Linear functors and modal logic

R.A.G. Seely

John Abbott College
& McGill University

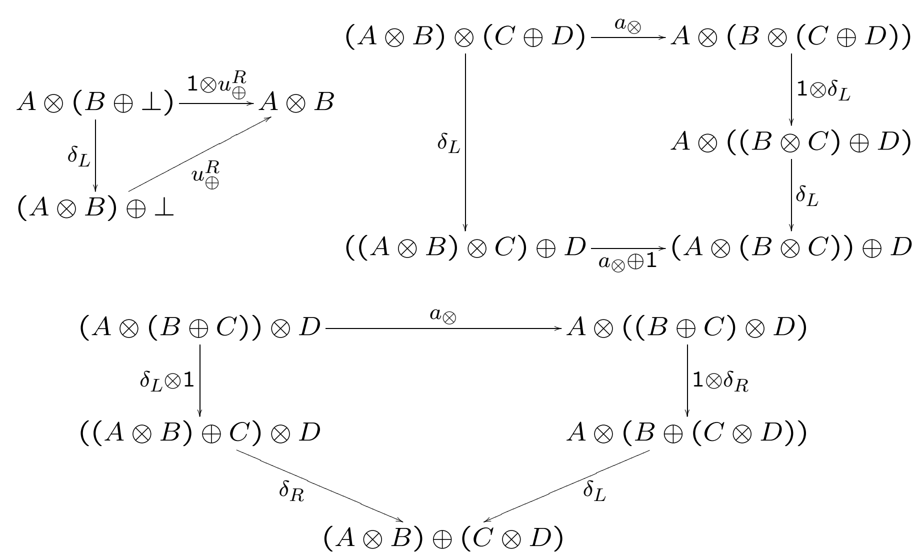
- An extension of an idea from a paper [Blute, Cockett, Seely; MSCS 2002]
  - Modal logic given by a linear functor (a special case of *"the logic of linear functors"*)
- Based on an "abandoned" project [Sadrzadeh, Cockett, Seely, 2009–2010, intended for MFPS 2010]
  - Adjoint modal pairs (think two varieties of "possibly" and "necessarily") (as given in "positive tense logic" of Prior)
  - Relational models of such modal logic (using some ideas of Hermida, IMLA 2002)

A linearly distributive category is a category $\mathbf{X}$ with two associative tensors $\otimes, \oplus$ (and their units $\top, \bot$) which are strong (costrong) with respect to each other, as indicated by these natural transformations:

$$\delta_L: A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C \quad \text{and}$$

$$\delta_R: (A \oplus B) \otimes C \longrightarrow A \oplus (B \otimes C)$$

subject to "obvious" coherence conditions (as is usual for tensorial strength, we want these strengths/linear distributions to be well-behaved with respect to the unit and associativity isos, as well as with each other):



Given linearly distributive categories $\mathbf{X}, \mathbf{Y}$, a linear functor $F\colon \mathbf{X} \longrightarrow \mathbf{Y}$ consists of:

- a pair of functors $F_\otimes, F_\oplus\colon \mathbf{X} \longrightarrow \mathbf{Y}$ so that $F_\otimes$ is monoidal with respect to $\otimes$, and $F_\oplus$ is comonoidal with respect to $\oplus$:

$$\begin{array}{ll} m_\top\colon \top \longrightarrow F_\otimes(\top) & m_\otimes\colon F_\otimes(A) \otimes F_\otimes(B) \longrightarrow F_\otimes(A \otimes B) \\ n_\bot\colon F_\oplus(\bot) \longrightarrow \bot & n_\oplus\colon F_\oplus(A \oplus B) \longrightarrow F_\oplus(A) \oplus F_\oplus(B) \end{array}$$

- natural transformations (called "linear strengths"):

$$\begin{array}{l} \nu_\otimes^R\colon F_\otimes(A \oplus B) \longrightarrow F_\oplus(A) \oplus F_\otimes(B) \\ \nu_\otimes^L\colon F_\otimes(A \oplus B) \longrightarrow F_\otimes(A) \oplus F_\oplus(B) \\ \nu_\oplus^R\colon F_\otimes(A) \otimes F_\oplus(B) \longrightarrow F_\oplus(A \otimes B) \\ \nu_\oplus^L\colon F_\oplus(A) \otimes F_\otimes(B) \longrightarrow F_\oplus(A \otimes B) \end{array}$$

satisfying the following coherence conditions:

$$\begin{array}{ccc} F_\otimes(\bot \oplus A) & \xrightarrow{F_\otimes(u_\oplus^L)} & F_\otimes(A) \\ \downarrow{\scriptstyle \nu_\otimes^R} & & \uparrow{\scriptstyle u_\oplus^L} \\ F_\oplus(\bot) \oplus F_\otimes(A) & \xrightarrow[n_\bot \oplus 1]{} & \bot \oplus F_\otimes(A) \end{array}$$

$$\begin{array}{ccc} F_\otimes((A \oplus B) \oplus C) & \xrightarrow{F_\otimes(a_\oplus)} & F_\otimes(A \oplus (B \oplus C)) \\ \downarrow{\scriptstyle \nu_\otimes^R} & & \downarrow{\scriptstyle \nu_\otimes^R} \\ F_\oplus(A \oplus B) \oplus F_\otimes(C) & & F_\oplus(A) \oplus F_\otimes(B \oplus C) \\ \downarrow{\scriptstyle n_\oplus \oplus 1} & & \downarrow{\scriptstyle 1 \oplus \nu_\otimes^R} \\ (F_\oplus(A) \oplus F_\oplus(B)) \oplus F_\otimes(C) & \xrightarrow[a_\oplus]{} & F_\oplus(A) \oplus (F_\oplus(B) \oplus F_\otimes(C)) \end{array}$$

$$\begin{array}{ccc} F_\otimes((A \oplus B) \oplus C) & \xrightarrow{F_\otimes(a_\oplus)} & F_\otimes(A \oplus (B \oplus C)) \\ \downarrow{\scriptstyle \nu_\otimes^L} & & \downarrow{\scriptstyle \nu_\otimes^R} \\ F_\otimes(A \oplus B) \oplus F_\oplus(C) & & F_\oplus(A) \oplus F_\otimes(B \oplus C) \\ \downarrow{\scriptstyle \nu_\otimes^R \oplus 1} & & \downarrow{\scriptstyle 1 \oplus \nu_\otimes^L} \\ (F_\oplus(A) \oplus F_\otimes(B)) \oplus F_\oplus(C) & \xrightarrow[a_\oplus]{} & F_\oplus(A) \oplus (F_\otimes(B) \oplus F_\oplus(C)) \end{array}$$

$$\begin{array}{ccc} F_\otimes(A) \otimes F_\otimes(B \oplus C) & \xrightarrow{1 \otimes \nu_\otimes^R} & F_\otimes(A) \otimes (F_\oplus(B) \oplus F_\otimes(C)) \\ \downarrow{\scriptstyle m_\otimes} & & \downarrow{\scriptstyle \delta_L} \\ F_\otimes(A \otimes (B \oplus C)) & & (F_\otimes(A) \otimes F_\oplus(B)) \oplus F_\otimes(C) \\ \downarrow{\scriptstyle F_\otimes(\delta_L)} & & \downarrow{\scriptstyle \nu_\oplus^R \oplus 1} \\ F_\otimes((A \otimes B) \oplus C) & \xrightarrow[\nu_\otimes^R]{} & F_\oplus(A \otimes B) \oplus F_\otimes(C) \end{array}$$

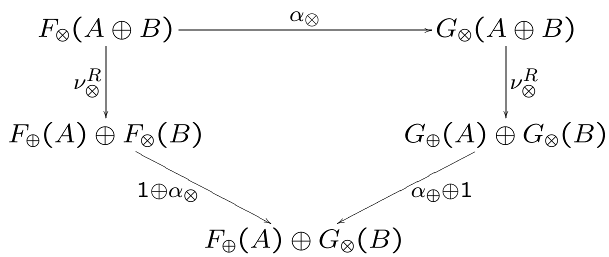
$$\begin{array}{ccc} F_\otimes(A) \otimes F_\otimes(B \oplus C) & \xrightarrow{1 \otimes \nu_\otimes^L} & F_\otimes(A) \otimes (F_\otimes(B) \oplus F_\oplus(C)) \\ \downarrow{\scriptstyle m_\otimes} & & \downarrow{\scriptstyle \delta_L} \\ F_\otimes(A \otimes (B \oplus C)) & & (F_\otimes(A) \otimes F_\otimes(B)) \oplus F_\oplus(C) \\ \downarrow{\scriptstyle F_\otimes(\delta_L)} & & \downarrow{\scriptstyle m_\otimes \oplus 1} \\ F_\otimes((A \otimes B) \oplus C) & \xrightarrow[\nu_\otimes^L]{} & F_\otimes(A \otimes B) \oplus F_\oplus(C) \end{array}$$

Of course, all this is much easier to "see" using a graphical calculus with "linear functor boxes", but for a change (!) I won't use them in this talk . . .

Given linear functors $F, G: \mathbf{X} \longrightarrow \mathbf{Y}$, a linear transformation $\alpha: F \longrightarrow G$ consists of a pair:

- $\alpha_\otimes$, a monoidal natural transformation $F_\otimes \longrightarrow G_\otimes$
- $\alpha_\oplus$, a comonoidal natural transformation $G_\oplus \longrightarrow F_\oplus$.

These must satisfy the "obvious" coherence conditions:



(and dual conditions)

Key example: Basic linear modal logic

Consider a linear functor $F: \mathbf{X} \longrightarrow \mathbf{X}$; we'll write $\Box$ for $F_\otimes$ and $\Diamond$ for $F_\oplus$. A complete description of the modal logic one obtains from this is in [BCS 2002], but here are some highlights:

$$\begin{array}{rcl} \nu^L_\otimes: \Box(A \oplus B) & \longrightarrow & \Box A \oplus \Diamond B \\ m_\otimes: \Box A \otimes \Box B & \longrightarrow & \Box(A \otimes B) \end{array}$$

In a classical setting, these would be equivalent to

$$\begin{array}{rcl} \Box(A \Rightarrow B) & \longrightarrow & (\Box A \Rightarrow \Box B) \\ \Box A \wedge \Box B & \longrightarrow & \Box(A \wedge B) \end{array}$$

the first being "normality" of the logic, and the second being one half (the linear half!) of the standard isomorphism

$$\Box A \wedge \Box B \longleftrightarrow \Box(A \wedge B)$$

In "the" process calculus (*e.g.* Hennessy & Milner 1985), the following rule is basic:

$$\frac{A_1, A_2, \cdots, A_m, B \vdash C_1, C_2, \cdots, C_n}{\Box A_1, \Box A_2, \cdots, \Box A_m, \Diamond B \vdash \Diamond C_1, \Diamond C_2, \cdots, \Diamond C_n}$$

This rule holds in basic linear modal logic.

Our intention now is to generalize this logic, to include a second pair of modalities, $\blacksquare, \blacklozenge$, corresponding to a second linear functor $G: \mathbf{X} \longrightarrow \mathbf{X}$, $G_\otimes = \blacksquare$, $G_\oplus = \blacklozenge$. (In fact, we could generalise the situation to $F: \mathbf{X} \longrightarrow \mathbf{Y}$ and $G: \mathbf{Y} \longrightarrow \mathbf{X}$, but for simplicity, we shall not do that now.)

The key idea is that of a linear adjunction: Given two linear functors $F: \mathbf{X} \longrightarrow \mathbf{Y}$ and $G: \mathbf{Y} \longrightarrow \mathbf{X}$

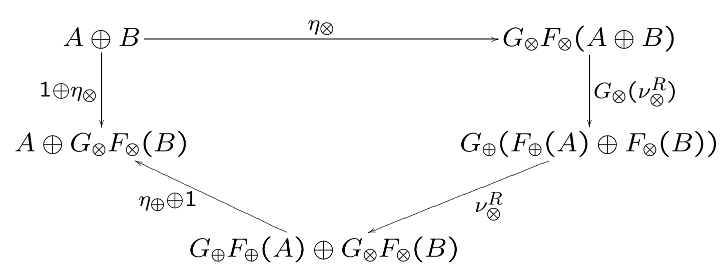
we say that F is left linear adjoint to G, $F \dashv G$ if this is so in the 2-categorical sense, in the 2-category **Lin** of linearly distributive categories, linear functors, and linear transformations.

In essence this means we have (ordinary) natural transformations

$$\eta_\otimes: A \longrightarrow G_\otimes F_\otimes(A) \quad \text{and} \quad \epsilon_\otimes: F_\otimes G_\otimes(A) \longrightarrow A$$

$$\eta_\oplus: G_\oplus F_\oplus(A) \longrightarrow A \quad \text{and} \quad \epsilon_\oplus: A \longrightarrow F_\oplus G_\oplus(A)$$

plus coherence conditions such as



In other words, we have ordinary adjunctions $F_\otimes \dashv G_\otimes$ and $G_\oplus \dashv F_\oplus$ which are "coherent" with respect to one another.

In terms of our modal logic, this gives us a logical structure with 4 modalities

$$\Box \dashv \blacksquare \quad \text{and} \quad \blacklozenge \dashv \Diamond$$

which have (among others) the following derivations:

Monoidal: $\Box A, \Box B \longrightarrow \Box(A \otimes B) \qquad \top \longrightarrow \Box\top$

and duals like: $\blacksquare A, \blacksquare B \longrightarrow \blacksquare(A \otimes B) \qquad \top \longrightarrow \blacksquare\top$

$\Diamond(A \oplus B) \longrightarrow \Diamond A, \Diamond B \qquad \Diamond\bot \longrightarrow \bot$

$\blacklozenge(A \oplus B) \longrightarrow \blacklozenge A, \blacklozenge B \qquad \blacklozenge\bot \longrightarrow \bot$

Strength: $\Box(A \oplus B) \longrightarrow \Diamond A, \Box B \qquad \blacklozenge A, \blacksquare B \longrightarrow \blacklozenge(A \otimes B)$ (etc)

Adjoints: $A \longrightarrow \blacksquare\Box A \qquad \Box\blacksquare A \longrightarrow A$

$\blacklozenge\Diamond A \longrightarrow A \qquad A \longrightarrow \Diamond\blacklozenge A$

All together: $\Box(\blacksquare A \otimes \blacksquare B) \longrightarrow A \otimes B \qquad A \oplus B \longrightarrow \blacksquare(\Diamond A \oplus \Diamond B)$

$\blacklozenge(\Diamond A \otimes \Box B) \longrightarrow A \otimes B \qquad A \oplus B \longrightarrow \blacksquare(\Diamond A \oplus \Diamond B)$

$\top \longleftrightarrow \Box\top \qquad \bot \longleftrightarrow \blacklozenge\bot$ (not iso)

Now - this is not what Mehrnoosh had in mind . . .

She wanted to find a semantics for positive tense logic, which required something like the following "twisted" adjoints:

Given a linear functor G, we can construct G^{op}, which has the same objects and arrows (but regarded in the opposite direction), and which switches the $\otimes$ and $\oplus$, including such things as $G_\otimes$ and $G_\oplus$ (so $G_\otimes^{\mathrm{op}} = G_\oplus$ and $G_\oplus^{\mathrm{op}} = G_\otimes$). Then what is now wanted is that G^{op} be left linear adjoint to F, so that $G_\oplus \dashv F_\otimes$ and $F_\oplus \dashv G_\otimes$, or in terms of the usual modal operators, that

$$\blacklozenge \dashv \Box \quad \text{and} \quad \Diamond \dashv \blacksquare$$

(There are some sticky "issues" with this, as one can see if one tries to insert this structure in the 2-category **Lin**. But we'll pass over this in silence for now . . .)

"Recall" that a linear bicategory (Cockett, Koslowski, Seely, MSCS 2000) is a bicategory whose 1- and 2-cells have linearly distributive structure "typed" by the 0-cells (so a 1-object linear bicategory is just a LDC). A $*$-linear bicategory is a linear bicategory which has a "nice" duality (this is actually a surprisingly subtle matter, and anyone interested in it should look up the CKS paper for the details).

In a linear bicategory, a linear adjunction between 1-cells: $A \dashv\!\!\!\dashv B$ for $A\colon X \longrightarrow Y, B\colon Y \longrightarrow X$, is given by 2-cells $\top_X \longrightarrow A \oplus B$ and $B \otimes A \longrightarrow \bot_Y$, satisfying obvious ("triangle") identities.

The *-linear bicategory we'll use is **Rel**, consisting of sets, relations (where tensor is relational composition, and par its deMorgan dual), ordered by inclusion. (This could be generalized, of course.)

In **Rel**, every 1-cell A has a 2-sided linear adjoint, which (today) we'll denote by A^*.

We recall the "subobject" fibration $\mathcal{P}$ (over **Sets**) of predicates $\varphi(x)$ over sets X (*i.e.* subsets of X). In a canonical way, this extends to a fibration $\mathcal{P}$ over **Rel**: for a set X, the fibres are still X-predicates; for a relation $R\colon X \not\to Y$, *i.e.* $X \xleftarrow{\pi} R \xrightarrow{\pi'} Y$, the "inverse image" map is $\pi'^{*}; \Sigma_{\pi}\colon \mathcal{P}(Y) \longrightarrow \mathcal{P}(R) \longrightarrow \mathcal{P}(X)$.

This takes a predicate $\varphi(y)$ over Y to the predicate $\exists y[xRy \wedge \varphi(y)]$ over X.

Following Hermida, we'll denote $\exists y[xRy \wedge \varphi(y)]$ over X by $\langle R\rangle\varphi$.

There is a dual operation $[R]\varphi = \forall y[xRy \longrightarrow \varphi(y)]$. This may be presented as $\pi'^{*}; \Pi_{\pi}\colon \mathcal{P}(Y) \longrightarrow \mathcal{P}(R) \longrightarrow \mathcal{P}(X)$.

In addition, we may do this with the converse relation R° ($xR^{\circ}y$ iff yRx) obtaining $\langle R^{\circ}\rangle, [R^{\circ}]\colon \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$.

[Here's an interpretation of these operators, in terms of a Kripke-style "possible worlds" semantics:]

$\langle R\rangle\varphi$	$\equiv$	φ will someday be true	$[R]\varphi$	$\equiv$	φ will always be true
$\langle R^{\circ}\rangle\varphi$	$\equiv$	φ was once true	$[R^{\circ}]\varphi$	$\equiv$	φ was always true

One can make this a bit more "concrete" as follows:

Note first that given a relation $R\colon X \not\to Y$ we can define its "lifting" to a function $\widetilde{R}\colon \mathcal{P}X \longrightarrow \mathcal{P}Y\colon A \mapsto \{y \in Y \mid \exists a \in A\, aRy\}$

Then, we can identify $\Diamond = \langle R\rangle$ with $\widetilde{R}$, and $\Box = [R]$ with its "Galois right adjoint". This is easily seen to exist; it is given by $[R](B) = \bigcup\{A \subseteq X \mid \widetilde{R}(A) \subseteq B\}$.

Similarly, we can identify $\blacklozenge = \langle R^{\circ}\rangle$ with $\widetilde{R^{\circ}}$, $\blacksquare = [R^{\circ}]$ with its Galois right adjoint (ditto).

It might be of interest to see what these operations are when R is a partial function f (so $dom(f) \subset X$ and $f\colon dom(f) \longrightarrow Y$). Then

$$(\langle f\rangle\varphi)(x) \equiv x \in dom(f) \wedge \varphi(f(x))$$

i.e. "$\varphi(f(x))$ if $x \in dom(f)$ and $\perp$ otherwise".

(The Scott-Fourman partial term substitution operator)

Similarly,

$$([f]\varphi)(x) \equiv x \in dom(f) \longrightarrow \varphi(f(x))$$

i.e. "$x \notin dom(f)$ or $\varphi(f(x))$".

(The Hoare weakest precondition operator)

The point? For any relation R, we have these adjunctions of modal operators:

$$\langle R \rangle \dashv [R^\circ] \quad \text{and} \quad \langle R^\circ \rangle \dashv [R]$$

and so it makes sense to make these identifications:

$\langle R \rangle$ with $\Diamond$, $[R]$ with $\Box$, $\blacklozenge$ with $\langle R^\circ \rangle$ and $\blacksquare$ with $[R^\circ]$

so $\Diamond \dashv \blacksquare$ and $\blacklozenge \dashv \Box$ (as is wanted for tense logic).

Why? This basically boils down to these facts:

$$\top \longrightarrow R^* \oplus R \quad \textit{i.e. } x = y \longrightarrow \forall z(\neg zRx \vee zRy)$$

and

$$R \otimes R^* \longrightarrow \bot \quad \textit{i.e. } \exists z(xRz \wedge \neg yRz) \longrightarrow x \neq y$$

(where xR^*y iff $\neg yRx$; $x\neg Ry$ iff $\neg xRy$, so $R^* = \neg R^\circ$)

So $R^* \dashv\!\vdash R$, which is the essence of $\langle R \rangle \dashv [R^\circ]$ in view of the following observation:

$$\langle R \rangle = R \otimes - \qquad [R^\circ] = \neg R^\circ \oplus -$$

and similarly:

$$\langle R^\circ \rangle = R^\circ \otimes - \qquad [R] = \neg R \oplus -$$

which is the clue as to how to generalize this to other $*$-linear bicategories.

But first, we note that there is a modification to what we have done, using structure less specific to **Rel**: since the operation "converse" R° is not generally available in $*$-linear bicategories, we notice that we could have also used the linear adjoints R^* instead. (This gives a slightly different "twisted" modal pair.)

So, there are two ways we could generalize our construction. First, we could build relational models on categories other than **Sets**. The construction above certainly extends to relations on a topos (though actually less is needed), as shown by Hermida.

But we can also use other $*$-linear bicategories than **Rel(S)** (for *e.g.* a topos **S**), if we slightly re-jig our example (using R and R^* as suggested above).

So for a $*$-linear bicategory **B**, and for any 1-cell A we can define modalities $\langle A \rangle = A \otimes -$ and $[A] = A \oplus -$. The key point then is that if A has a 2-sided adjoint A^* then $\langle A \rangle \dashv [A^*]$ and $\langle A^* \rangle \dashv [A]$.

Why? For the same reason as before, with **Rel**: we want a unit $I \longrightarrow [A^*]\langle A \rangle$, which for arbitrary X gives $X \longrightarrow A^* \oplus (A \otimes X)$ as follows:

$$X \longrightarrow \top \otimes X \longrightarrow (A^* \oplus A) \otimes X \longrightarrow A^* \oplus (A \otimes X)$$

using the unit of the linear adjunction (and linear distributivity).

Dually, we have the counit of the adjunction from the counit of the linear adjunction:

$$A \otimes (A^* \oplus X) \longrightarrow (A \otimes A^*) \oplus X \longrightarrow \bot \oplus X \longrightarrow X$$

[Coherence? An exercise for the audience . . .]

So there should be lots of examples of such "twisted" modal pairs, coming from linearly adjoint 1-cells in $*$-linear bicategories.