A Differential Model Theory for Resource Lambda Calculi - Part II

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Outline

1. Cartesian (Closed) Differential Categories [BEM'10, Man'11]
   - Typed and untyped models of differential/resource λ-calculus
   - Soundness & Completeness
   - Examples
2. Building Differential Categories [LMM'11]
   - A general recipe:
     From SMC(C) $\mapsto$ Differential SMC(C) $\mapsto$ Differential C(C)C
3. Game Semantics [LMM'11]
   - Instances: Categories of Games, The relational semantics
   - Full abstraction for relational semantics w.r.t. Resource PCF
4. Conclusions

Differential Categories

The differential λ-calculus inspired researchers working on category theory.

- Aim: Axiomatize a differential operator $D(-)$ categorically.

Differential categories

Blute, Cockett and Seely proposed:

- BCS'06: (monoidal) differential categories
  - point of view too fine
- BCS'09: Cartesian differential categories
  - sound and complete for their term calculus,
  - lack of higher order functions!

Not enough for modeling the differential λ-calculus!!!

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Left Additive Categories

We have sums of λ-terms and the application is **left**-linear:

$$(\sum_i s_i)t = \sum_i s_i t \qquad s(\sum_i t_i) \neq \sum_i s t_i$$

We need a left-additive sum on morphisms!

A category **C** is **left-additive** if:

- each homset has a structure of commutative monoid $(\mathbf{C}(A, B), +_{AB}, 0_{AB})$,
- $f; (g + h) = (f; g) + (f; h)$ and $f; 0 = 0$.

When f satisfies also $(g + h); f = (g; f) + (h; f)$ and $0; f = 0$ it is called **additive**. (weak form of *linearity*)

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Cartesian (Closed) Left-additive Categories

A category **C** is **Cartesian left-additive** if:

- **C** is a left-additive category,
- it is Cartesian (= it has products),
- all projections and pairings of additive maps are additive.

A category **C** is **Cartesian closed left-additive** if:

- **C** is Cartesian left-additive,
- it is a ccc ($\Lambda(-) =$ curry, $ev =$eval),
- it satisfies $\Lambda(f+g) = \Lambda(f) + \Lambda(g)$ and $\Lambda(0) = 0$.
  (implies $\langle f+g, h\rangle; ev = \langle f, h\rangle; ev + \langle g, h\rangle; ev$)

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> The Cartesian (closed) structure does not behaves automatically well with the left-additive enrichment!

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Cartesian Differential Categories

Cartesian differential operator:

$$D^{\times}\ \frac{f : A \to B}{D^{\times}(f) : \underline{A} \times A \to B}$$

Satisfying:

D1. $D^{\times}(f+g) = D^{\times}(f) + D^{\times}(g)$ and $D^{\times}(0) = 0$

D2. $\langle h+k, v\rangle ; D^{\times}(f) = \langle h, v\rangle ; D^{\times}(f) + \langle k, v\rangle ; D^{\times}(f)$ and $\langle 0, v\rangle ; D^{\times}(f) = 0$

D3. $D^{\times}(Id) = \pi_1$, $D^{\times}(\pi_1) = \pi_1 ; \pi_1$ and $D^{\times}(\pi_2) = \pi_1 ; \pi_2$

D4. $D^{\times}(\langle f, g\rangle) = \langle D^{\times}(f), D^{\times}(g)\rangle$

D5. $D^{\times}(g; f) = \langle D(g), \pi_2 ; g\rangle ; D^{\times}(f)$

D6. $\langle\langle g, 0\rangle, \langle h, k\rangle\rangle ; D^{\times}(D^{\times}(f)) = \langle g, k\rangle ; D^{\times}(f)$

D7. $\langle\langle 0, h\rangle, \langle g, k\rangle\rangle ; D^{\times}(D^{\times}(f)) = \langle\langle 0, g\rangle, \langle h, k\rangle\rangle ; D^{\times}(D^{\times}(f))$

Subcategory of Linear Morphisms

Linear morphisms

A morphism f is *linear* if its differential is constant: $D^{\times}(f) = \pi_1; f$.

$$f \text{ linear} \Rightarrow f \text{ additive}$$
$$f \text{ linear} \not\Leftarrow f \text{ additive}$$

Partial differentiation

Imagine we just want to differentiate $f : C \times A \to B$ on C.

- $D^{\times}(f) : \underline{(C \times A)} \times (C \times A) \to B$,
- we can obtain the partial derivative $D^{\times}_{C}(f) : \underline{C} \times (C \times A)$ by "zeroing out" the A component,

$$\underline{C} \times (C \times A) \xrightarrow{\langle Id_C, 0^C_A \rangle \times Id_{C \times A}} \underline{(C \times A)} \times (C \times A) \xrightarrow{D^{\times}(f)} B$$

$D^{\times}_{C}(f) = \langle Id_C, 0^C_A \rangle \times Id_{C \times A}; D^{\times}(f)$

Cartesian closed differential category

Cartesian closed differential category [Bucciarelli-Ehrhard-Manzonetto'10]

C is a *Cartesian closed differential category* if:

- **C** is a Cartesian differential category,
- it is Cartesian closed left-additive,
- it satisfies the following rule:

For all $f : C \times A \to B$:

$$D^{\times}(\Lambda(f)) = \Lambda(\langle \pi_1 \times 0_A, \pi_2 \times Id_A \rangle; D^{\times}(f))$$

Intuitively, the following methods for partial derivatives are equivalent:

1. Do $\Lambda(f) : C \to [A \to B]$ then apply $D^{\times}(\cdot)$,
2. Use the trick by "zeroing out" the A component as before.

Categorical Interpretation (simply typed)

Define $f \star g = \langle\langle 0, \pi_1; g\rangle, Id\rangle; D^\times(f)$:

$$\star\frac{f : C \times A \to B \quad g : C \to A}{f \star g : C \times A \to B}$$

Define $[\![\Gamma \vdash_D s : \sigma]\!] = [\![s^\sigma]\!]_\Gamma : [\![\Gamma]\!] \to [\![\sigma]\!]$ by:

- $[\![x^\sigma]\!]_{\Gamma;x:\sigma} = \pi_2$,
- $[\![y^\tau]\!]_{\Gamma;x:\sigma} = \pi_1; [\![y^\tau]\!]_\Gamma$,
- $[\![(sT)^\tau]\!]_\Gamma = \langle [\![s^{\sigma\to\tau}]\!]_\Gamma, [\![T^\sigma]\!]_\Gamma\rangle; ev$,
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Soundness

If **C** is a Cartesian closed differential category, then

$$Th_D(\mathbf{C}) = \{s = t \mid \Gamma \vdash_D s : \sigma \quad \Gamma \vdash_D t : \sigma \quad [\![s^\sigma]\!]_\Gamma = [\![t^\sigma]\!]_\Gamma\}$$

is a *differential* λ*-theory* (i.e., it contains $=_D$ and it is contextual).

Soundness Theorem [BEM'10]

Cartesian closed differential categories are sound models for:

- Simply Typed Differential λ-calculus

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We can interpret the Resource Calculus by translation:

$$[\![\Gamma \vdash_R M : \sigma]\!] = [\![(M^\circ)^\sigma]\!]_\Gamma$$

we get that $Th_R(\mathbf{C})$ is a *resource* λ*-theory.*

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- Simply Typed Resource Calculus (by translation $(-)^\circ$)

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Every model of the differential λ-calculus is also a model of the resource calculus

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Cartesian closed differential categories are sound models for:

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- Simply Typed Resource Calculus (by translation $(-)^\circ$)

Modelling the Untyped Differential λ-Calculus

As in the λ-calculus we need a *reflexive object* U (i.e., $[U \Rightarrow U] \lhd U$) in a Cartesian closed differential category, **but it is not enough!**

Linear reflexive object

A reflexive object $[U \Rightarrow U] \lhd U$ is *linear* if

$$\mathcal{A} : U \to [U \Rightarrow U], \qquad \lambda : [U \Rightarrow U] \to U$$

are both *linear* maps.

We modify the interpretation in the obvious way:

- $[\![x_i]\!]_{\vec{x}} = \pi_i$,
- $[\![sT]\!]_{\vec{x}} = \langle \mathcal{A} \circ [\![s]\!]_{\vec{x}}, [\![T]\!]_{\vec{x}} \rangle ; \mathrm{ev}$,
- $[\![\lambda y.s]\!]_{\vec{x}} = \Lambda([\![s]\!]_{\vec{x},y}) ; \lambda$, with $y \notin \vec{x}$
- $[\![D(s) \cdot t]\!]_{\vec{x}} = \Lambda(\Lambda^-([\![s]\!]_{\vec{x}} ; \mathcal{A}) \star [\![t]\!]_{\vec{x}}) ; \lambda$.

Soundness and Completeness

Theorem [Manzonetto'11]

Cartesian closed differential categories are *sound* and *complete* models of the differential λ-calculus.

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Completeness Theorem for λ-calculus [Scott $\|$ Koymans]

Every λ-theories T is the theory of a reflexive object in a suitable CCC $\mathbf{C}_T$.

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Scott-Koymans completeness proof for regular λ-calculus (take T):

$$\Lambda/T \xrightarrow{\text{Karoubi envelope}} \mathbf{C}_T = \mathcal{K}(\Lambda/T) \qquad \ni \mathbf{I} \text{ reflexive obj.}$$

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Theorem [Manzonetto'11]

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The category $\mathbf{C}_T$ is described as follows (where $M; N = \lambda z.N(Mz)$):

Objects:	$\{A \in \Lambda/T \mid A; A = A\}$
Hom(A,B):	$\{f \in \Lambda/T \mid A; f; B = f\}$
Identity:	$\mathrm{Id}_A = A$
Composition:	$f; g$

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Idea: encode the categorical constructions via λ-terms:

Products:

$$\langle f, g\rangle = \lambda z.[fz, gz] \qquad\qquad \pi_i^{A_1,A_2} = p_i; A_i$$

where $[M, N] = \lambda y.yMN$, $p_1 = \lambda x.xK$ and $p_2 = \lambda x.xK^*$ (Church encoding).

Exponents:

$$\begin{array}{lcl} [A \Rightarrow B] & = & \lambda z.A; z; B \\ \mathrm{ev}_{A,B} & = & \lambda z.B(p_1 z(A(p_2 z))) \\ \Lambda(f) & = & \lambda xy.f[x, y] \end{array}$$

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Hom(A,B):	$\{f \in \Lambda/T \mid A; f; B = f\}$
Identity:	$\mathrm{Id}_A = A$
Composition:	$f; g$

Additive structure: the sum in the category is the sum of terms.

Completeness Theorem for differential λ-calculus [Manzonetto'11]

Every **differential** λ-theory T is the theory of a **linear** reflexive object in a suitable Cartesian closed **differential** category $\mathbf{C}_T$.

Soundness and Completeness

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Problem: Church's pairing is not additive!

$$[M{+}M', N{+}N'] = \lambda y.y(M{+}M')(N{+}N') \neq \lambda y.yMN{+}\lambda y.yM'N' = [M, N]{+}[M', N']$$

$$\Rightarrow \langle f + f', g + g' \rangle \neq \langle f, g \rangle + \langle f', g' \rangle$$

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Solution: set-like encoding exploiting resource consciousness

$$[M, N] = \lambda y.M + \lambda y.D(y) \cdot N \qquad p_1 = \lambda x.x0 \qquad p_2 = \lambda x.(D(x) \cdot \mathbf{I})00$$

We need to restrict to theories with idempotent sum!

Completeness Theorem for differential λ-calculus [Manzonetto'11]

Every **differential** λ-theory T with idempotent sum is the theory of a **linear** reflexive object in a suitable Cartesian closed **differential** category $\mathbf{C}_T$.

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The encoding of the differential operator is straightforward:

$$D^{\times}(f) = \lambda z.B((D(f) \cdot (A(p_1 z)))(A(p_2 z))) : A \times A \to B$$

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Completeness proof for differential λ-calculus (take T):

$$\Lambda_d/T \xrightarrow{\text{Karoubi envelope}} \mathbf{C}_T = \mathcal{K}(\Lambda_d/T) \qquad \ni \mathbf{I} \text{ linear refl. obj.}$$

$$M \in \Lambda_d^o \xmapsto{[\![\cdot]\!]} [\lambda x.\widetilde{M}]_T$$

$\widetilde{(\cdot)} : D(M) \cdot N \mapsto \lambda y.(D(\widetilde{M}) \cdot \widetilde{N})y$, the identity otherwise.

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We add equations $\mathrm{Th}(\mathbf{I}) \vdash D(M) \cdot N = \lambda y.(D(M) \cdot N)y$
(differentially extensional axiom)

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We restrict to differentially extensional T

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All models arising naturally are differentionally extensional, also the non-extensional ones!

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Examples

1. **MFin**: Finiteness Spaces. [Ehrhard] [No reflexive objects in it!]
2. **MRel**: The Relational Semantics. The coKleisli of **Rel** $+ M_f(-)$.
3. Its variations with “Infinite Multiplicities” [Carraro-Ehrhard-Salibra]
4. Convenient differential category [Blute-Ehrhard-Tasson]
5. Categories of games. . . [Laird-Manzonetto-MCusker] (see later)

Main Example: The Relational Semantics

MRel

- Objects: sets,
- Morphisms: $\mathbf{MRel}(A, B) = \mathcal{P}(M_f(A) \times B)$ (relations between $M_f(A)$ and B).

Given $f : A \to B$ we can define:

$$D(f) = \{((([a], m), b) \mid (m \uplus [a], b) \in f\} : A \times A \to B.$$

Theorem [BEM'10]

The category **MRel** is a Cartesian closed differential category.

Corollary

MRel is a model of the simply typed differential λ-calculus.

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Untyped Models in **MRel**

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  Construction:

$$D_0 = \emptyset \qquad D_{n+1} = M_f(D_n)^{(\omega)} \qquad \mathcal{D} = \cup_{n\in\omega} D_n$$

$$(m_1, m_2, m_3, \ldots) \in \mathcal{D} \iff (m_1, (m_2, m_3, \ldots)) \in M_f(\mathcal{D}) \times \mathcal{D} = [\mathcal{D} \Rightarrow \mathcal{D}]$$

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Full Abstraction [Bucciarelli-Carraro-Ehrhard-Manzonetto'11]

$\mathcal{D}$ is a fully abstract model of the resource calculus with tests.

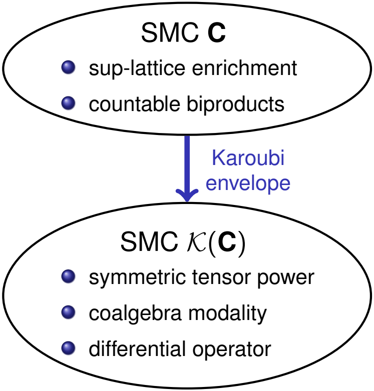
$$[\![M]\!] \subseteq [\![N]\!] \iff [\forall C(\cdot)\ C(M) \Downarrow \Rightarrow C(N) \Downarrow]$$

Building Differential Categories

SMC **C**

- sup-lattice enrichment
- countable biproducts

Building Differential Categories



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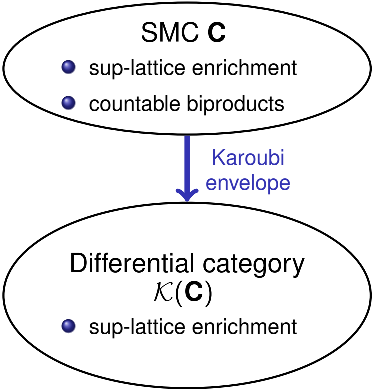

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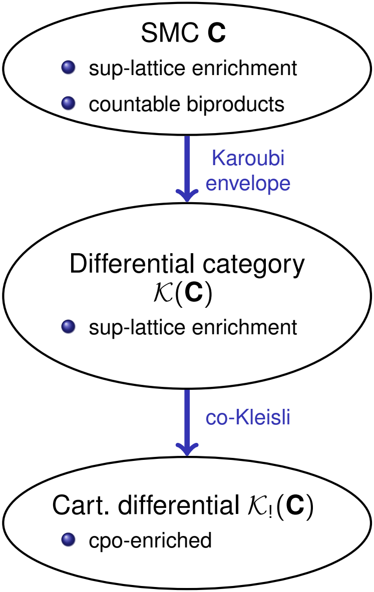

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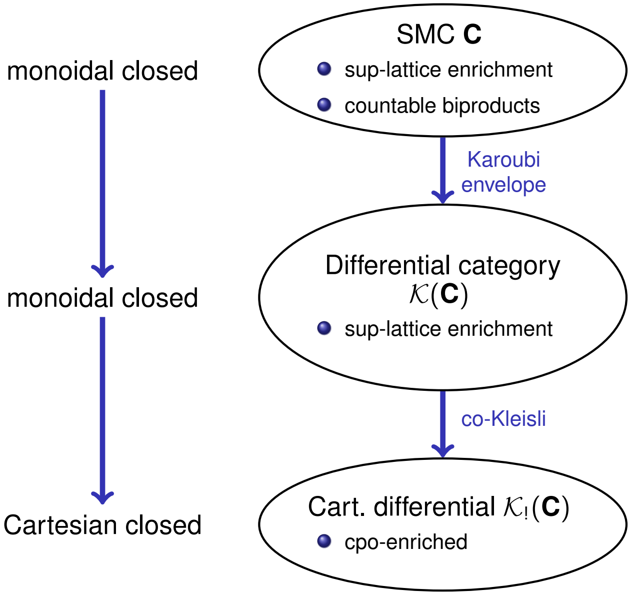

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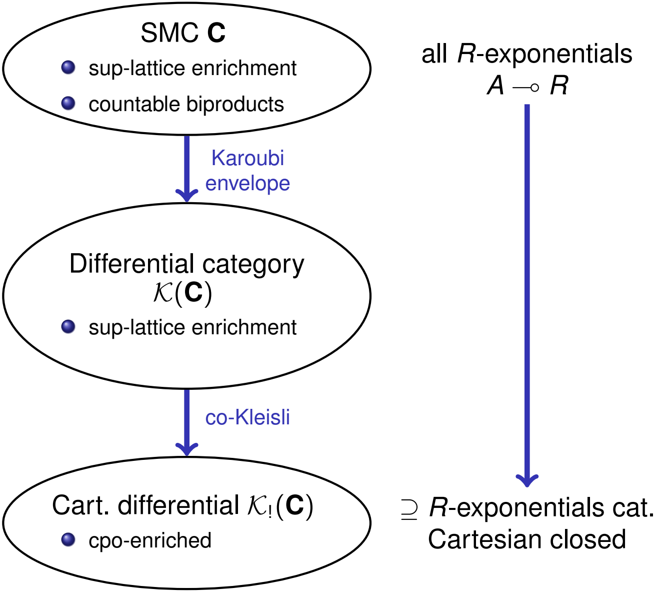

General recipe for coalgebra modalities

Take an SMC. In presence of the equalizer A^n (as in many models of LL):

$$A^{\otimes n} \overset{\text{equalizer}}{\dashrightarrow} A^{\otimes n} \underset{\vdots\ n!\ \text{permutations}}{\rightrightarrows} A^{\otimes n}$$

A^n gives the n-th layer of the free commutative monoid $!A$.

Coalgebra modality

$$!A = \prod_{n\in\omega} A^n$$

This works for symmetric monoidal categories where the tensor distributes over the infinite product:

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Always works in the Karoubi Envelope

$\mathcal{K}(\mathbf{C})$ (for sup-semilattice enriched **C** with infinite biproducts):

Obj:	(A, f), $A \in \mathbf{C}$, $f : A \to A$ idempotent
Hom((A,f),(B,g)):	$h \in \mathbf{C}(A, B)$, such that $f; h = h \ \wedge\ g; h = h$

We can construct the *symmetric tensor product* A^n using $\Theta_{A,n} = \sum_{\sigma \in \mathfrak{S}_n} \sigma$

$$(A^{\otimes n}, f^{\otimes n}; \Theta_{A,n}) \xrightarrow{f^{\otimes n}; \Theta_{A,n}} (A^{\otimes n}, f^{\otimes n}) \ \vdots\ n! \text{ permutations} \ (A^{\otimes n}, f^{\otimes n})$$

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What if I don't have sup-lattice enrichment or infinite biproducts?

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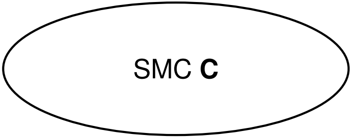

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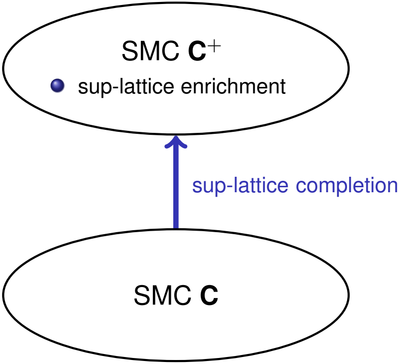

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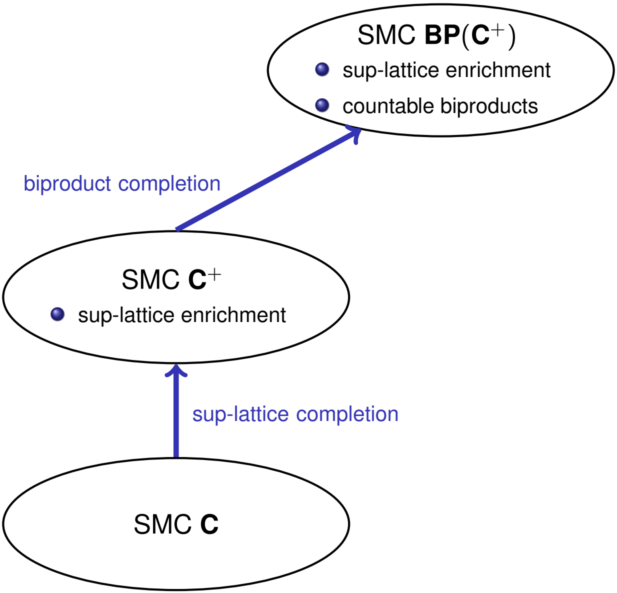

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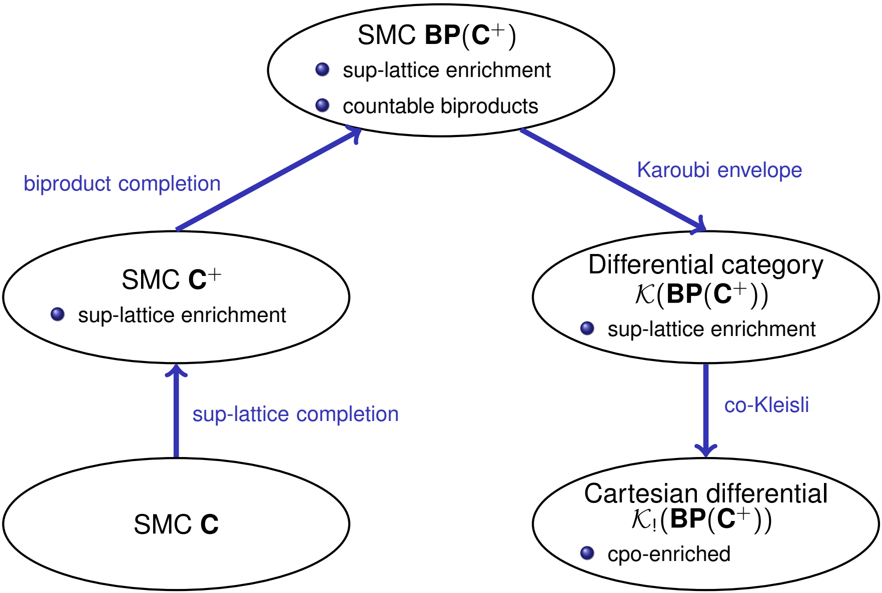

Easy way to Build Models of Resource PCF

Resource PCF

- Simply Typed Resource Calculus
- + Constants for natural numbers (of ground type *Nat*)
- + Fixed Point Combinator **Y**,
- + "If-zero?" instruction.

Operational semantics: Linear Head Reduction.

Denotational Models

- Cartesian Closed Differential Categories,
- + Fixpoints,
- + (weak) Natural number object.

Remark on the construction

We recover **MRel** and build categories of games.

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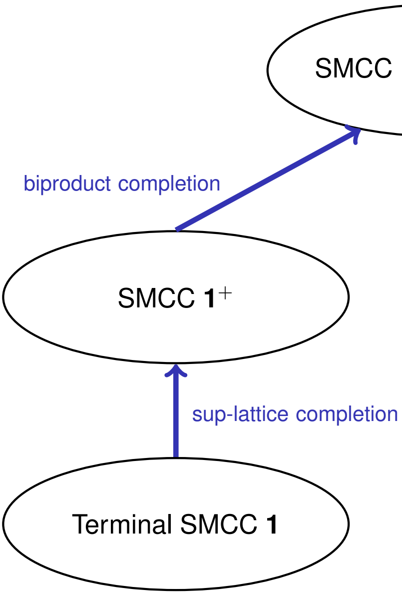

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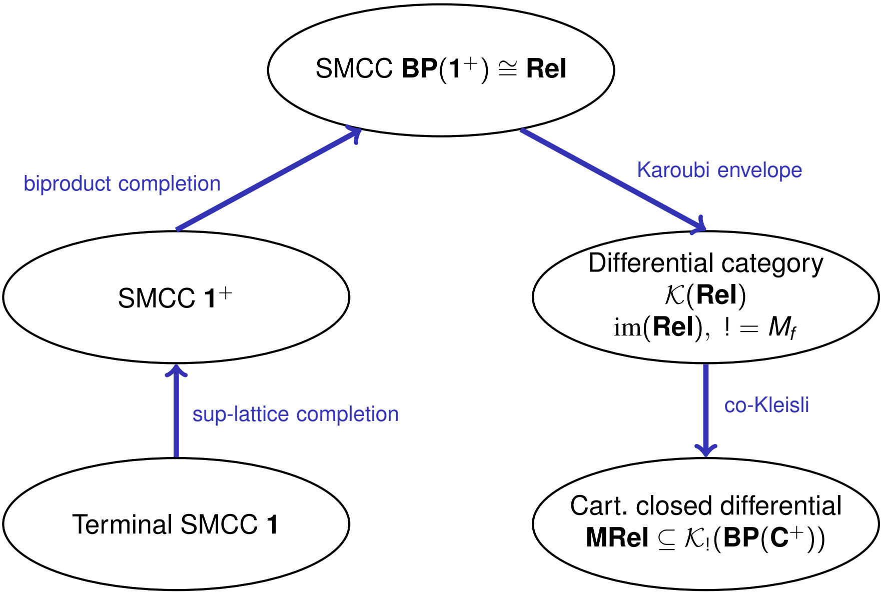

Playing with games. . .

Arena

An *arena* A is given by:

- a finite bipartite forest over two sets of *moves* M_A^P and M_A^O,
- an edge relation $\vdash$ (enabling),
- a labelling function (*Q*uestions and *A*nswers).

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Given an arena A, a **strategy** over A is:

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Category **G** of games [Harmer-McCusker]

The category **G**:

- Objects: arenas whose roots are all *O*-moves,
- Hom(A,B): strategies on $A^\perp \uplus B$,
- Composition: usual "parallel composition plus hiding" construction,
- Identities: copycat strategies.
- $\otimes$ = disjoint union,
- $A \multimap B$ = Arena B with a copy of $A^\perp$ attached below each initial move (to maintain the forest structure)

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The subcategory $\mathbf{G}^{\otimes}$

Remark

Every object of **G** can be endowed with a comonoid structure

Comonoid homomorphisms = maps whose choice of move at any stage depends only on the current thread.

Subcategory of comonoid homomorphisms

Cartesian closed category $\mathbf{G}^{\otimes}$.

Theorem [Harmer-McCusker'99]

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Given a complete play (playing in $A^{\perp}$ at least once):

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Derivative combinator

$D^{\times}(\sigma) = \{s' \in \mathrm{comp}(A^{\perp} \uplus A^{\perp} \uplus B) \mid s' \text{ is a derivative of some } s \in \sigma\}$

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Analysis of $\mathbf{G}^{\otimes}$

Path:

- non-repeating enumeration of all moves,
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- the first move is by O,
- moves alternate polarity thereafter.

Exhausting strategy: set of even paths satisfying P-visibility.

Category of Exhausting games

The category **EG**:

- Objects: finite O-rooted arenas,
- Hom(A,B): exhausting strategy on $A^{\perp} \uplus B$,
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- has all R-exponentials $A \multimap R$, where R = arena with a single O-move.

Let's apply (the second part of) our construction…

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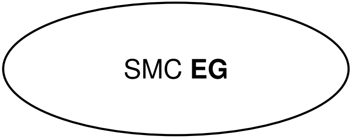

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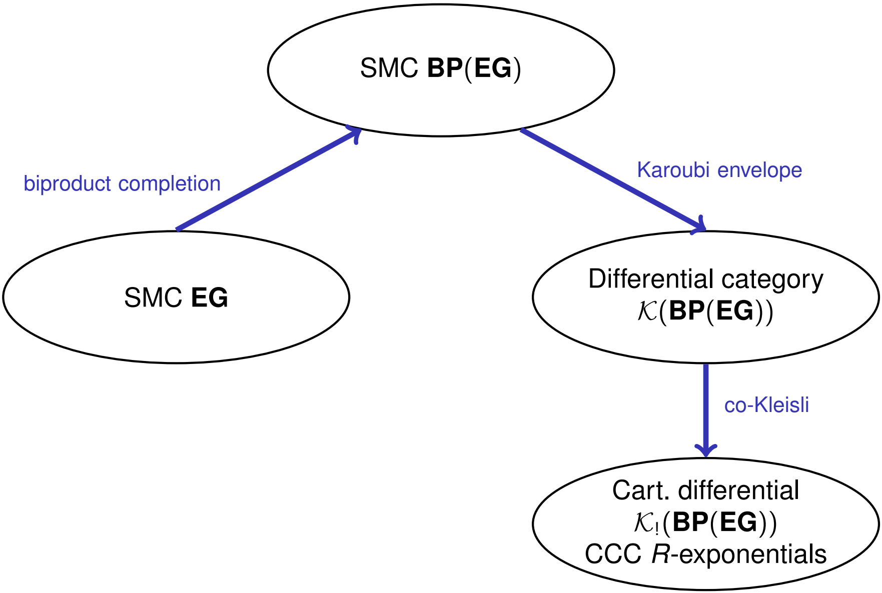

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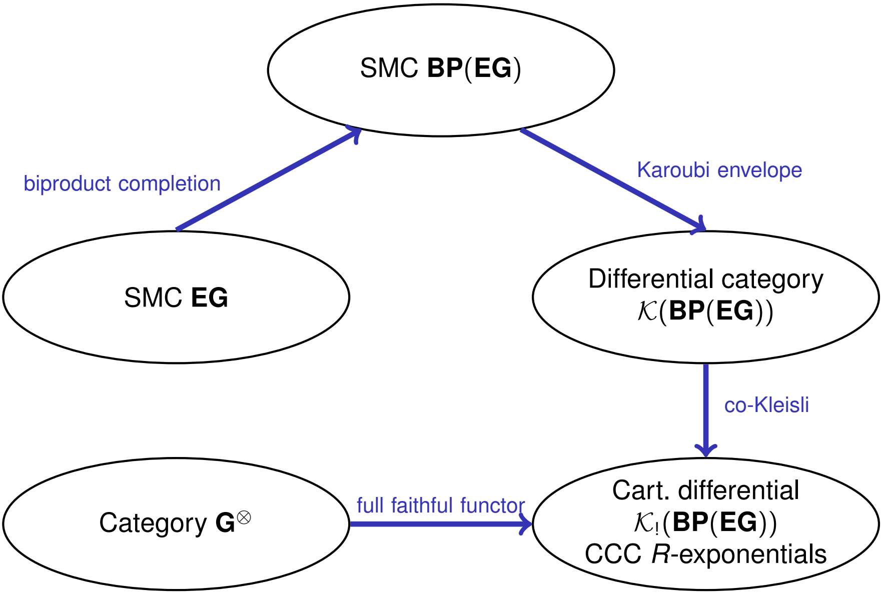

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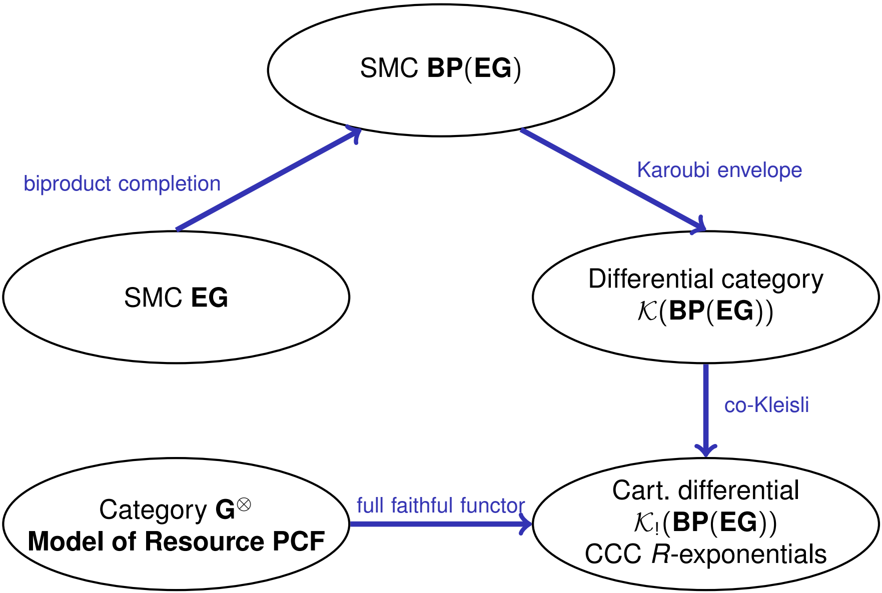

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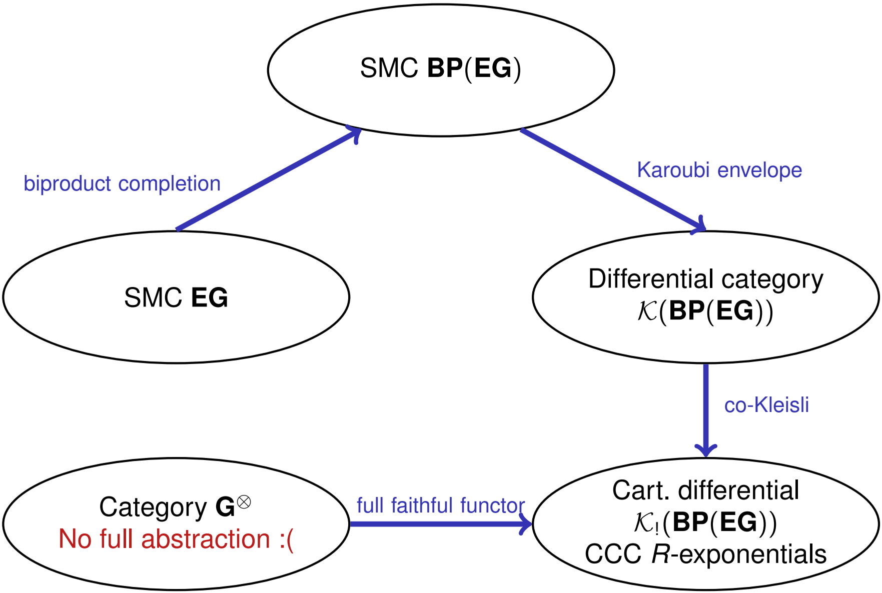


Refining

We consider a **switch** equivalence relation:

$$\sim \;=\; \text{least equivalence relation } s \cdot o \cdot p \cdot o' \cdot p' \cdot t \sim s \cdot o' \cdot p' \cdot o \cdot p \cdot t$$

$\sim$-closed strategies = causal independent strategies

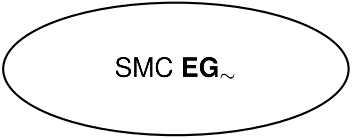
The category $\mathbf{EG}_{\sim}$

- Objects: O-rooted arenas,
- Hom(A,B): deterministic $\sim$-closed strategies.

Remark

Nondeterminism arise with the sup-lattice completion.

Let's try again. . .



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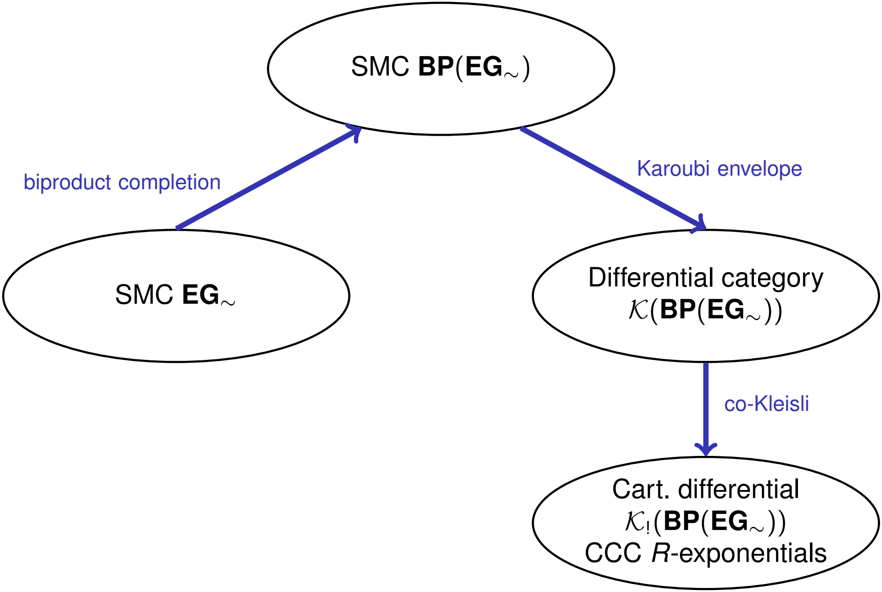

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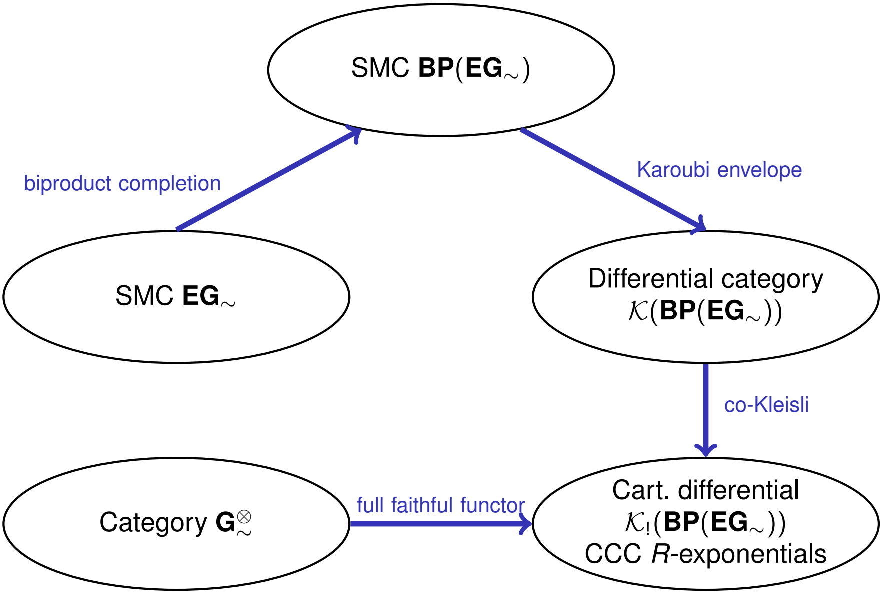

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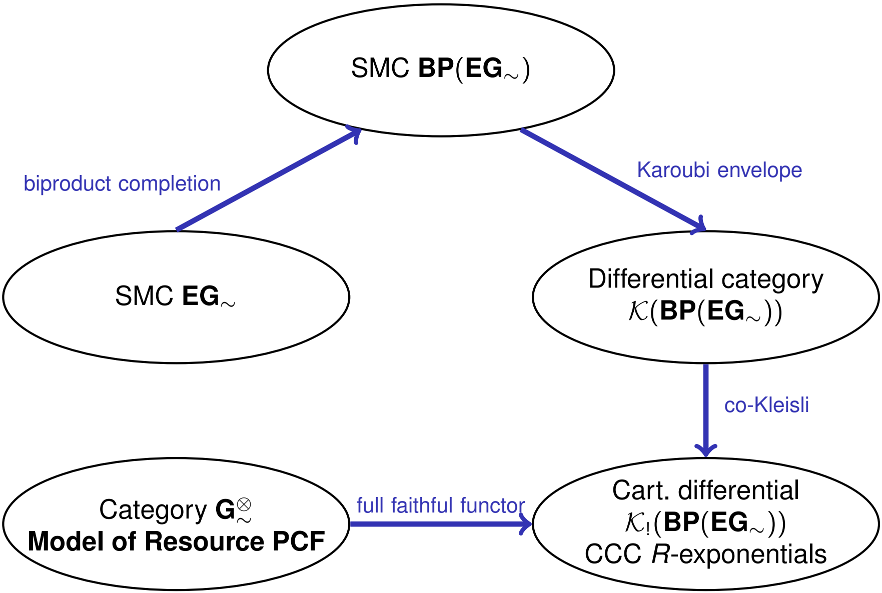

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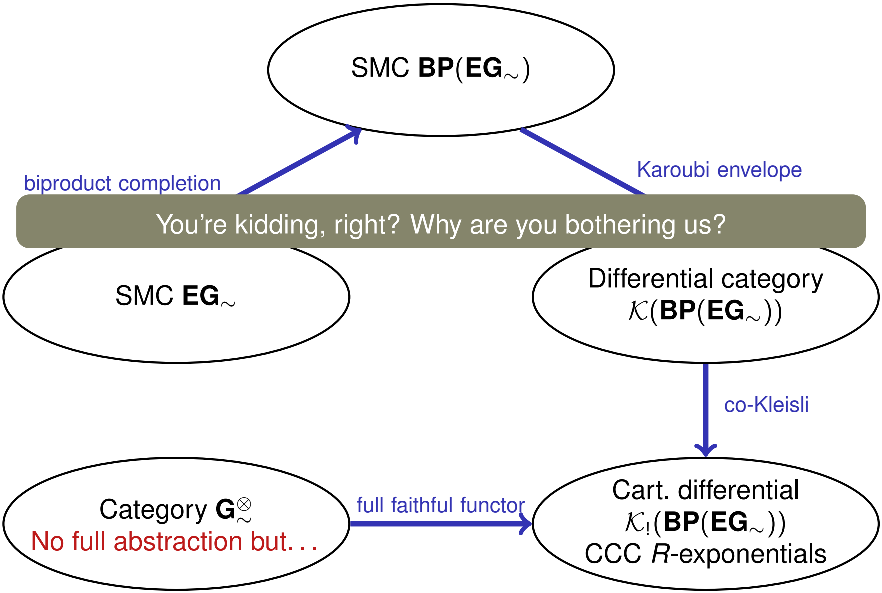

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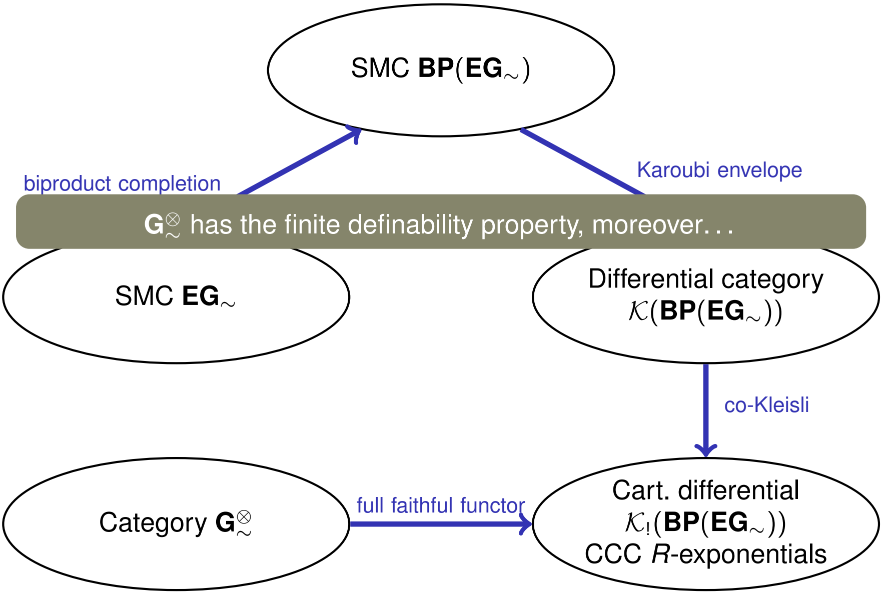

Full Abstraction for Resource PCF

$\mathbf{EG}^{QA}_{\sim}$: Arenas where each question enables a unique answer

(Unique) Full Functor $\top : \mathbf{EG}^{QA}_{\sim} \to \mathbf{1}$

Full Abstraction for Resource PCF

$\mathbf{G}^{\otimes}_{\sim}$ embeds in $\mathcal{K}_!(\mathbf{BP}((\mathbf{EG}^{QA}_{\sim})^+))$ by construction!

(Unique) Full Functor $\top : \mathbf{EG}^{QA}_{\sim} \to \mathbf{1}$

Full Functor $F : \mathcal{K}_!(\mathbf{BP}((\mathbf{EG}^{QA}_{\sim})^+)) \to \mathcal{K}_!(\mathbf{Rel})$ (lifted)

Full Abstraction for Resource PCF

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Finite Definability Property

Both $\mathbf{G}^{\otimes}_{\sim}$ and **MRel** have the finite definability property. (def. $\mathbf{G}^{\otimes}_{\sim} \Rightarrow$ def. **MRel**)

Corollary: MRel is Fully Abstract for Resource PCF

Let M, N of type A.

$$[\![M]\!] \subseteq [\![N]\!] \iff [\forall C(\cdot)\ C(M) \Downarrow \Rightarrow C(N) \Downarrow]$$

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Finite Definability Property

Both $\mathbf{G}^{\otimes}_{\sim}$ and **MRel** have the finite definability property. (def. $\mathbf{G}^{\otimes}_{\sim} \Rightarrow$ def. **MRel**)

Corollary: **MRel** is Fully Abstract for Resource PCF

Let M, N of type A.

$$[\![M]\!] \subseteq [\![N]\!] \iff [\forall C(\cdot)\ C(M) \Downarrow \Rightarrow C(N) \Downarrow]$$

Proof (sketch)

- Suppose $\exists a \in [\![M]\!] - [\![N]\!]$,
- By finite definability $\{(a, 0)\} : A \to \mathrm{Nat}$ denotes some term $x : A \vdash C(x)$,
- Hence $[\![C(M)]\!] = [\![\mathrm{zero}]\!]$ and $[\![C(N)]\!] = \emptyset$,
- Thus $C(M) \Downarrow$ but $C(N) \not\Downarrow$ □

Conclusions

We have:

- Defined Cartesian closed differential categories,
- Shown they are sound and complete models of untyped differential λ-calculus,
- Provided a construction for building differential categories,
- Applied it to categories of games and categories of sets and relations,
- shown that **MRel** is fully abstract for Resource PCF.

Thanks for your attention!

Questions?