If syntax were a chicken and semantic an egg?

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introduction

D_∞ is one of the first fully abstract model of the lambda calculus
first proof: Wadsworth in 1976 (see Manzonneto thesis for a new and more general version)
It was the construction of successive fully abstract models (η-expanced bohm-trees, $\mathcal{H}*$) in order to reach our model.

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It was the construction of successive fully abstract models (η-expanced bohm-trees, $\mathcal{H}*$) in order to reach our model. New here: we are using opposite direction: construction a new system from our model

Définition: stability

if $s \rightarrow t$ then $[\![s]\!] = [\![t]\!]$

Définition: stability

if $s \to t$ then $[\![s]\!] = [\![t]\!]$

Définition: abstraction

t is head normalizable iff $[\![t]\!]$ is empty

full abstraction

Définition: operational equivalence

$s \equiv_o t$ iff for all context $C(\!|.|\!)$, $C(\!|s|\!)$ is head normalizable iff $C(\!|t|\!)$ is head normalizable

full abstraction

Définition: operational equivalence

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Définition: full abstraction

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Définition: definability

for all α finitarry in your model, there is a terme t such that $[\![t]\!] = \alpha$

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Stronger in the case that you have the adequation lemma, and that the application can distinguish both the observational equivalence classes (context lemma) and the elements of the model.

Preorders: model of linear logic

Définition: lolipop

$a :: \alpha \leq_{e \multimap \epsilon} b :: \beta$ iff $a \geq_e b$ and $\alpha \leq_\epsilon \beta$

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finites multisets with the order: $a \leq_{!\epsilon} b$ iff for all $\alpha \in a$ we can find a $\beta \in b$ such that $\alpha \leq \beta$

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Définition: orthogonal

$a \leq_{e^\perp} b$ iff $a \geq_e b$

D_∞

Définition: D_∞

$$D_\infty = ((!D_\infty)^{(\omega)})^\perp$$

D_∞

Définition: D_∞

$$D_\infty = ((!D_\infty)^{(\omega)})^\perp$$

Définition: initial segment

$A \subseteq D_inf$ such that $a \leq_{D_\infty} b \in A \Rightarrow a \in A$

Its properties

Proposition : top

$* = [] :: *$ is the first element created by our process and is becoming a top in D_∞

Proposition : complete lattice

finite intersection and infinite union are allowed

λ-calculus with tests

Définition: tests

operators:

$$\begin{array}{rll} (\text{terms}) & M, N: & \lambda x.M \mid M\ N \mid \bar{\tau}(Q) \\ (\text{test}) & Q, R: & \epsilon \mid \tau(M) \end{array}$$

rules:

$$\begin{array}{lll} (\beta) & (\lambda x.M)N & \to M[N/x] \\ (\kappa) & \tau\bar{\tau}(Q) & \to Q \\ (\tau) & \tau(\lambda x.M) & \to \tau(M[\Omega/x]) \\ (\bar{\tau}) & (\bar{\tau}(Q))\ N & \to \bar{\tau}(Q) \end{array}$$

it's interpretation

$$
\begin{aligned}
[\![y_i]\!]^{\bar{y}} &= \downarrow\{(\bar{u};\alpha)|u_i = [\alpha]!\} \\
[\![\lambda x.M]\!]^{\bar{y}} &= \downarrow\{(\bar{u}; v{::}\alpha)|(v.\bar{u};\alpha) \in [\![M]\!]^{x.\bar{y}}\} \\
[\![M\ N]\!]^{\bar{y}} &= \downarrow\{(\bar{u};\alpha)|\exists\beta_1\cdots\beta_n, (\bar{u};[\beta_1\cdots\beta_n]{::}\alpha) \in [\![M]\!]^{\bar{y}} \bigwedge_i (\bar{u};\beta_i) \in [\![
\\
[\![\bar{\tau}(Q)]\!]^{\bar{y}} &= \downarrow\{(\bar{u};*)|\bar{u} \in [\![Q]\!]^{\bar{y}}\} \\
[\![\tau(M)]\!]^{\bar{y}} &= \downarrow\{\bar{u}|(\bar{u};*) \in [\![M]\!]^{\bar{y}}\}
\end{aligned}
$$

operational equivalence

Proposition :

For every pairs of terms of the λ-calculus that are separated with a context $C(\!|.|\!)$ of the λ-calculus with tests, we can find a new context without tests that can separate them.

lambda-calculus with tests

Définition: product

we can add the following test operator:

$$(test)Q, R : Q|R$$

with the rules:

$$\begin{array}{lll} (\epsilon_|) & Q|\epsilon & \rightarrow Q \\ (\epsilon'_|) & \epsilon|Q & \rightarrow Q \end{array}$$

lambda-calculus with tests

Définition: sum

and its dual:

$$(test)Q, R : Q|R$$

$$\begin{array}{lll} (\epsilon_+) & Q + \epsilon & \to \epsilon \\ (\epsilon'_+) & \epsilon + Q & \to \epsilon \end{array}$$

it's interpretation

$$\begin{aligned} [\![Q|R]\!]^{\bar{y}} &= [\![Q]\!]^{\bar{y}} \cap [\![R]\!]^{\bar{y}} \\ [\![Q+R]\!]^{\bar{y}} &= [\![Q]\!]^{\bar{y}} \cup [\![R]\!]^{\bar{y}} \end{aligned}$$

operational equivalence

Proposition :

For every pairs of terms of the λ-calculus that are separated with a context $C(|.|)$ from the λ-calculus with tests, we can find a new context without sum nor product that can separate them.

operational equivalence

Theoreme :

the restriction to the λ-calculus of the operational equivalence is identical when working with test or not. In particular the full abstraction for the λ-calculus with test imply the full abstraction for the usual one.

definissability

Définition: α^+ and α^-

Given $\alpha = u_1 :: \cdots :: u_r :: *$ with $u_i = [\alpha^i_1 \cdots \alpha^1_{k_i}]$ we can define:

$$\alpha^- = \lambda \bar{x}^r . \bar{\tau}[\ (||_i||_j(\alpha^i_j)^+ (\!| x_i |\!))\ ]$$

$$\alpha^+ (\!| . |\!) = \tau((\!| . |\!)\ (\Sigma_j(\alpha^1_j)^-) \cdots\ \ (\Sigma_j(\alpha^r_j)^-))$$

Proposition :

For all $\alpha \in D_\infty$:

- $\vdash \alpha^- : a \Leftrightarrow a \sqsubseteq_{D_\infty} \alpha$
- $x : u \vdash \alpha^+ (\!| x |\!) \Leftrightarrow u \sqsupseteq_{!(D_\infty)} [\alpha]$

conclusion

Other directions:

- relational model and the λ-calculus with resources
- coherent model and the λ-calculus
- relation with extensional collapse