Exponentiability in Cat and Top via Double Categories

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Motivation

SBN, 1978 (PhD Thesis) TFAE for $p\colon Y \to B$ in Top

(1) p is exponentiable in Top/B

(2) $(\bigsqcup_b \mathcal{O}Y_b) \times_B Y \to \mathbb{2}$ is continuous, where $\bigsqcup_b \mathcal{O}Y_b$ has the "fiberwise" Scott topology

(3) For all $y \in V_b$ and $V_b \in \mathcal{O}Y_b$, there exists $H \subseteq \bigsqcup_b \mathcal{O}Y_b$ such that H is fiberwise Scott open, $V_b \in H_b$, and $y \in (\bigsqcup_b (\cap H_b))^\circ$

Street, 2001 (Powerful Functors) Using $\mathrm{Cat}/B \simeq \mathrm{Lax}_{\mathrm{N}}(B, \mathbb{C}\mathrm{at})$

$Y \to B$ is exp in Cat/B iff $B \to \mathbb{C}\mathrm{at}$ is pseudo

SBN, 2010 (CT10 paper) $\mathrm{Top}/B \simeq \mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{op})$, B finite poset

Goal: $Y \to B$ is exp in Top/B iff $B \to \mathbb{T}\mathrm{op}$ is pseudo and ?

The Fiberwise Scott Topology

Let $p\colon Y \twoheadrightarrow B$. Then $H \subseteq \bigsqcup_b \mathcal{O}Y_b$ is called *fiberwise Scott open* if each H_b is Scott open in $\mathcal{O}Y_b$ and for all $U \in \mathcal{O}Y$

$$\{b \in B \mid U_b \in H_b\} \in \mathcal{O}B \qquad (\star)$$

If B is a poset with the $\downarrow$-topology, then $(\star)$ says

$$U_c \in H_c, b < c \Rightarrow U_b \in H_b$$

Recall For a complete lattice L, $H \subseteq L$ is *Scott open* if $\uparrow H = H$ and $\bigvee S \in H \Rightarrow \bigvee F \in H$, for some finite $F \subseteq S$.

L is *continuous* if $\forall y$, $y = \{x \in L \mid x << y\}$, where $x << y$ if $y \leq \bigvee S \Rightarrow x \leq \bigvee F$, for some finite $F \subseteq S$.

Exponentiability

Suppose $\mathcal{C}$ has finite limits. An object Y is called *exponentiable* if $- \times Y \colon \mathcal{C} \to \mathcal{C}$ has a right adjoint (denoted by $(\ )^Y$).

A morphism $Y \to B$ is *exponentiable* if it is exponentiable in $\mathcal{C}/B$.

Y is exponentiable in Top iff $\mathcal{O}Y$ is continuous (Day/Kelly 1970).

The exponential $Z^Y = \mathit{Top}(Y, Z)$, with the topology generated by

$$\langle H, W \rangle = \{ f \mid f^{-1}W \in H \}$$

where H is Scott open in $\mathcal{O}Y$ and W is open in Z. In particular, $2^Y \cong \mathcal{O}Y$, with the Scott topology.

Double Categories

A (pseudo) double category $\mathbb{D}$ is a (pseudo) category object in Cat

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \;\begin{array}{c} \xrightarrow{\;\pi_2\;} \\ \xrightarrow{\;\mu\;} \\ \xrightarrow[\;\pi_1\;]{} \end{array}\; \mathbb{D}_1 \;\begin{array}{c} \xrightarrow{\;d_0\;} \\ \xleftarrow{\;\Delta\;} \\ \xrightarrow[\;d_1\;]{} \end{array}\; \mathbb{D}_0$$

Objects: objects of $\mathbb{D}_0$

Horizontal morphisms: morphisms of $\mathbb{D}_0$, $f : X \longrightarrow Y$

Vertical morphism: objects of $\mathbb{D}_1$, $m : X_0 \mathrel{\bullet\!\!\rightarrow} X_1$

Cells: morphisms of $\mathbb{D}_1$,

$$\begin{array}{ccc} X_0 & \xrightarrow{\;f_0\;} & Y_0 \\ {\scriptstyle m}\big\downarrow\bullet & \varphi & \bullet\big\downarrow{\scriptstyle n} \\ X_1 & \xrightarrow[\;f_1\;]{} & Y_1 \end{array}$$

The Double Category $\mathbb{T}\mathrm{op}$

The other morphisms between spaces

Glueing Maps $m\colon X_0 \multimap\!\!\!\!\rightarrow X_1$

$$m\colon \mathcal{O}X_0 \rightarrow \mathcal{O}X_1 \text{ s.t. } mX_0 = X_1,\ m(U_0 \cap V_0) = mU_0 \cap mV_0$$

Example For $f: X_0 \rightarrow X_1$ continuous, get $f^{-1}\colon X_1 \multimap\!\!\!\!\rightarrow X_0$, and its right adjoint $f_*\colon X_0 \multimap\!\!\!\!\rightarrow X_1$, $f_* U_0 = [X_1 \backslash f(X_0 \setminus U_0)]^\circ$

Morphisms of Glueing Maps "glueing squares"

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ m \downarrow & \supseteq & \downarrow n \\ X_1 & \xrightarrow[f_1]{} & Y_1 \end{array} \qquad \text{i.e., } \frac{f_1^{-1} n \subseteq m f_0^{-1}}{n f_{0*} \subseteq f_{1*} m}$$

Notation $\mathbb{T}\mathrm{op}_1$, the category of glueing maps

Normal Lax Functors

$\mathrm{Lax_N}(B, \mathbb{T}\mathrm{op})$, for a poset B

Objects: vertical normal lax functors $X \colon B \longrightarrow \mathbb{T}\mathrm{op}$, i.e.,

X_b, for all $b \in B$, and $m_{bc} \colon X_b \xrightarrow{\bullet} X_c$, for all $b < c$,

s.t. $m_{cd}\, m_{bc} \supseteq m_{bd}$, for all $b < c < d$

Morphisms: horizontal lax transformations $f \colon X \longrightarrow Y$, i.e.,

$$X_b \xrightarrow{f_b} Y_b, \text{ for all } b \in B, \text{ s.t. } \begin{array}{ccc} X_b & \xrightarrow{f_b} & Y_b \\ {\scriptstyle m_{bc}} \big\downarrow \bullet & \supseteq & \bullet \big\downarrow {\scriptstyle n_{bc}} \\ X_c & \xrightarrow[f_c]{} & Y_c \end{array} \text{ for all } b < c$$

Example T s.t. $T_b = 1$ with $id_{\mathcal{O}1} \colon 1 \xrightarrow{\bullet} 1$ is a terminal object

Note $\mathrm{Lax_N}(\mathbb{1}, \mathbb{T}\mathrm{op}) \cong \mathrm{Top}$ and $\mathrm{Lax_N}(\mathbb{2}, \mathbb{T}\mathrm{op}) \cong \mathbb{T}\mathrm{op}_1$

The Glueing Functor

The constant functor $\mathrm{Top} \xrightarrow{\Delta} \mathrm{Lax_N}(B, \mathbb{T}\mathrm{op})$ has a left adjoint Γ

$\Gamma X = \bigsqcup_b X_b$ with $U \subseteq \bigsqcup_b X_b$ *open* if $U_b \in \mathcal{O}X_b$ and

$U_c \subseteq m_{bc} U_b$, where $m_{bc} \colon X_b \twoheadrightarrow X_c$, for $b < c$

When $B = \mathbb{2}$, write $X_0 +_m X_1$ or X_{01} for $\Gamma(X_0 \xrightarrow{m} X_1)$

Then $U \subseteq X_{01}$ is open if $U_0 \in \mathcal{O}X_0$, $U_1 \in \mathcal{O}X_1$, and $U_1 \subseteq mU_0$

Remarks

1. $\Gamma T = B$ with the Alexandroff topology
2. Γ induces a functor $\Gamma_B \colon \mathrm{Lax_N}(B, \mathbb{T}\mathrm{op}) \to \mathrm{Top}/B$
3. Γ_B is an equivalence when B is finite (CT2010)
4. To prove:

Theorem $Y \colon B \to \mathbb{T}\mathrm{op}$ is exponentiable in $\mathrm{Lax_N}(B, \mathbb{T}\mathrm{op}) \iff$

Y is pseudo and $Y_b \twoheadrightarrow Y_c$ is exponentiable in $\mathbb{T}\mathrm{op}_1, \forall b < c$

Preliminaries

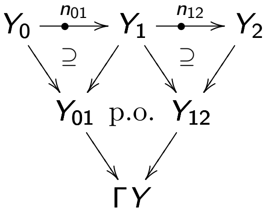
Lemma 1 If A is a subposet of B, then the restriction functor

$$(\ )_A \colon \mathrm{Lax_N}(B, \mathbb{T}\mathrm{op}) \to \mathrm{Lax_N}(A, \mathbb{T}\mathrm{op})$$

has a left adjoint L_A such that $(\ )_A \circ L_A = id$.

Proof Take $(L_A X)_b = \begin{cases} X_b & b \in A \\ \emptyset & b \notin A \end{cases}$

Lemma 2 $Y \colon \mathfrak{Z} \to \mathbb{T}\mathrm{op}$ is pseudo $\iff$



Proof An exercise in lax colimits

Theorem $Y\colon B \rightarrow \mathbb{T}\mathrm{op}$ is exponentiable in $\mathrm{Lax_N}(B, \mathbb{T}\mathrm{op}) \iff$
Y is pseudo and $n_{bc}\colon Y_b \dashrightarrow Y_c$ is exponentiable in $\mathbb{T}\mathrm{op}_1, \forall b < c$

Proof ($\Leftarrow$) By Lemma 1, $(\ )_b$ preserves products, and so the counit for n_{bc} is given by

$$\begin{array}{ccc}
Z_b^{Y_b} \times Y_b & \xrightarrow{ev_b} & Z_b \\
\downarrow\bullet & \supseteq & \downarrow\bullet \\
Z_c^{Y_c} \times Y_c & \xrightarrow[ev_c]{} & Z_c
\end{array}$$

Thus, it suffices to show that $b \mapsto Z_b^{Y_b}$ is a normal lax functor

$$\begin{array}{ccc}
Z_b^{Y_b} & \xrightarrow{id} & Z_b^{Y_b} \\
u\,\downarrow\bullet & & \\
Z_c^{Y_c} & \supseteq & \downarrow\bullet \\
v\,\downarrow\bullet & & \\
Z_d^{Y_d} & \xrightarrow[id]{} & Z_d^{Y_d}
\end{array}
\quad\longleftrightarrow\quad
\begin{array}{ccccccccc}
Z_b^{Y_b} \times Y_b & \xrightarrow{id} & Z_b^{Y_b} \times Y_b & \xrightarrow{ev_b} & Z_b & \xrightarrow{id} & Z_b \\
 & & u \times n_{bc}\downarrow\bullet & \supseteq & \downarrow\bullet & & \\
vu \times n_{bd}\downarrow\bullet & =^* & Z_c^{Y_c} \times Y_c & \xrightarrow{ev_c} & Z_c & \supseteq & \downarrow\bullet \\
 & & v \times n_{cd}\downarrow\bullet & \supseteq & \downarrow\bullet & & \\
Z_d^{Y_d} \times Y_d & \xrightarrow[id]{} & Z_d^{Y_d} \times Y_d & \xrightarrow[ev_d]{} & Z_d & \xrightarrow[id]{} & Z_d
\end{array}$$

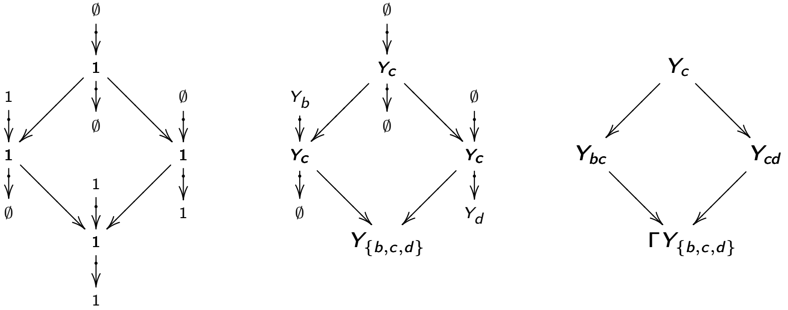
* since Y is pseudo

($\Rightarrow$) Suppose $Y\colon B \to \mathbb{T}\mathrm{op}$ is exponentiable in $\mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{op})$.

Then $Y_b \to Y_c$ is exponentiable in $\mathbb{T}\mathrm{op}_1$ since

$$\frac{\dfrac{\dfrac{\dfrac{X \times Y_{\{b,c\}} \to Z = (L_{\{b,c\}} Z)_{\{b,c\}}}{L_{\{b,c\}}(X \times Y_{\{b,c\}}) \to L_{\{b,c\}} Z}}{L_{\{b,c\}} X \times Y \to L_{\{b,c\}} Z}}{L_{\{b,c\}} X \to (L_{\{b,c\}} Z)^Y}}{X \to \left((L_{\{b,c\}} Z)^Y\right)_{\{b,c\}}}$$

Similarly, $Y_{\{b,c,d\}}$ is exp in $\mathrm{Lax}_{\mathrm{N}}(\mathfrak{3}, \mathbb{T}\mathrm{op})$, for all $b < c < d$, and so pb along $Y_{\{b,c,d\}}$ takes the pushout on the left to the one in the middle. Applying Γ, the diagram on the right is a pushout, and it follows that Y is pseudo, by Lemma 2.



Corollary If B is a finite T_0 space, then $Y \to B$ is exponentiable in $\mathrm{Top}/B \iff$ the normal lax functor $B \to \mathbb{T}\mathrm{op}$ is a pseudo-functor and $Y \times_B 2 \to 2$ is exponentiable in $\mathrm{Top}/2$, for all $2 \to B$.

Exponentiability in $\mathbb{T}\mathrm{op}_1 \simeq \mathrm{Top}/2$

Given $n\colon Y_0 \twoheadrightarrow Y_1$, consider $\tilde{n}\colon \mathcal{O}Y_0 \twoheadrightarrow \mathcal{O}Y_1$, defined by

$$\tilde{n}H_0 = \bigcup\{H_1 \mid n^{-1}H_1 \subseteq H_0\}$$

where $\mathcal{O}Y_0$ and $\mathcal{O}Y_1$ are given the Scott topology.

Note $H_1 \subseteq \tilde{n}H_0$ iff $n^{-1}H_1 \subseteq H_0$, but $n^{-1}H_1$ need not be Scott open

Write $U_1 << V_1 \pmod{Y_0}$ if $\exists\, H_0 \subseteq \mathcal{O}Y_0$ Scott open such that

$$V_1 \in \tilde{n}H_0 \text{ and } U_1 \subseteq n(\wedge H_0),$$

We say $n\colon Y_0 \twoheadrightarrow Y_1$ is *doubly continuous* if $\mathcal{O}Y_0$ and $\mathcal{O}Y_1$ are continuous lattices and

$$V_1 = \bigcup\{U_1 \mid U_1 << V_1 \pmod{Y_0}\}$$

for all $V_1 \in \mathcal{O}Y_1$

Proposition Suppose $n\colon Y_0 \twoheadrightarrow Y_1$. Then n is exponentiable in $\mathbb{T}\mathrm{op}_1$ $\iff$ n is doubly continuous.

Proof Show:

$$\begin{array}{ccc} \mathcal{O}Y_0 \times Y_0 & \to & 2 \\ {\scriptstyle \tilde{n}\times n}\big\downarrow & \supseteq & \big\downarrow{\scriptstyle id} \\ \mathcal{O}Y_1 \times Y_1 & \to & 2 \end{array} \iff m \text{ is doubly continuous}$$

Corollary If Y is a T_1-space and $p\colon Y \to 2$ is exponentiable in $\mathrm{Top}/2$, then $Y = Y_0 + Y_1$. If, in addition, Y is a connected, then p is constant.

Exercise If $n\colon Y_0 \twoheadrightarrow Y_1$ is exponentiable, then $n\colon \mathcal{O}Y_0 \to \mathcal{O}Y_1$ is continuous in the Scott topology, but not conversely.

Alternate Proof of Theorem ($\Leftarrow$)

To show $Y: B \to \mathbb{T}\mathrm{op}$ is exponentiable in $\mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{op})$, it suffices to show that the exponential $(\Delta \mathbb{2})^Y$ exists in $\mathrm{Lax}_{\mathrm{N}}(B, \mathbb{T}\mathrm{op})$. Since $Y_b \twoheadrightarrow Y_c$ is exponentiable,

$$\begin{array}{ccc} \mathcal{O}Y_b \times Y_b & \to & \mathbb{2} \\ {\scriptstyle \tilde{n}_{bc} \times n_{bc}} \downarrow & \supseteq & \downarrow {\scriptstyle id} \\ \mathcal{O}Y_c \times Y_c & \to & \mathbb{2} \end{array}$$

Thus, it suffices to show $\tilde{n}$ is a lax functor, i.e., $\tilde{n}_{cd}\tilde{n}_{bc} \supseteq \tilde{n}_{bd}$, for all $b < c < d$. But, $H_d \subseteq \tilde{n}_{bd}H_b \iff n_{bd}^{-1}H_d \subseteq H_b \underset{\text{pseudo}}{\iff} n_{bc}^{-1}n_{cd}^{-1}H_d \subseteq H_b \underset{\substack{n_{cd} \text{ cont} \\ \text{since exp}}}{\iff} n_{cd}^{-1}H_d \subseteq \tilde{n}_{bc}H_b \iff H_d \subseteq \tilde{n}_{cd}\tilde{n}_{bc}H_b$

Note If H_b and H_c are Scott open, $H_c \subseteq \tilde{n}_{bc}H_b$ iff $n_{bc}^{-1}H_c \subseteq H_b$, but $n_{cd}^{-1}H_d$ is not, in general, Scott open. However, it is by the Exercise on the previous slide.