Algebraic Models of Topological Dynamics: Coproduct Preserving Monads and Semigroups

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Broad Goals Use

- Universal algebra (operations and equations);
- Coproduct preserving monads;
- Semigroup theory.

all to study *almost periodicity* in topological dynamics.

Part I. The proximal relation;

Part II. The role of semigroup theory;

Part III. Generalizing compact metric spaces;

Part IV. Finding examples.

Part I. The proximal relation.

A **transformation monoid** is a continuous $M \times X \to X$, $(t, x) \mapsto tx$ with M a topological monoid, X a topological space subject to the **action equations**

$$\begin{aligned} 1x &= x \\ (tu)x &= t(ux) \end{aligned}$$

Say that M is **classical** if M is one of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$, $\mathbb{R}$.

Mainstream topological dynamics: M classical, X compact metric.

$x, y \in X$ are **proximal** if there exists a sequence $t_n \in M$ with $\lim\ t_n x = \lim\ t_n y$.

x, y are **distal** if x, y are not proximal.

X is **distal** if all $x, y \in X$ are.

X is **equicontinous** if the family $\pi_t : X \to X$ (where $\pi_t x = tx$) is equicontinuous.

If M is a group, equicontinuous $\Rightarrow$ distal.

The set $Mx = \{tx : t \in M\}$ is the **orbit** of x.

$A \subset X$ is **invariant** if $x \in A,\ t \in M \ \Rightarrow\ tx \in A$. $A \subset X$ is **minimal** if it is nonempty, closed and invariant and contains no proper subset with those properties.

Definition

- For $A \subset X,\ t \in M,\ t^{-1}A = \{x \in X : tx \in A\}$.
- $A \subset M$ is **syndetic** if there exists compact $K \subset M$ with $M = K^{-1}A$.
- $x \in X$ is **periodic** $\{t \in M : tx = x\}$ is syndetic.
- $x \in X$ is **almost periodic** if for every neighborhood U of x, $\{t : tx \in U\}$ is syndetic.
- $x \in X$ is **discretely almost periodic** if it is almost periodic with M discrete.

Proposition Assume M is one of $\boldsymbol{Z}$, $\boldsymbol{R}$, X is Hausdorff and $x \in X$ is periodic. Then the orbit Mx is compact and minimal.
Proof Let $A = \{t : tx = x\}$. Then $M = KA$ with K compact. Thus $Mx = KAx = Kx$ is compact as $t \mapsto tx$ is continuous. As Mx is closed, being compact, and M is a group, Mx is minimal. $\square$

Theorem (G. D. Birkhoff, 1912) Let M be classical or a group and let X be locally compact, Hausdorff. For $x \in X$, the following are equivalent.

1. x is almost periodic;
2. x is discretely almost periodic ($M = KA$ with K finite);
3. $\overline{Mx}$ is minimal and compact.

Theorem If X is compact Hausdorff, X contains a minimal set.

Proof Use Zorn's lemma. The intersection of a chain of non-empty closed invariant sets is non-empty.

The differential equation $x' = -x$ induces the flow $\boldsymbol{R}^+ \times [0,1] \to [0,1]$, $tx_0 = x_0\, e^{-t}$.

Checking the action equations:

$$\begin{aligned} 0\, x_0 &= x_0 e^0 \;=\; x_0 \\ (t+u)x_0 &= x_0\, e^{-(t+u)} \;=\; (x_0\, e^{-u})\, e^{-t} \;=\; t(ux) \end{aligned}$$

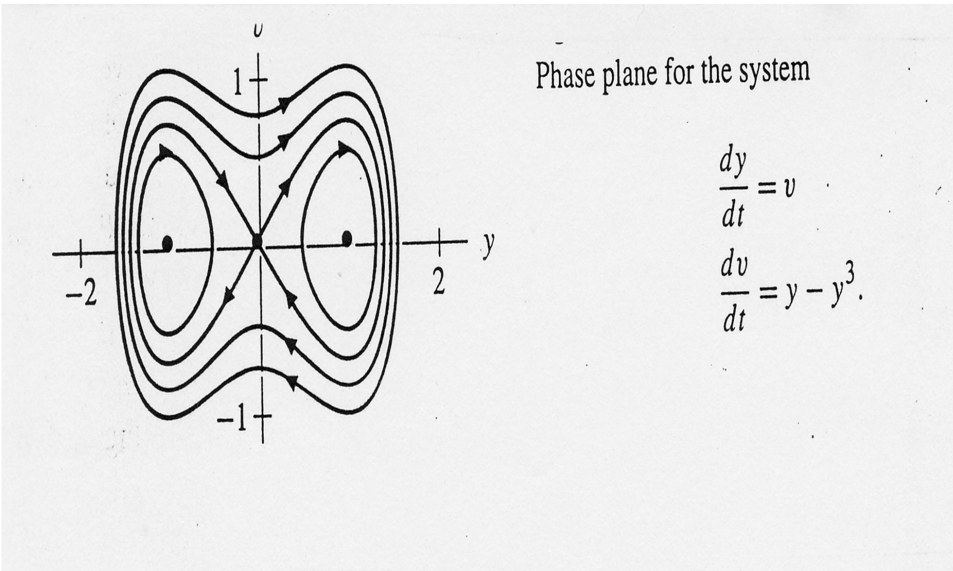
As $\lim_{t\to\infty} tx_0 = 0 = \lim_{t\to\infty} ty_0$, every x_0, y_0 are proximal.

What are the minimal sets?

Poincaré-Bendixon theorem If X is a compact subset of $\mathbb{R}^2$, every almost periodic point of an action $\mathbb{R} \times X \to X$ is periodic.

The following plot shows orbits of **Duffing's equation** in the plane.

$$y'' = y - y^3$$



See Exercise 15 of the handout.

The Lorenz attractor

This famous system is given the the system

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y-x)\\ \frac{dy}{dt} &= x(\rho - z) - y\\ \frac{dz}{dt} &= xy - \beta z\end{aligned}$$

For $\rho \neq 28$, all orbits are periodic. For $\rho = 28$ the other parameters may be adjusted so that some pairs of nearby points are strongly not proximal –this is **chaos**.

I am unable to figure out if non-periodic almost periodicity exists.

Example The differential equation $y'''' + 3y'' + 2y = 0$ induces a flow $\boldsymbol{R} \times \boldsymbol{R}^4 \to \boldsymbol{R}^4$.

All solutions are almost periodic and only a few are periodic (see Exercise 16 of the handout).

The solution curve with initial state $x_0 = (1, 1, -2, -1)$ is

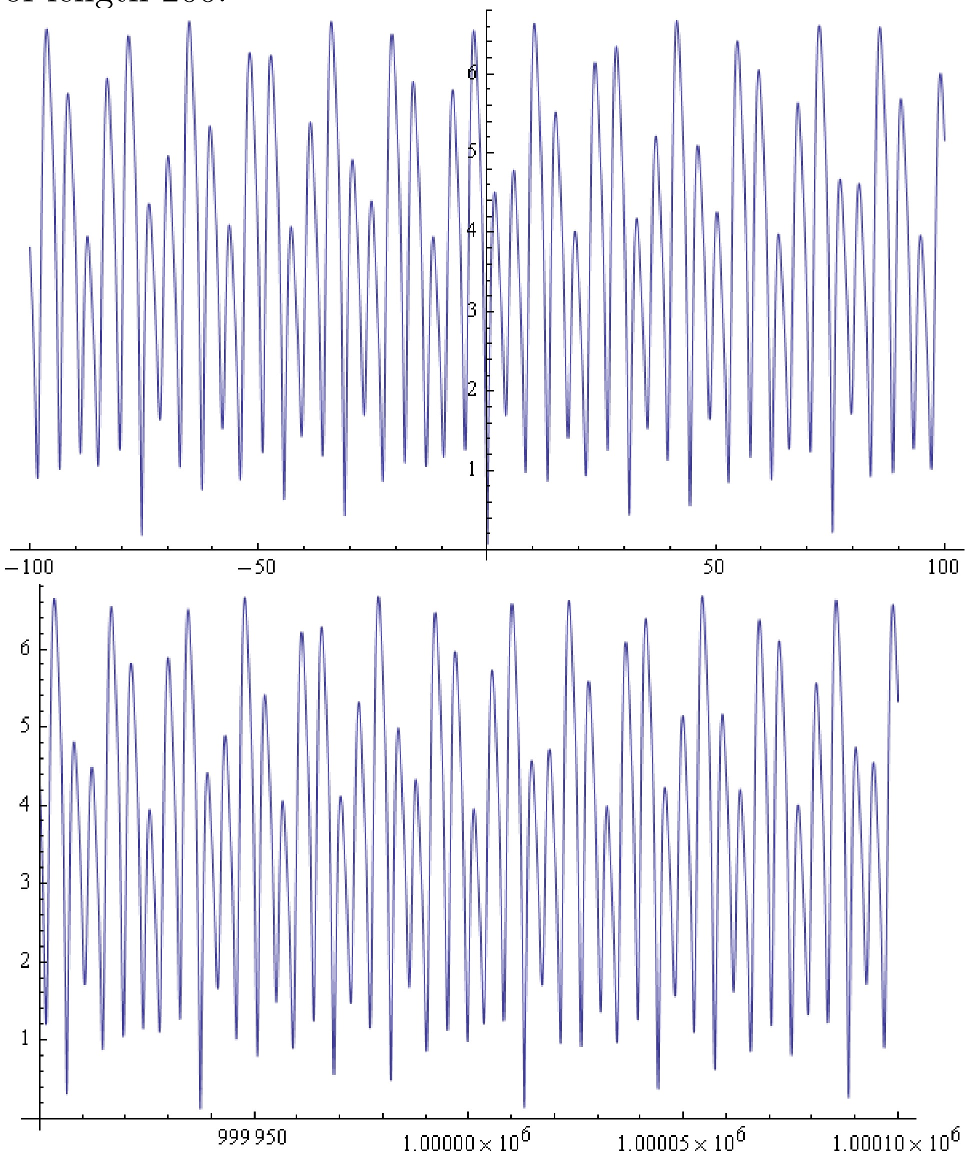
$$y(t) \;=\; sin\, t + cos\, \sqrt{2}\, t$$

so that the orbit is

$$\{(y(t), y'(t), y''(t), y'''(t)) : t \in \boldsymbol{R}\}$$

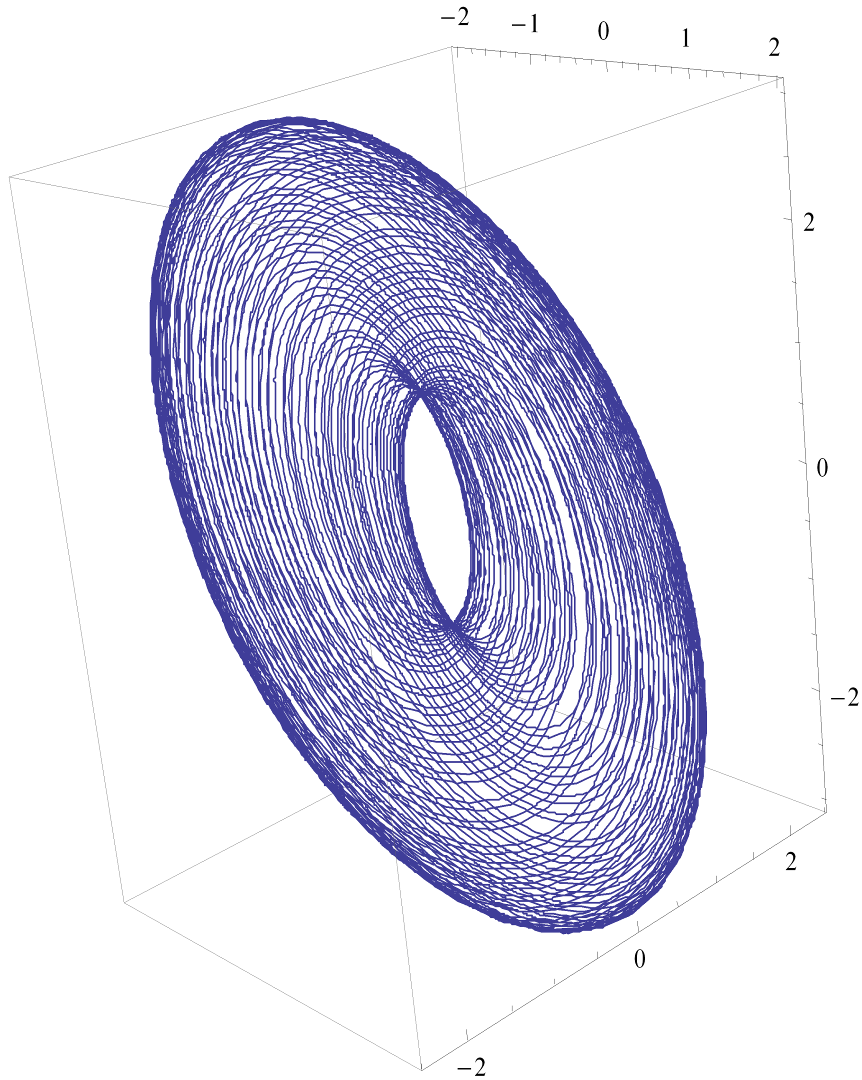
x_0 is not periodic, but is almost periodic as we explore on the next slide.

Plot of $||(y(t), y'(t), y''(t), y'''(t)) - (1, 1, -2, -1)||$ on two different ranges of length 200.



The following is a plot of $(y(t), y'(t), y''(t))$ for $-200 \leq t \leq 200$.

Out[5]=



Part II. The role of semigroup theory.

Definition

- A semigroup is a **left group** (A. H. Clifford, 1933) if $\forall\, x, y\ \exists!\, z$ with $zx = y$.
- A (non-empty) left ideal of a semigroup is an lg**-ideal** if, as a semigroup, it is a left group.
- An lg**-semigroup** is a semigroup which has an lg-ideal.

Dually, **right group**, rg**-ideal**, rg**-semigroup**.

Observe that if I, J are (non-empty, two-sided) ideals in a semigroup then IJ is again an ideal. Thus a semigroup can have at most one minimal ideal. If S has a minimal ideal it is called the **kernel** of S, written $K(S)$.

Green's equivalence relations in a semigroup:

- $x\,\mathcal{L}\,y$ if $x = y$ or $\exists\, t, u$ with $tx = y$, $uy = x$;
- $x\,\mathcal{R}\,y$ if $x = y$ or $\exists\, t, u$ with $xt = y$, $yu = x$;
- $\mathcal{D} = \mathcal{L} \vee \mathcal{R} = \mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L}$;
- $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

The **Clifford-Preston** eggbox picture of a $\mathcal{D}$-class:

a		c
d		b

The columns are the $\mathcal{L}$-classes. The rows are the $\mathcal{R}$-classes. The cells are the $\mathcal{H}$-classes. Every $\mathcal{H}$-class with an idempotent is a group.

Theorem The following hold in any lg-semigroup:

1. K(S) exists and is a single $\mathcal{D}$-class.
2. In the eggbox of $K(S)$, the columns are the lg-ideals of S and the rows are the rg-ideals of S. Each two lg-ideals and each two rg-ideals are isomorphic as semigroups.
3. All the cells (= $\mathcal{H}$-classes) are groups and all these groups are isomorphic.
4. If R is a row and L is a column, $RL = R \cap L$ is a group, whereas $LR = K(S)$.

In particular, every lg-semigroup is an rg-semigroup. The group of $K(S)$ defines the **Ellis group** of S.

Paragroups

The term "paragroup" is due to Hofmann and Mostert (1966). Other names include *Rees matrix semigroup* and *completely simple semigroup.*

Definition A **paragroup** is a semigroup isomorphic to $J \times G \times \Lambda$ where J, Λ are non-empty sets, G is a group, $[\cdot,\cdot] : \Lambda \times J \to G$ is any function and the multiplication is given by

$$(j, g, \lambda)\,(k, h, \mu) \;=\; (j,\; g\,[\lambda, k]\,h,\; \mu)$$

Theorem Let P be a semigroup. Then P is a paragroup if and only if there exists an lg-semigroup S with $P \cong K(S)$.

The construction of a paragroup structure of $K(S)$ for lg-semigroup S is as follows. Let $e^2 = e \in K(S)$. Define

$$\begin{aligned} J &= L_e, \quad \Lambda \;=\; R_e, \quad G = H_e \\ [\lambda, j] &= \lambda j \end{aligned}$$

The isomorphism $\psi : K(S) \to J \times G \times \Lambda$ is given by $\psi a = (u, eae, v)$ where u is the unit of H_{ae} and v is the unit of H_{ea}.

lg**-monads**

For any monad **D** in **Set**, the free algebra $D1$ on one generator is a monoid. The multiplication is

$$xy \;=\; (D1 \xrightarrow{y^{\#}} D1)\, x$$

Definition Monad **D** is an lg**-monad** if $D\emptyset = \emptyset$ and if $D1$ is an lg-monoid.

Definition An object in a category $\mathcal{C}$ is **coalescent** if each of its endomorphisms is an isomorphism. If $\mathcal{P}$ is a subcategory, $U \in \mathcal{P}$ is a **universal** $\mathcal{P}$-object if U is coalescent and each object in $\mathcal{P}$ admits a $\mathcal{P}$-morphism from U.

Evidentally, such U is unique up to isomorphism.

Theorem (R. Ellis, Universal Minimal, 1960). For an lg-monad $\mathbf{D}$, if U is an lg-ideal of $D1$ then U is a $\mathbf{D}$-subalgebra of $D1$ and U is the universal minimal $\mathbf{D}$-algebra.

Corollary Every non-empty $\mathbf{D}$-algebra for lg-monad $\mathbf{D}$ has a minimal subalgebra.

Proof Let X be an algebra, $x \in X$. Let $\psi : D1 \to X$ map the free generator to x. If $U \subset D1$ is the universal minimal algebra, $\psi(U)$ is minimal.

Proposition Let $\mathbf{D}$ be any monad and let X be one of its algebras. Then the unique homomorphism $D1 \to X^X$ to the product algebra mapping the free generator to id_X is a monoid homomorphism.

Thus the image of the above homomorphism, $E(X) \subset X^X$ is both a subalgebra and a submonoid. It is called the **enveloping semigroup** of X.

Theorem Let $\mathbf{D}$ be a monad, X an algebra and let $\mathbf{Set}^{\mathbf{E}}$ be the variety generated by X. The following hold.

1. If $\mathbf{D}$ is lg, so is $\mathbf{E}$.
2. (Manes, 1969) $E(X)$ is the free $\mathbf{E}$-algebra on one generator. Thus $E(X)$ is an lg-semigroup.

The following are due to Ellis.

Definition Let $\mathbf{D}$ be an lg-monad, X an algebra. Then

- X is **distal** if $E(X)$ is a group;
- X is **equicontinuous** if, on the product algebra $X \times X$

  $$(x, y)\, R\, (a, b) \;\Leftrightarrow\; (x, y) \in <(a, b)>$$

  is a congruence.
- $x, y \in X$ are **proximal**, xPy, if there exists $t \in K(E(X))$ with $tx = ty$.

If xPy, $\{t \in K(E(X)) : tx = ty\}$ always contains an lg-ideal, an idempotent in particular.

Theorem (Ellis 1960, J. Auslander 1960) The following are equivalent.

1. Proximal is an equivalence relation on X.
2. E(X) has a unique lg-ideal.
3. If xPy then $tx = ty$ for all $t \in K(E(X))$.

Theorem (J. Auslander, 1960) If $Y \subset <x>$ with Y minimal then there exists $y \in Y$ with $x\,P\,y$.

Theorem (J. Auslander, 1960) Let $<x>$ be minimal for all x. Let J be the set of idempotents of $K(E(X))$. Then for all x, $\{y : x\,P\,y\} = Jx$.

Part III. Generalizing compact Metric spaces.

Varieties

A **variety** is an equationally-definable class, e.g.

semigroups: $\Omega_2 = \{\cdot\}$, $x(yz) = (xy)z$.

monoid actions: $\Omega_1 = M$, $1x = x$, $(tu)x = t(ux)$.

Theorem (G. Birkhoff, 1938) If $\mathcal{V}$ is a variety and $\mathcal{A} \subset \mathcal{V}$ then $\mathcal{A}$ arises as a variety by imposing further equations (in the same operations) if and only if $\mathcal{A}$ is closed under subalgebras, products and quotients.

The variety generated by $\mathcal{A}$ is $QSP(\mathcal{A}$ and the equations are precisely those satisfied by $\mathcal{A}$.

Theorem (Zürich triples gang, 1960s) Varieties are precisely $\mathbf{Set}^{\mathbf{T}}$ for $\mathbf{T}$ a monad in **set**.

Compact metric spaces via universal algebra.

The most highly developed form of universal algebra is the *finitary* case –operations have finitely many variables. However, many important structures can be described using infinitary operations, not always in an obvious way.

For example, Isbell showed in 1982 that real commutatative C^*-algebras (with algebra maps of norm at most 1 and with the unit disc as the "underlying set") could be equationally described with five operations on the unit disc, the only infinitary one being $\sum 2^{-n} x^n$. The equations are precisely those satisfied by $\boldsymbol{R}$.

For compact metric spaces, every sequence has a convergent subsequence. How can we choose one?

Let an ultrafilter choose for us

Let βX be the set of ultrafilters on the set X. β is an endofunctor of the category **Set** of sets and functions: for $f : X \to Y$, let $(\beta f)\mathcal{U} = \{B \subset Y : f^{-1}B \in \mathcal{U}\}$.

Topologists write $f\mathcal{U}$ for $(\beta f)\mathcal{U}$ and we will too.

Fix a non-principal ultrafilter $r \in \beta\omega\backslash\omega$. Such r will do the choosing for us. If X is any compact Hausdorff space, define an operation $\delta_r : X^\omega \to X$ by

$$fr \rightharpoondown \delta_r(f)$$

Thus X is a Σ-algebra where the signature Σ has exactly one operation, the ω-ary operation δ_r.

Definition Let $\mathcal{V}_r$ be the variety of Σ-algebras generated by all compact metric spaces.

To learn more about $\mathcal{V}_r$ we must enter the murky world of . . .

Coproduct preserving monads

Definition A functor $C : \mathbf{Set} \to \mathbf{Set}$ is CP (Coproduct Preserving for *binary* coproducts) if whenever $A \subset X$ with complement A',

$$CA \to CX \leftarrow CA'$$

is a coproduct.

Monad $\mathbf{C} = (C, \eta, \mu)$ is CP if C is.

Example Any subfunctor of β is CP.

Theorem For an algebra of a CP monad, the subalgebras form the closed sets of a topology.

- The clopen subsets of (CX, μ_X) are $\{CA : A \subset X\}$.
- Every compact Hausdorff space arises this way.
- $\boldsymbol{R}$ cannot arise this way.

For $\mathbf{T} \subset \boldsymbol{\beta}$ we set out to identify the $\mathbf{T}$-algebras.

Definition For T a subfunctor of β and X a topological space,

- $A \subset X$ is T**-closed** if $A \in \mathcal{U} \in TX$, $\mathcal{U} \to x \;\Rightarrow\; x \in A$.
- X is a T**-space** if every T-closed set is closed.
- X is T**-compact** if each ultrafilter in TX converges.
- X is T**-Hausdorff** if each ultrafilter in TX converges to at most one point.

Theorem (Manes 2010) For $\mathbf{T} \subset \boldsymbol{\beta}$, $\mathbf{Set}^{\mathbf{T}}$ is a full subcategory of topological spaces, namely is the T-compact, T-Hausdorff T-spaces. The closed sets of an algebra coincide with the subalgebras.

Countable tightness

Recall that a space is **countably tight** if whenever $x \in \overline{A}$ there exists countable $C \subset A$ with $x \in \overline{C}$.

Metrizable $\Rightarrow$ first countable $\Rightarrow$ Fréchet $\Rightarrow$ sequential $\Rightarrow$ countably tight.

For a compact Hausdorff space, metrizable $\Leftrightarrow$ second countable.

From Robert Ellis, *The Furstenberg structure theorem*, 1978: "It has been fourteen years since Furstenberg proved his beautiful structure theorem for metrizable minimal distal flows. Since then there have been many attempts to do without the assumption that the phase space of the flow be metrizable. These have only been partially successful; some sort of countability assumption has always seemed necessary...".

Example $\beta_\omega X = \{\mathcal{U} \in \beta X : \exists \text{ countable } C \in \mathcal{U}\}$ is a submonad.

- β_ω-space = countably tight space.
- β_ω-compact spaces were called *ultracompact* by (Bernstein 1970). We note that a space is β_ω-compact if and only if for each open cover, each countable subset has a finite subcover.
- For every submonad **T** of $\boldsymbol{\beta}_\omega$, every compact metric space is a **T**-algebra and every **T**-algebra is countably tight.

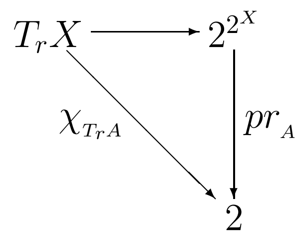
Observe that for $S \subset T \subset \beta$, every S-space is a T-space.

Our model for "generalized compact metrizable spaces", then, is **T**-algebras for **T** a submonad of $\boldsymbol{\beta}_\omega$.

The monad $\mathbf{T}_r$.

Theorem (Manes 2011) The variety $\mathcal{V}_r$ of Σ-algebras generated by the r-convergence of compact metric spaces is precisely $\mathbf{Set}^{\mathbf{T}_r}$ where $\mathbf{T}_r$ is the submonad of $\boldsymbol{\beta}_\omega$ generated by r.

Corollary The equations defining $\mathcal{V}_r$ are precisely those satisfied by the operation $\chi_r : 2^\omega \to 2$.
Proof



Theorem (R. Börger, 1987) For CP C there exists a unique natural transformation $C \to \beta$ which is a monad map if C is a monad.

For a proof, see Exercise 18.

It follows that no two distinct submonads of β can be isomorphic.

Theorem There are 2^{2^ω} non-isomorphic varieties $\mathcal{V}_r$.

First proof This past March, Neil Hindman told me that whatever one wants to count in $\beta\omega$, the answer is 2^{2^ω}.

Second proof Garcia-Fereirra (1993) showed that there are 2^{2^ω} Comfort types. It turns out that a Comfort type is no more and no less than a submonad of form $\mathbf{T}_r$. Different submonads are not isomorphic by Börger's theorem. Thus their categories of algebras are not isomorphic (over **Set**).

Dynamic monads

Given a submonad $\mathbf{T}$ of $\boldsymbol{\beta}$ and a discrete monoid M, the class of all continuous left actions $M \times X \to X$ with X a $\mathbf{T}$-algebra is the algebras of a monad $\mathbf{T}_M$ with $T_M X = T(M \times X)$. This is well known and arises from a Beck distributive law, a detail which is important in some of the proofs.

We are interested in CP varieties of $\mathbf{T}_M$-algebras. By "CP variety" we mean that the monad for such a variety is itself CP.

Definition A monad is **pre-dynamic** if its algebras form a CP variety of some $\mathbf{T}_M$. It is **dynamic** if it is pre-dynamic and lg.

Examples of CP varieties of dynamic monads

- Distal;
- Equicontinuous;
- Putting a topology on M.

For the last one, M should be a T-space and $M \otimes X \to X$ is continuous where $\otimes$ is the product in the category of T spaces.

What are the exponentiable T-spaces? A continuous monoid homomorphism $M \to X^X$ is a natural idea.

Characterizing pre-dynamic monads

Definition For CP $\mathbf{C}$, factor the unique map into its image as

$$\mathbf{C} \xrightarrow{\rho} \mathbf{T} \xrightarrow{i} \boldsymbol{\beta}$$

The submonad $\mathbf{T}$ of $\boldsymbol{\beta}$ is called the **topological part** of $\mathbf{C}$.

Proposition Say that a topological space is **completely separated** if distinct points can be separated by clopen sets. Then for each set X, $\rho_X : CX \to TX$ is the completely separated reflection of CX and is a quotient mapping.

See Exercises 20, 23.

Theorem A monad is pre-dynamic if and only if it is CP and admits a monad map from its topological part.

Open question Is every CP monad pre-dynamic?

Part IV. Finding examples.

It is high time to show dynamic monads exist! We want the phase space to be countably tight which does not happen classically.

Definition

- A semigroup S is **compactible** if it is non-empty and admits a compact Hausdorff topology with all right translations $\rho_t u = ut$ continuous.
- A semigroup S is **tight** if it is non-empty and if for every function $f : \Gamma \to S$, $\{\Gamma \xrightarrow{f} S \xrightarrow{\lambda_t} S : t \in S\}$ is closed in the topology of pointwise convergence induced by discrete S. ($\lambda_t u = tu$).
- Right cancellable $\Rightarrow$ tight.

Definition In a semigroup S, $\Delta \subset S$ is a **division set** if

$$\forall\, x \in S\ \forall\, y \in S^1\ \exists z \in S\ \forall\, \delta \in \Delta\ \ zx\delta = y\delta$$

Hierarchy theorem (Manes 1969) Compactible $\Rightarrow$ tight $\Rightarrow$ there exists a minimal left ideal and a maximal division set $\Rightarrow$ lg.

No implications are reversible. A countable group is tight but not compactible.

It is well known that for any semigroup S, βS is canonically a compactible semigroup. This is major in Ellis' work and is the subject of a whole book (N. Hindman and D. Strauss, 1998).

Now let $\mathbf{D}$ be any monad in **Set**. Given $m : X \times X \to X$, freeness induces a natural lift

$$X \times DX \to DX$$

$$DX \times DX \xrightarrow{\widehat{m}} DX$$

Note that $\widehat{m}(\cdot, v)$ is a $\mathbf{D}$-homomorphism for each $v \in DX$.

In a 2007 paper on distributive laws, Mulry and I observed that when $\mathbf{D}$ is a commutative monad, every *linear* equation (e.g. $x(yz) = (xy)z$, $xy = yx$, same variables both sides, no repetition) which holds for m also holds for $\widehat{m}$.

But $\boldsymbol{\beta}$ is not commutative. Remarkably,

the associative law lifts for all monads.

βS is rarely a commutative semigroup, even if S is.

If $\mathbf{E}$ is a submonad of $\mathbf{D}$ then ES is a subsemigroup of DS.

For the particular case of the pre-dynamic monad $\mathbf{T}_M$,

$$TM = T(M \times 1) = T_M 1$$

is already known to be a monoid and it is the same monoid as above.

Thus $\boldsymbol{\beta}_M$ is a dynamic monad, the standard case.

A dynamic monad with all spaces countably tight

In any semigroup, the idempotents are partially ordered by

$$e \leq f \quad \Leftrightarrow \quad ef = e = fe$$

A minimal idempotent in this order is called **primitive**.

Theorem In an lg-semigroup S, the primitive idempotents are precisely the units of the $\mathcal{H}$-classes of $K(S)$.

Definition A semigroup is **weakly left cancellative** if for all x, y the set $\{z : xz = y\}$ is finite (possibly empty).

The next result can be used to show that many dynamic monads exist.

Theorem If M is weakly left cancellative then there exists a primitive idempotent $p \in \beta M \backslash M$ and the monad $(\mathbf{T}_p)_M$ is a dynamic monad.

β_ω-actions have almost periodic points

Theorem Let $\boldsymbol{R} \times X \to X$ be a continuous left action with X a non-empty $\boldsymbol{\beta}_\omega$-algebra. Then X has an almost periodic point.

Proof comments:

$\beta_\omega \boldsymbol{Z}$ is different topological space than $\beta \boldsymbol{Z}$ but, as TM is always a submonoid of βM, the two are exactly the same monoid.

As $\beta_\omega \boldsymbol{Z}$ is free on one generator and an lg-semigroup, it has a universal minimal set U, namely any lg-ideal.

Restrict the given $\boldsymbol{R} \times X \to X$ to $\boldsymbol{Z} \times X \to X$. Using U, there exists x with $\overline{\boldsymbol{Z}x}$ minimal.

$\overline{\boldsymbol{Z}x}$ minimal $\Rightarrow$ x is discretely almost periodic under $\boldsymbol{Z}$ –this uses the covering property of β_ω-compactness.

$\boldsymbol{Z} \subset \boldsymbol{R}$ is a closed, syndetic normal subgroup so, by a well-known "inheritance theorem", x is almost periodic under $\boldsymbol{R}$.

Three papers

- Varieties generated by compact metric spaces (submitted)
- r-algebras (submitted)
- Monads in topology (2010)

I'm wearing them around my neck.

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Mega-congratulations

for surviving

yet another

Manes tutorial!