cos\,\theta)\, \left(\frac{\rho}{\alpha}\right)^n$$

Since $deg(P_n) = n$, the P_n are linearly independent in $C[a, b]$. Can we find $[a, b]$ such that the P_n are orthogonal? If so, one suspects it would be an orthogonal basis for the Hilbert space $C[a, b]$ by the Stone-Weierstrass theorem, and this can be shown rigorously.

Toward finding $[a, b]$, differentiate!

$$\begin{aligned} y &= (x^2-1)^n \\ y' &= 2nx\,(x^2-1)^{n-1} \end{aligned}$$

so...

$$\begin{aligned} (x^2-1)y' - 2nxy &= 0 \\ (x^2-1)y'' - 2x(n-1)y' + 2ny &= 0 \\ &\cdots \\ (x^2-1)y^{(n+2)} + 2xy^{(n+1)} - n(n+1)y^{(n)} &= 0 \end{aligned}$$

Thus the P_n satisfy **Legendre's differential equation**

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$$

Multiplying the equations for $P_n,\ P_m$ By $P_m,\ P_n$ and subtracting gives

$$\frac{d}{dx}(1-x^2)\,(P_n^{'}P_m - P_m^{'}P_n) \;=\; (n(n+1)-m(m+1))P_mP_n$$

so that

$$\int_a^b P_m(x)\,P_n(x)\,dx \;=\; \frac{1-x^2}{n(n+1)-m(m+1)}(P_mP_n^{'}-P_m^{'}P_n)\,|_a^b$$

So we choose $[a,b]=[-1,1]$ since $1-x^2$ has roots $1,\ -1$.

Thus for $f \in C[-1,1]$, $f = \Sigma_{k=0}^{\infty}\, a_n\, P_n$ (both mean and pointwise convergence) where

$$a_n \;=\; \frac{<f,P_n>}{||P_n||^2} \;=\; \frac{2}{2n+1}\int_{-1}^{1} f(x)\,P_n(x)\,dx$$

Facts about Legendre polynomials

- $P_{n+1} \;=\; \frac{2n+1}{n+1}\, x\, P_n - \frac{n}{n+1}\, P_{n-1}$
- P_n has n distinct roots in $[-1, 1]$.
- The kernel function for the nth partial sum $S_{f,n}(x)$ in the Legendre series for f is $K_n(t, x) = \sum_{k=0}^{n} \frac{2k+1}{2}\, P_k(t)\, P_k(x)$, so that
$$S_{f,n}(x) \;=\; \int_{-1}^{1} K_n(t, x)\, f(t)\, dt$$

Let $f(x) = \frac{e^x}{8}\, sin\, 10x$.

The degree-15 Taylor expansion of f about 0 is

$$\frac{5x}{4} + \frac{5x^2}{4} - \frac{485x^3}{24} - \frac{165x^4}{8} + \frac{9005x^5}{96} + \frac{29003x^6}{288} - \frac{793493x^7}{4032} - \frac{14773x^8}{64} + \frac{65251609x^9}{290304} + \frac{88250801x^{10}}{290304} - \frac{4825396489x^{11}}{31933440} - \frac{83151601x^{12}}{322560} + \frac{289796841413x^{13}}{4981616640} + \frac{5278393991807x^{14}}{34871316480} - \frac{1631181003097x^{15}}{209227898880}$$

The degree-15 Legendre approximant of f on $[-1, 1]$ is

$$0.000338286822094260\,86 + 1.249714795203168\, x + 1.20272761705514\, x^2 - 20.194127413740084\, x^3 - 19.53972960572615\, x^4 + 93.59841403204769\, x^5 + 91.15027962415643\, x^6 - 195.51192014695948\, x^7 - 188.91521865333604\, x^8 + 220.59378321239416\, x^9 + 201.2424491124213\, x^{10} - 143.95071152099672\, x^{11} - 109.83398669386563\, x^{12} + 52.66100043325236\, x^{13} + 24.614622828224174\, x^{14} - 8.551179016273345\, x^{15}$$

The Taylor approximation is only good near 0:

The Legendre approximation is awesome. Plotted is the difference between the two functions.

A Dirichlet problem

Find the steady state temperature $u(\rho, \phi)$ (spherical coordinates (ρ, θ, ϕ)) on and inside the unit ball $\rho = 1$, assuming independence of θ and given the temperature on the surface. Thus

$$\begin{aligned} \bigtriangledown u(\rho, \phi) &= 0 \\ u(1, \phi) &= f(\phi) \end{aligned}$$

It turns out the solution is as follows and involves Legendre polynomials.

Expand $f(arc\,cos\,x) = \Sigma_{k=0}^{\infty} a_k\, P_k(x)$ on $[-1, 1]$. Then

$$u(\rho, \phi) = \sum_{k=0}^{\infty} a_k\, \rho^k\, P_k(cos\,\phi)$$

10 The Basel problem

What is $\zeta(2) = \Sigma_{k=1}^{\infty} \frac{1}{k^2}$?

Some background:

- Nicole Oresme (1323–1382): $\Sigma \frac{1}{k}$ diverges.
- $H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$ is the maximum overhang distance of n cards of length 2.
- Euler (1707–1783) $\gamma = lim(H_n - ln\, n) \sim 0.5772156....$
- Abel (1828) showed $\Sigma \frac{1}{k\; ln\, k}$ diverges.
- Mathematica: For $S_n = \frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{n^2}$
  - $S_{100} = 1.6349...$
  - $S_{10000} = 1.6448...$
  - Johann Bernoulli (1667–1748): $lim\, S_n < 2$
  - The true first four places are 1.6449.

Fourier series

$C[-\pi, \pi]$ with $< f, g >= \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$ is a real Hilbert space with orthogonal basis

$$1,\ cos\, x,\ sin\, x,\ cos\, 2x,\ sin\, 2x,\ \ldots$$

so that continuous $f : [-\pi, \pi] \to \mathbb{R}$ has **Fourier series**

$$f = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\, cos\, kx \,+\, b_k\, sin\, kx$$

- The Fourier series of a differentiable function f converges pointwise to f everywhere. This is not true for Taylor series!
- A. Kolmogorov (1903–1987): There exists an L^1 function whose Fourier series does not converge pointwise anywhere.
- L. Carleson (1966): The Fourier series of an L^2 function f converges pointwise to f almost everywhere.
- R. Hunt (1968): For $1 < p \leq \infty$, the Fourier series of an L^p function converges pointwise to f almost everywhere.

To solve the Basel problem we look for $f \in C[-\pi, \pi]$ with Fourier coefficients a constant multiple of

$$\epsilon_1 \frac{1}{1},\ \epsilon_2 \frac{1}{2},\ \epsilon_3 \frac{1}{3}, \ldots \quad \text{with } \epsilon_k \in \{-1, 1\}$$

with the idea of applying Parsival's equality.

The simplest example $f(x) = x$ does the job. As $f(-x) = -f(x)$ is odd, $a_k = 0$, whereas

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, cos \, kx \, dx = (-1)^k \frac{2}{k}$$

On the other hand,

$$||f||^2 = \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{3} x^3 |_{-\pi}^{\pi} = \frac{2}{3} \pi^3$$

This gives

$$\frac{2}{3} \pi^3 = \sum_{k=1}^{\infty} (\sqrt{\pi} \, (-1)^k \, \frac{2}{k})^2$$

from which we have the desired result

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

11 Finding ζ of an even positive integer

Euler solved the Basel problem at the beginning of his career and toward the end found $\zeta(2n) = \Sigma_{k=1}^{\infty} \frac{1}{k^{2n}}$ in general.

His calculation, which uses the Bernoulli numbers, appears with few changes in modern advanced texts.

The **Bernoulli numbers** B_k may be defined through their exponential generating function by

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}$$

Properties of the Bernoulli numbers

- The B_k are rational and $B_3, B_5, B_7, \ldots = 0$. The sequence B_{2k} alternates in sign. $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, ….
- The numbers are named after Jakob Bernoulli who showed
$$1^m + 2^m + \cdots + (n-1)^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k\, n^{m-k+1}$$
- Kummer (1850): A prime p is **regular** if it does not divide the numerator of any $B_2, \ldots, B_{p-1}$. For regular p, $x^p + y^p = z^p$ has no integer solutions.

We began with the Fourier series of $cos\,\alpha x$, α not an integer. Since this is an even function, the $sin\,kx$ coefficients are zero.

$$cos\,\alpha x \;=\; \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\,cos\,kx$$

We have

$$a_0 \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} cos\,\alpha x\,dx \;=\; 2\,\frac{sin\,\alpha\pi}{\alpha\pi}$$

and, for $k > 0$,

$$a_k \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} cos\,\alpha x\,cos\,kx\,dx$$

An antiderivative is

$$\frac{sin\,(x(k-\alpha))}{2(k-\alpha)} + \frac{sin\,(x(k+\alpha))}{2(k+\alpha)}$$

so that for $k > 0$,

$$a_k \;=\; \frac{2\,\alpha\,sin\,\alpha\pi}{\pi}\,\frac{(-1)^{k+1}}{k^2-\alpha^2}$$

So far:

$$cos\,\alpha x \;=\; \frac{sin\,\alpha\pi}{\pi}\left(\frac{1}{\alpha} + 2\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2-\alpha^2}\,cos\,kx\right)$$

Letting $x = \pi$,

$$cot\,\alpha\pi \;=\; \frac{cos\,\alpha\pi}{sin\,\alpha\pi} \;=\; \frac{1}{\alpha\pi} - 2\sum_{k=1}^{\infty}\frac{\alpha}{k^2-\alpha^2}$$

But now, letting $u = \alpha\pi$ for $0 < u < \pi$ (so that α is never an integer) we get

$$u\,cot\,u \;=\; 1 - 2\sum_{k=1}^{\infty}\frac{u^2}{k^2\pi^2-u^2}$$

Note: Expansions of this type result from what is called the **Mittag-Leffler expansion theorem**; using Fourier series is not the usual route.

So what on earth does this have to do with $\zeta(2n)$?????

Here's the idea: noting $|\frac{u}{k\pi}| < 1$,

$$\begin{aligned}
\frac{1 - u\,cot\,u}{2} = \sum_{k=1}^{\infty} \frac{u^2}{k^2\pi^2 - u^2} &= \sum_{k=1}^{\infty} \frac{u^2}{k^2\pi^2}\,\frac{1}{1-(\frac{u}{k\pi})^2} \\
&= \sum_{k=1}^{\infty} \frac{u^2}{k^2\pi^2}\,\sum_{n=0}^{\infty}(\frac{u}{k\pi})^{2n} \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\frac{u}{k\pi})^{2n} \\
&= \sum_{n=1}^{\infty}\frac{u^{2n}}{\pi^{2n}}\,\sum_{k=1}^{\infty}(\frac{1}{k})^{2n} \\
&= \sum_{n=1}^{\infty}\frac{u^{2n}}{\pi^{2n}}\,\zeta(2n)
\end{aligned}$$

a power series!

Thus to find $\zeta(2n)$ we seek another power series for $u\,cot\,u$ and equate coefficients.

We know $e^{iu} = cos\, u + i\, sin\, u$, $e^{-iu} = cos\, u - i\, sin\, u$ which allows us to solve for $sin\, u$, $cos\, u$ in terms of e^{iu}, e^{-iu}. Recalling the generating function defining the Bernoulli numbers, we get

$$u\, cot\, u \;=\; i\, u\, \frac{e^{2iu}}{e^{2iu}-1} + \frac{iu}{e^{2iu}-1} \;=\; 1 + \sum_{k=2}^{\infty} B_k\, \frac{(2iu)^k}{k!}$$

Thus equating coefficients leads to the amazing formula

$$\zeta(2n) = \sum_{k-1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}$$

Here are the first ten values:

$$\frac{\pi^2}{6}, \frac{\pi^4}{90}, \frac{\pi^6}{945}, \frac{\pi^8}{9450}, \frac{\pi^{10}}{93555}, \frac{691\pi^{12}}{638512875},$$

$$\frac{2\pi^{14}}{18243225}, \frac{3617\pi^{16}}{325641566250}, \frac{43867\pi^{18}}{38979295480125},$$

$$\frac{174611\pi^{20}}{1531329465290625}$$

What about $\zeta(2n+1)$?

$1.08232 < \zeta(3) < 1.64493$

The exact value of $\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}$ is unknown to this day!

That's it!