Convenient Vector Spaces, Convenient Manifolds and Differential Linear Logic

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ongoing discussions with
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Goals

- Develop a theory of (smooth) manifolds based on differential linear logic. Or perhaps develop a differential linear logic based on manifolds.
- Convenient vector spaces were recently shown to be a model.
- There is a well-developed theory of convenient manifolds, including infinite-dimensional manifolds.
- Convenient manifolds reveal additional structure not seen in finite dimensions. In particular, the notion of tangent space is much more complex.
- Synthetic differential geometry should also provide information. Convenient vector spaces embed into an extremely good model.

Convenient vector spaces (Frölicher,Kriegl)

Definition

A vector space is *locally convex* if it is equipped with a topology such that each point has a neighborhood basis of convex sets, and addition and scalar multiplication are continuous.

- Locally convex spaces are the most well-behaved topological vector spaces, and most studied in functional analysis.
- Note that in any topological vector space, one can take limits and hence talk about derivatives of curves. A curve is *smooth* if it has derivatives of all orders.
- The analogue of Cauchy sequences in locally convex spaces are called *Mackey-Cauchy sequences.*
- The convergence of Mackey-Cauchy sequences implies the convergence of all Mackey-Cauchy nets.

The following is taken from a long list of equivalences.

Convenient vector spaces II: Definition

Theorem

Let E be a locally convex vector space. The following statements are equivalent:

- *If $c\colon \mathbb{R} \to E$ is a curve such that $\ell \circ c\colon \mathbb{R} \to \mathbb{R}$ is smooth for every linear, continuous $\ell\colon E \to \mathbb{R}$, then c is smooth.*
- *Every Mackey-Cauchy sequence converges.*
- *Any smooth curve $c\colon \mathbb{R} \to E$ has a smooth antiderivative.*

Definition

A vector space satisfying any of these conditions is called a *convenient* vector space.

Convenient vector spaces III: Bornology

The theory of bornological spaces axiomatizes the notion of bounded sets.

Definition

A *convex bornology* on a vector space V is a set of subsets $\mathcal{B}$ (the bounded sets) such that

- $\mathcal{B}$ is closed under finite unions.
- $\mathcal{B}$ is downward closed with respect to inclusion.
- $\mathcal{B}$ contains all singletons.
- If $B \in \mathcal{B}$, then so are $2B$ and $-B$.
- $\mathcal{B}$ is closed under the convex hull operation.

A map between two such spaces is *bornological* if it takes bounded sets to bounded sets.

Convenient vector spaces IV: More bornology

- To any locally convex vector space V, we associate the *von Neumann bornology*. $B \subseteq V$ is bounded if for every neighborhood U of 0, there is a real number λ such that $B \subseteq \lambda U$.
- This is part of an adjunction between locally convex topological vector spaces and convex bornological vector spaces. The topology associated to a convex bornology is generated by *bornivorous disks*.

Theorem

Convenient vector spaces can also be defined as the fixed points of these two operations, which satisfy Mackey-Cauchy completeness and a separation axiom.

Convenient vector spaces V: Frölicher Spaces

Yet another way to define convenient vector spaces:

Definition

Let X be a set. Let $\mathcal{C}_X \subseteq Hom(\mathbb{R}, X)$ be a set of functions, called the *smooth curves* into X. Let $\mathcal{F}_X \subseteq Hom(X, \mathbb{R})$ be another set, called the *functionals* on X. These determine each other in the sense that:

$$\mathcal{C}_X = \{f \colon \mathbb{R} \to X | \forall g \in \mathcal{F}_X, g \circ f : \mathbb{R} \to \mathbb{R} \text{ is smooth.}\}$$

$$\mathcal{F}_X = \{g \colon X \to \mathbb{R} | \forall f \in \mathcal{C}_X, g \circ f : \mathbb{R} \to \mathbb{R} \text{ is smooth.}\}$$

The triple $(X, \mathcal{C}_X, \mathcal{F}_X)$ is called a *Frölicher space*.

Let X and Y be Frölicher spaces. A function $f \colon X \to Y$ is a *map of Frölicher spaces* if $f(\mathcal{C}_X) \subseteq \mathcal{C}_Y$. This is equivalent to requiring $f^*(\mathcal{F}_Y) \subseteq \mathcal{F}_X$.

Theorem (Frölicher, Kriegl)

The category of Frolicher spaces and maps is cartesian closed.

A Frölicher space inherits a bornology from its space of functionals. $U \subseteq X$ is bounded if and only if $f(U) \subseteq \mathbb{R}$ is bounded for all $f \in \mathcal{F}_X$.

Theorem

Convenient vector spaces can also be defined as internal vector spaces in the category of Frölicher spaces satisfying a completeness condition.

Convenient vector spaces VI: Key points

- The category Con of convenient vector spaces and continuous linear maps forms a symmetric monoidal closed category. The tensor is a completion of the algebraic tensor. There is a convenient structure on the space of linear, continuous maps giving the **internal hom**.
- Since these are topological vector spaces, one can define smooth curves into them.

Definition

A function $f : E \longrightarrow F$ with E, F being convenient vector spaces is *smooth* if it takes smooth curves in E to smooth curves in F.

Convenient vector spaces VII: More key points

- The category of convenient vector spaces and smooth maps is cartesian closed. This is an enormous advantage over Euclidean space, as it allows us to consider function spaces.
- There is a comonad on Con such that the smooth maps form the coKleisli category:

We have a map δ as follows, with $C^\infty(E)$ being the set of smooth, real-valued maps.:

$$\delta \colon E \to \mathrm{Con}(C^\infty(E), \mathbb{R}) \quad \delta(x)(f) = f(x)$$

Then we define $\mathbf{!}E$ to be the closure of the span of the set $\delta(E)$.

Theorem (Frölicher,Kriegl)

- $\mathbf{!}$ *is a comonad.*
- $\mathbf{!}(E \oplus F) \cong \mathbf{!}E \otimes \mathbf{!}F.$
- *Each object* $\mathbf{!}E$ *has canonical bialgebra structure.*

Theorem (Frölicher,Kriegl)

The category of convenient vector spaces and smooth maps is the coKleisli category of the comonad **!** .

One can then prove:

Theorem (RB, Ehrhard, Tasson)

Con *is a model of differential linear logic. In particular, it has a codereliction map given by:*

$$coder(v) = \lim_{t \to 0} \frac{\delta(tv) - \delta(0)}{t}$$

Convenient vector spaces IX: Codereliction

Using this codereliction map, we can build a more general differentiation operator by precomposition:

Consider $f\colon\ !E \to F$ then define $df\colon E \otimes\ !E \to F$ as the composite:

$$E \otimes\ !E \xrightarrow{coder \otimes id}\ !E \otimes\ !E \xrightarrow{\nabla}\ !E \xrightarrow{f} F$$

Theorem (Frölicher,Kriegl)

Let E and F be convenient vector spaces. The differentiation operator

$$d\colon \mathcal{C}^\infty(E,F) \to \mathcal{C}^\infty(E, \mathrm{Con}(E,F))$$

defined as

$$df(x)(v) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

is linear and bounded. In particular, this limit exists and is linear in the variable v.

A convenient differential category

The above results show that Con really is an optimal differential category.

- The differential inference rule is really modelled by a directional derivative.
- The coKleisli category really is a category of smooth maps.
- Both the base category and the coKleisli category are closed, so we can consider function spaces.

This seems to be a great place to consider manifolds. There is a well-established theory.

Kriegl, Michor-*The convenient setting for global analysis*

Convenient manifolds

Definition

- A *chart* (U, u) on a set M is a bijection $u\colon U \to u(U) \subseteq E$ where E is a fixed convenient vector space, and $u(U)$ is an open subset.
- Given two charts (U_α, u_α) and (U_β, u_β), the mapping $u_{\alpha\beta} = u_\alpha \circ u_\beta^{-1}$ is called a *chart-changing*.
- An *atlas* or *smooth atlas* is a family of charts whose union is all of M and all of whose chart-changings are smooth.
- A *(convenient) manifold* is a set M with an equivalence class of smooth atlases.
- Smooth maps are defined as usual.

Lemma

A function between convenient manifolds is smooth if and only if it takes smooth curves to smooth curves.

This is a complicated subject.

Definition

A manifold M is *smoothly hausdorff* if smooth real-valued functions separate points.

Note that this implies:

- M is hausdorff in its usual topology, **which implies:**
- The diagonal is closed in the manifold $M \times M$.

These three notions are equivalent in finite-dimensions. In the convenient setting, the reverse implications are open. Note that the product topology on $M \times M$ is different than the manifold topology! Also:

Lemma

There are smooth functions that are not continuous. (Seriously.)

Smooth real-compactness

We have a map:

$$\delta\colon E \longrightarrow Hom_{Alg}(C^\infty(E), \mathbb{R})$$

Theorem

For finite-dimensional vector spaces and in fact any finite-dimensional manifolds, this map is a bijection.

It may or may not be a bijection for more general manifolds. We say:

Definition

A convenient vector space is *smoothly real-compact*, if the above map is a bijection.

Theorem (Arias-de-Reyna,Kriegl,Michor)

Lots of spaces are smoothly real-compact. Lots are not.

The many equivalent notions of tangent in finite-dimensions now become distinct. See Kriegl-Michor.

Definition

Let E be a convenient vector space, and let $a \in E$. A *kinematic tangent vector* at a is a pair (a, X) with $X \in E$. Let $T_a E = E$ be the space of all kinematic tangent vectors at a.

The above should be thought of as the set of all tangent vectors at a of all curves through the point a.

For the second definition, let $C_a^\infty(E)$ be the quotient of $C^\infty(E)$ by the ideal of those smooth functions vanishing on a neighborhood of a. Then:

Tangent spaces II

Definition

An *operational tangent vector* at a is a continuous derivation, i.e. a map

$$\partial \colon C^\infty_a(E) \to \mathbb{R}$$

such that

$$\partial(f \circ g) = \partial(f)g(a) + f(a)\partial(g)$$

Note that every kinematic tangent vector induces an operational one via the formula

$$X_a(f) = df(a)(X)$$

where d is the directional derivative operator. Let D_aE be the space of all such derivations.

Tangent spaces III

In finite dimensions, the above definitions are equivalent and the described operation provides the isomorphism. That is no longer the case here.

Let $Y \in E''$, the second dual space. Y canonically induces an element of D_aE by the formula $Y_a(f) = Y(df(a))$. This gives us an injective map $E'' \to D_aE$. So we have:

$$T_aE \hookrightarrow E'' \hookrightarrow D_aE$$

Definition

E satisfies the *approximation property* if $E' \otimes E$ is dense in $Con(E, E)$ (This is basically the MIX map.).

Theorem (Kriegl,Michor)

If E satisfies the approximation property, then $E'' \cong D_aE$. If E is also reflexive, then $T_aE \cong D_aE$.

Embedding into models of SDG

- Convenient vector spaces embed nicely into a well-behaved model of **synthetic differential geometry**.
- In **SDG**, the (kinematic) tangent bundle takes on a particularly simple form. It is an exponential.
- A model of **SDG** is, roughly speaking, a universe (a topos) in which all functions are smooth, and yet the category is cartesian closed. So the motivation is very much the same as ours.
- The model in question is called the *Cahiers topos*, and is due to E. Dubuc. The embedding is based on the notion of *Weil Prolongation*, due to A. Kock, and the final steps in the embedding are due to A. Kock and G. Reyes.

Embedding into models of SDG II

The difference between **DG** and **SDG** is the existence of infinitesimals. Weil prolongation is a way of adding them. The nLab calls this *thickening by infinitesimals*.

Definition

A *Weil algebra* is a $\mathbb{R}$-bilinear map $\mu \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ making $\mathbb{R}^n$ into a commutative algebra such that the element $(1, 0, 0, \ldots, 0)$ is the unit and the set $I = \{(0, r_1, r_2, \ldots, r_n)\}$ is a nilpotent ideal with nilpotence degree less than or equal to n.

The primary example is the ring $\mathbb{R}[x]/(x^2)$ or $\mathbb{R}[\varepsilon]$ where $\varepsilon^2 = 0$, the ring of dual numbers.

More examples of Weil algebras

Generalizing our previous example, $\mathbb{R}[x]/(x^n)$ is a Weil algebra.

Theorem

Let A be an $\mathbb{R}$-algebra. The following are equivalent, with m being the relevant maximal ideal.

- *A is a Weil algebra.*
- *A is of the form $\mathbb{R}[x_1, x_2, \ldots, x_n]/I$, where for each variable x_i there is a natural number n with $X_i^n \in I$.*
- *A is isomorphic to $\mathbb{R}[[x_1, x_2, \ldots, x_n]]/I$, with I a power of the unique maximal ideal.*
- *A is isomorphic to a ring $\mathcal{C}_0^\infty(\mathbb{R}^n)/I$ which is finite-dimensional as a real vector space.*

Weil prolongation

Definition

In the following X is a convenient vector space, and let X' be its linear, continuous dual space. Let I be an ideal in the ring $\mathcal{C}^\infty(\mathbb{R}^n)$.

Suppose $f, g \in \mathcal{C}^\infty(\mathbb{R}^n, X)$.

Say that $f \sim_I g$ if $\varphi \circ f - \varphi \circ g \in I$ for all $\varphi \in X'$. This is an equivalence relation on the set $\mathcal{C}^\infty(\mathbb{R}^n, X)$.

An equivalence class is called a *mod I jet into X*. We denote the set of equivalence classes by $X \otimes W$.

In the following, let CVS denote the category of convenient vector spaces, and *smooth* maps. Let We denote the category of Weil algebras and homomorphisms.

Weil prolongation II

Theorem (Kock)

The Weil prolongation process gives a functor $- \otimes - : \mathsf{CVS} \times \mathsf{We} \to \mathsf{CVS}$. *Furthermore, the action of the monoidal category* We *on* CVS *is associative, in the sense that there is a natural isomorphism*

$$X \otimes (W_1 \otimes W_2) \cong (X \otimes W_1) \otimes W_2$$

compatible with all relevant structure.

The Cahiers topos is a *Grothendieck topos*, i.e. a category of sheaves for a (very generalized) notion of topology. Instead of a topological space, one has a category called the *site of definition* equipped with a *Grothendieck topology.*

Weil prolongation III-Skipping many details

For this topos, the site of definition $\mathcal{D}$ has objects of the form $\mathcal{C}^\infty(\mathbb{R}^n) \otimes W$, with W a Weil algebra.

Theorem (Kock-Reyes)

The above action lifts to an action $- \otimes -\colon \mathsf{CVS} \times \mathcal{D} \to \mathsf{CVS}$

Now given such an action, we consider the exponential transpose of the composite:

$$\mathsf{CVS} \times \mathcal{D} \longrightarrow \mathsf{CVS} \longrightarrow \mathsf{Set}$$

This is a functor $J\colon \mathsf{CVS} \to \mathsf{Set}^{\mathcal{D}}$.

Theorem (Kock-Reyes)

- *For all convenient vector spaces X, the functor $J(X)$ is a sheaf with respect to the Grothendieck topology.*
- *The functor J is full and faithful.*
- *J preserves all finite limits, and the exponential structure.*

Conclusion

- Lifting the embedding to convenient manifolds?
- If this works, does the construction preserve the tangent bundle for either notion of tangent bundle?
- But are convenient manifolds the right thing? That category is not cartesian closed.
- Nishimura argues one should forget manifolds and generalize to some other class of Frölicher spaces. He has a specific proposal on the right class, but the existence of the embedding depends on a conjecture he hasn't managed to prove.
- Do any of these structures shed any light on the idea of differential linear logic for manifolds?