Proof nets for sum–product logic

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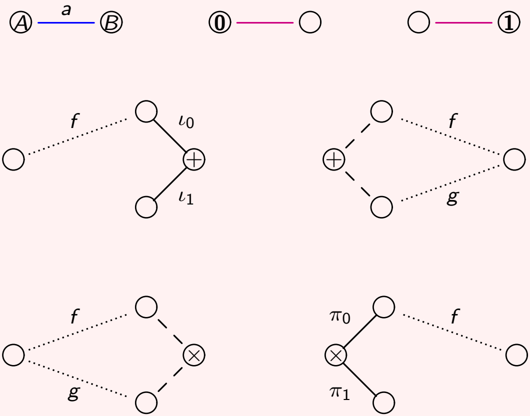
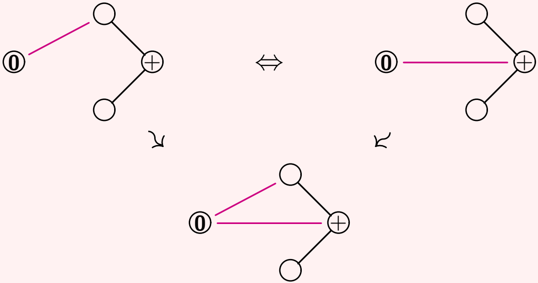
In part 2

- Recap
- The soundness proof

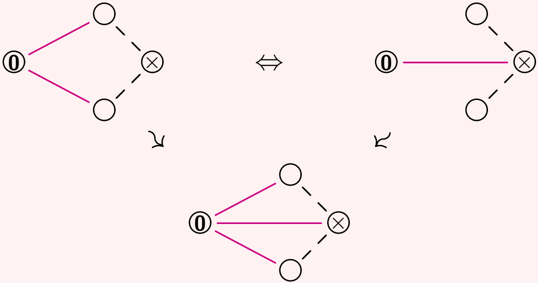
The full net calculus



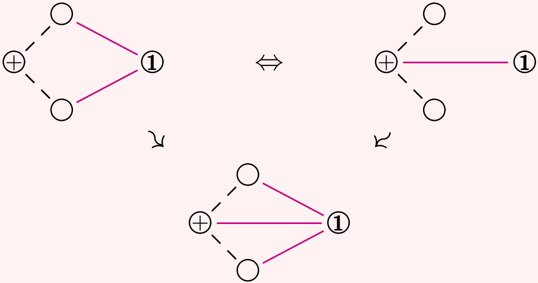
Equivalence and saturation



Equivalence and saturation



Equivalence and saturation



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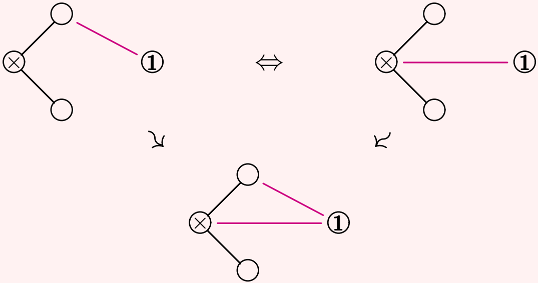

The soundness proof

To show:

$$\sigma R \; = \; \sigma S \qquad \Rightarrow \qquad R \; \Leftrightarrow \; S$$

The soundness proof: first intuition

Saturation allows induction on paths in $(\rightsquigarrow^*)$

$$R \rightsquigarrow R' \rightsquigarrow R'' \rightsquigarrow \ldots \rightsquigarrow \sigma R = \sigma S \leftsquigarrow \ldots \leftsquigarrow S'' \leftsquigarrow S' \leftsquigarrow S$$

For each step in $(\rightsquigarrow)$ there is a corresponding one in $(\Leftrightarrow)$

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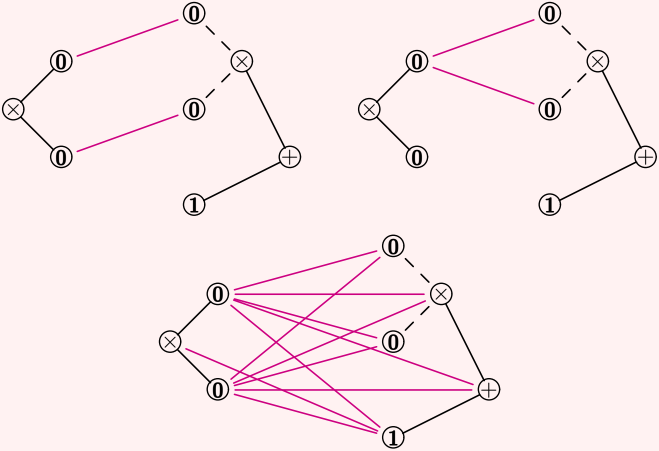
For each step in $(\rightsquigarrow)$ there is a corresponding one in $(\Leftrightarrow)$

$$R \Leftrightarrow R_0 \Leftrightarrow R_1 \Leftrightarrow \ldots \Leftrightarrow R_m \quad ?? \quad S_n \Leftrightarrow \ldots \Leftrightarrow S_1 \Leftrightarrow S_0 \Leftrightarrow S$$

But this only shifts the problem:
how to show that $\sigma R = \sigma S$ gives $R_m \Leftrightarrow S_n$?

'Desaturation' is not trivial

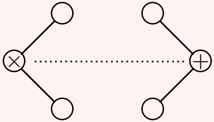


$S \subseteq \sigma R$ does not necessarily mean $\sigma S = \sigma R$

The soundness proof

To prove: $X \xrightarrow{R} Y \Leftrightarrow X \xrightarrow{S} Y$ given $\sigma R = \sigma S$

- One of X and Y is an atom or unit
- X is a coproduct or Y a product
- X is a product and Y a coproduct



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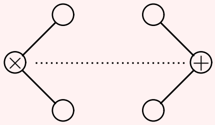

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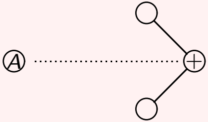
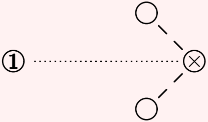
Atoms and units

$$A \xrightarrow{R,S} Y \qquad X \xrightarrow{R,S} A$$

$$\mathbf{1} \xrightarrow{R,S} Y \qquad X \xrightarrow{R,S} \mathbf{0}$$

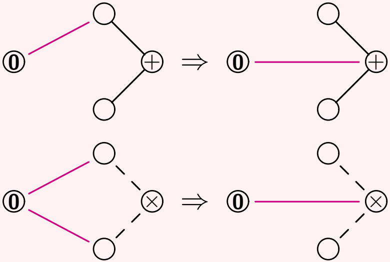




In these cases no rewrite rules apply, and $R = \sigma R = \sigma S = S$

Atoms and units

Nets corresponding to initial and terminal maps.



If $X = \mathbf{0}$ then

$X \xrightarrow{R} Y \quad \Rightarrow^*$  $^*\Leftarrow \quad X \xrightarrow{S} Y$

Similar for $Y = \mathbf{1}$

The soundness proof

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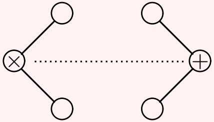

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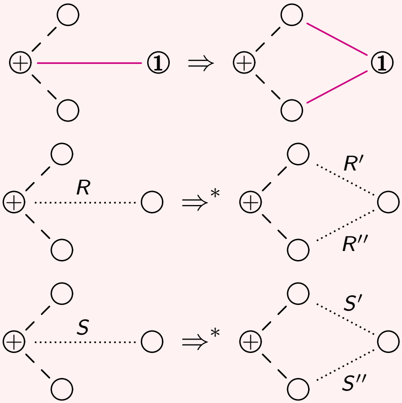
Coproduct source or product target



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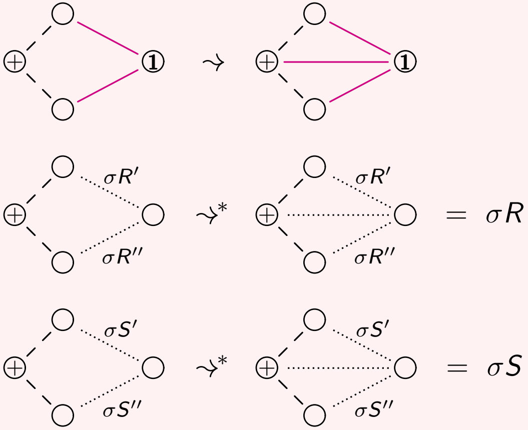
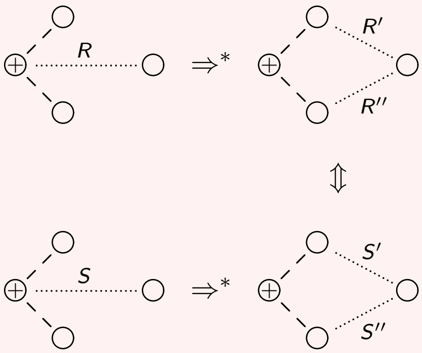


$\sigma R = \sigma S$ means that $\sigma R' = \sigma S'$ and $\sigma R'' = \sigma S''$

Coproduct source or product target

$$\sigma R = \sigma S \quad \text{means that} \quad \sigma R' = \sigma S' \quad \text{and} \quad \sigma R'' = \sigma S''$$



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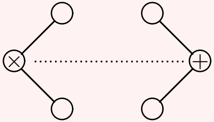

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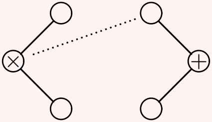
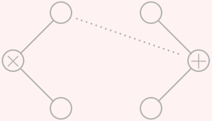
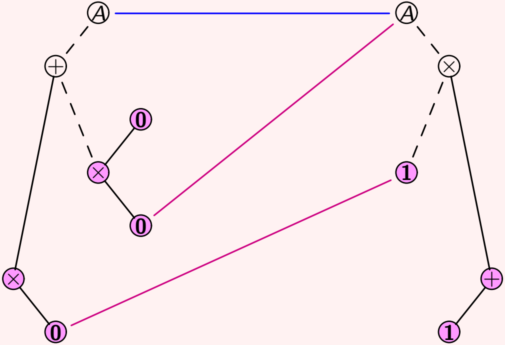
Nets from a product into a coproduct







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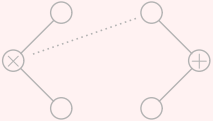

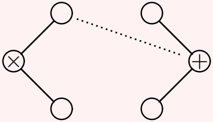

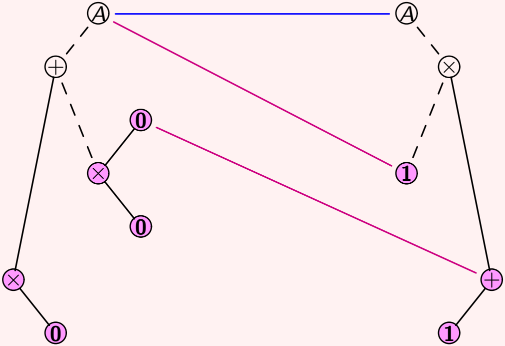

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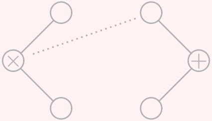

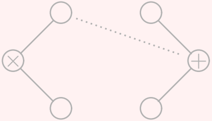

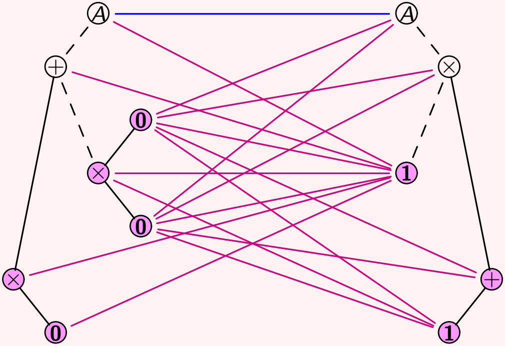

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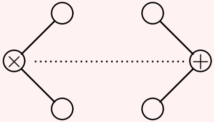

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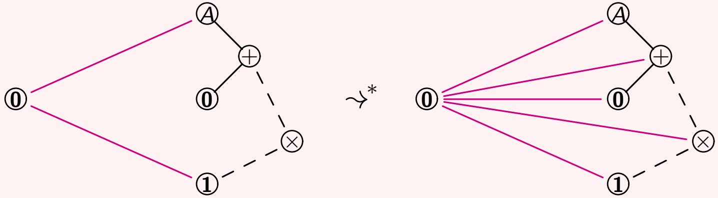
Initial and terminal nets

Call nets $\mathbf{0} \xrightarrow{R} Y$ initial and $X \xrightarrow{S} \mathbf{1}$ terminal



σR and σS are full: they have all possible unit links (but no atomic links)

Points and copoints

Points and copoints are maps out of $\mathbf{1}$ and into $\mathbf{0}$ respectively

$$X \xrightarrow{!} \mathbf{1} \xrightarrow{p} P \qquad\qquad Q \xrightarrow{q} \mathbf{0} \xrightarrow{?} Y$$

(co)pointed maps are those that factor through a (co)point

(co)pointed objects are those that admit (co)points

$$P \;:=\; \mathbf{1} \mid P{\times}P \mid P{+}Y \mid Y{+}P \qquad Q \;:=\; \mathbf{0} \mid Q{+}Q \mid Q{\times}X \mid X{\times}Q$$

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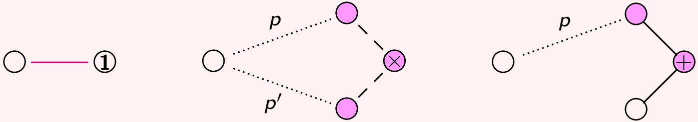
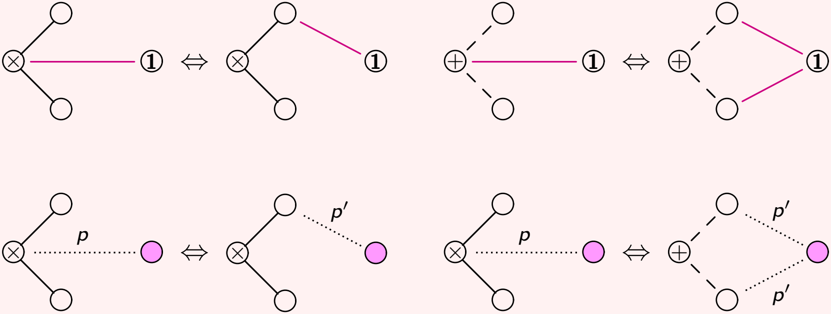
$$P \ := \ \mathbf{1} \mid P \times P \mid P + Y \mid Y + P \qquad Q \ := \ \mathbf{0} \mid Q + Q \mid Q \times X \mid X \times Q$$

(co)pointed nets will be those consisting of rooted unit links



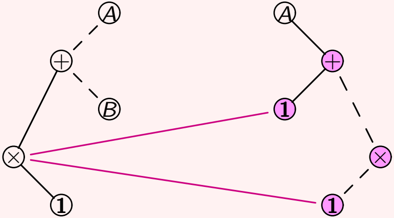
Points and copoints

Pointed nets may rewrite by moving their links in parallel



(all links in p and p' connect to the left root)

Points and copoints



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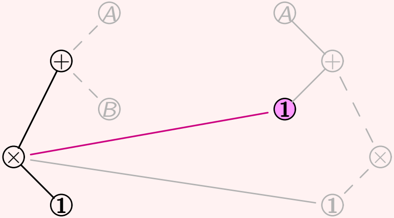

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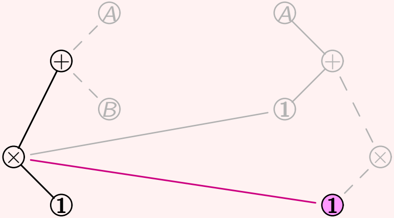

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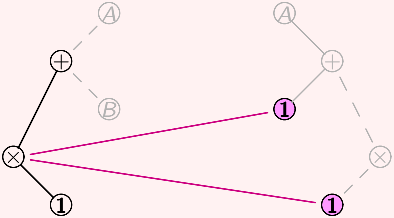

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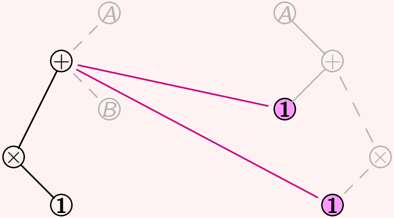

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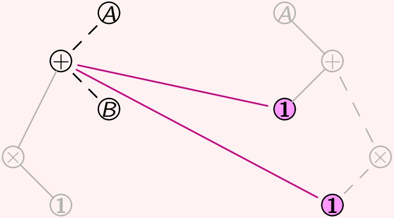

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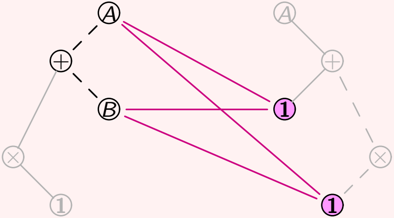

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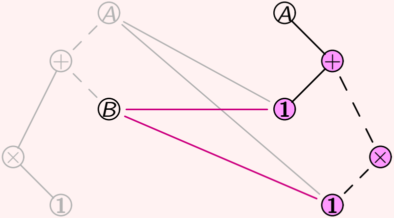

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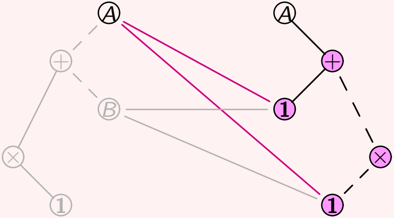
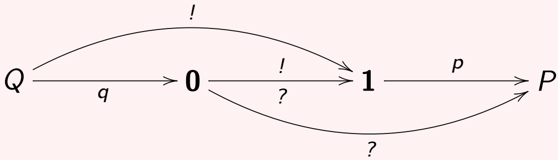


Bipointed nets

Bipointed maps (or disconnects) are both pointed and copointed.

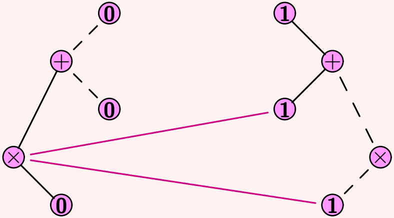
There is exactly one $b : Q \to P$ for copointed Q and pointed P, and none for other X, Y.



A bipointed net is one $Q \xrightarrow{q} P$ (copointed) or $Q \xrightarrow{p} P$ (pointed)

Bipointed nets

Two parallel bipointed nets are always equivalent



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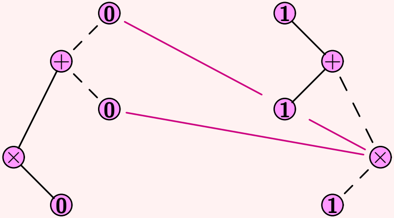

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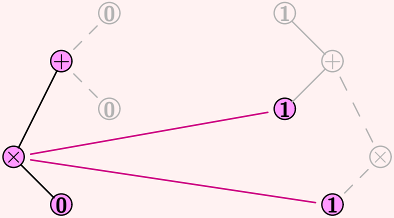

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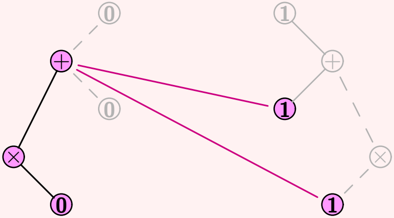

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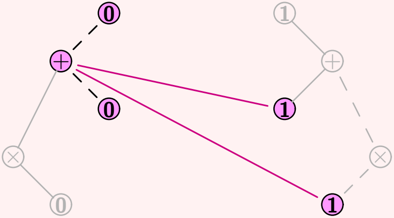

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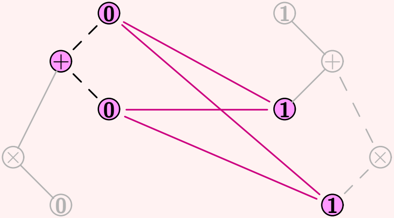

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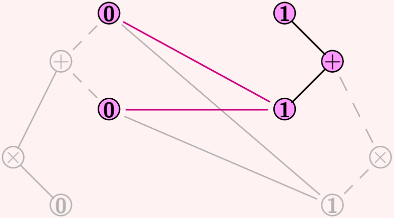

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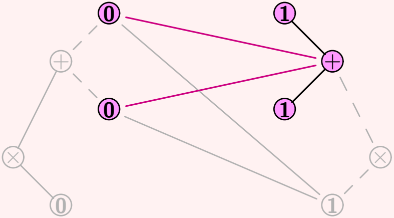

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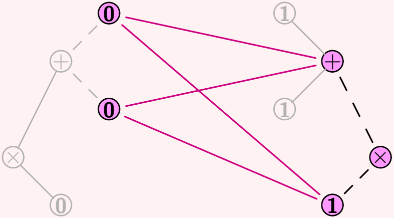

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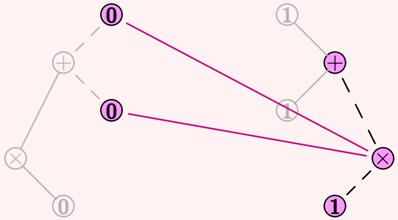
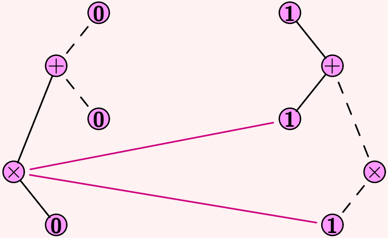


Bipointed nets

The saturation of a bipointed net is full



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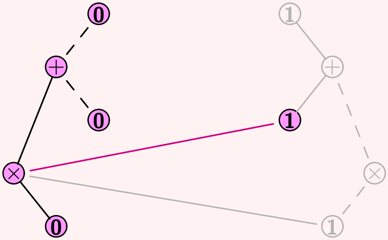

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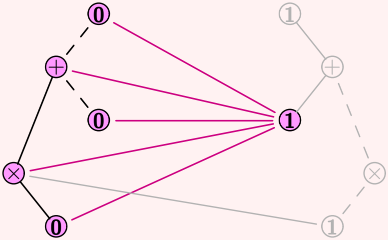

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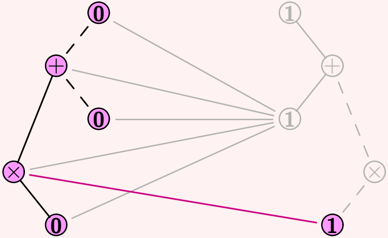

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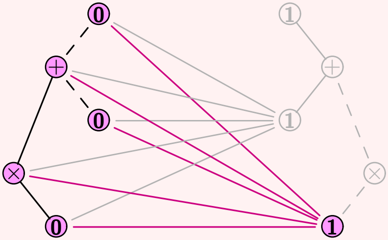

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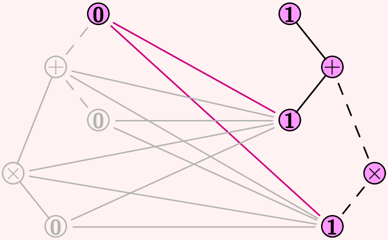

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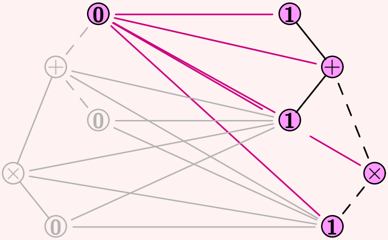

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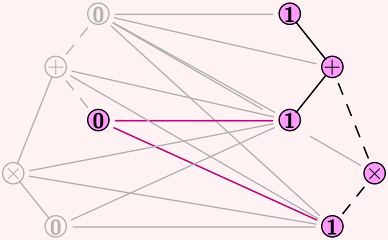
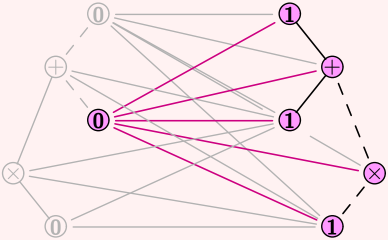
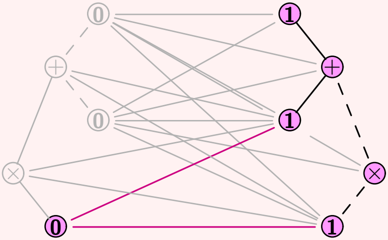


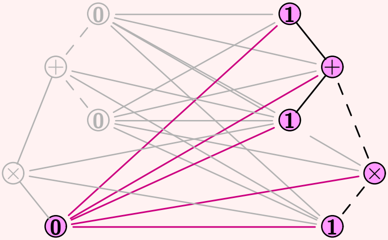
Bipointed nets



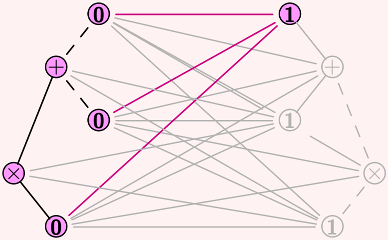
Bipointed nets



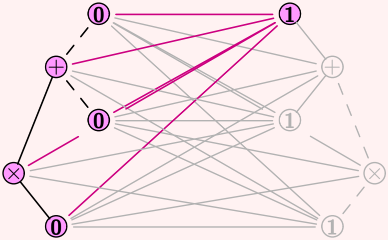
Bipointed nets



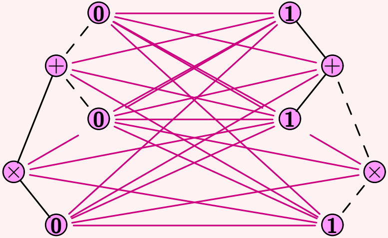
Bipointed nets



Bipointed nets



Bipointed nets



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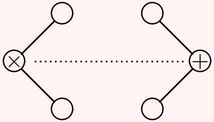

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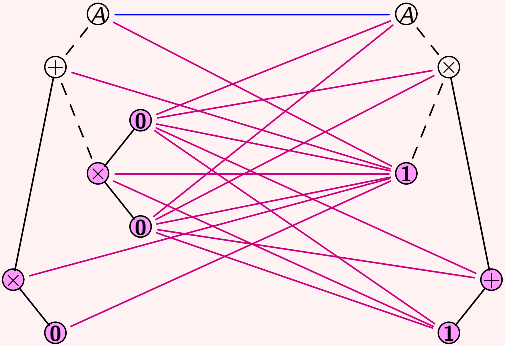
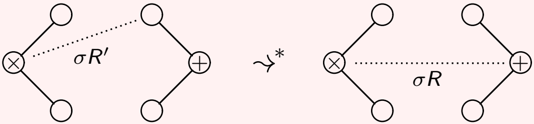
Inductive saturation

Saturated nets need not factor through injections/projections



Inductive saturation

Saturation of $\quad X \xrightarrow{R'} Y \xrightarrow{\iota_0} Y + Z \quad = \quad X \xrightarrow{R} Y + Z$



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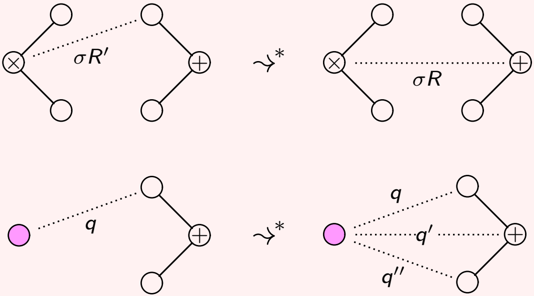

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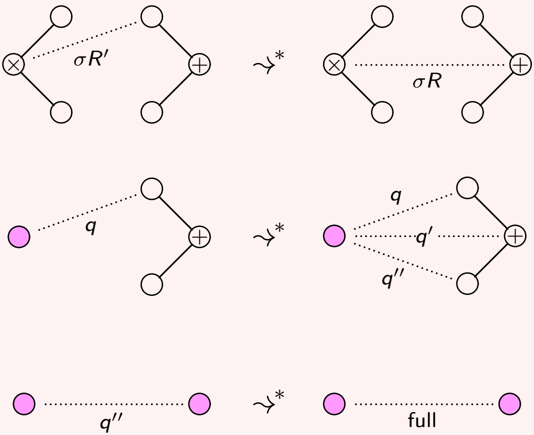
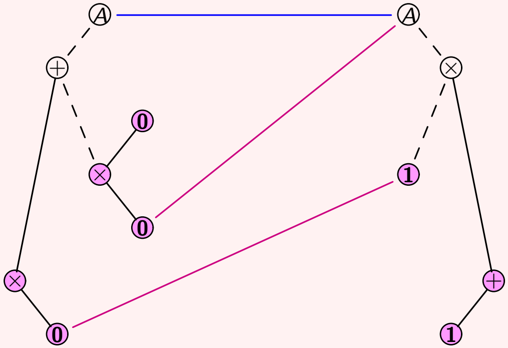
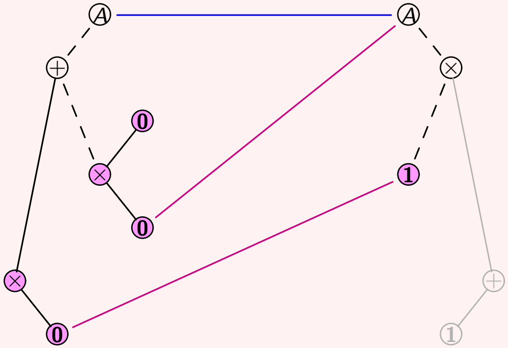


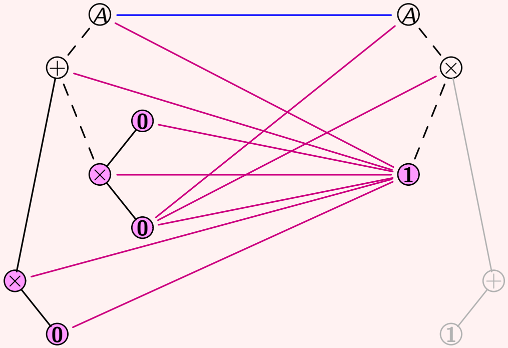
Inductive saturation



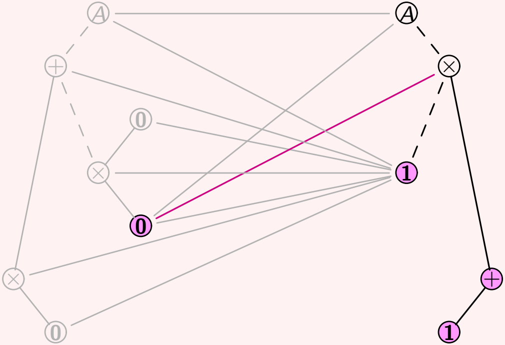
Inductive saturation



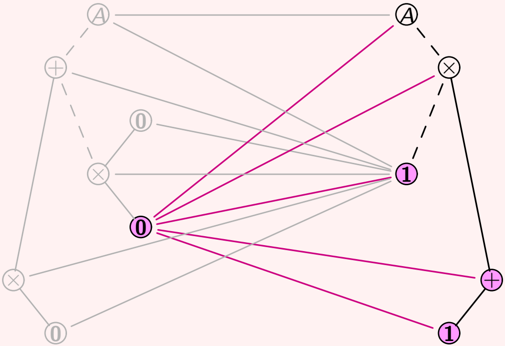
Inductive saturation



Inductive saturation



Inductive saturation



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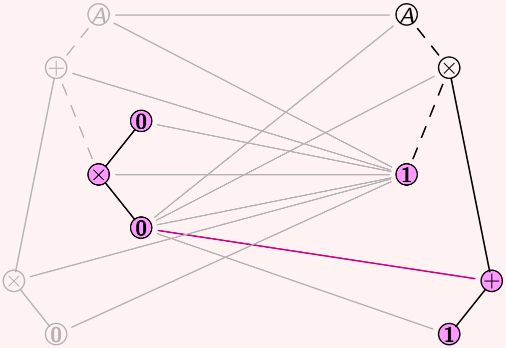
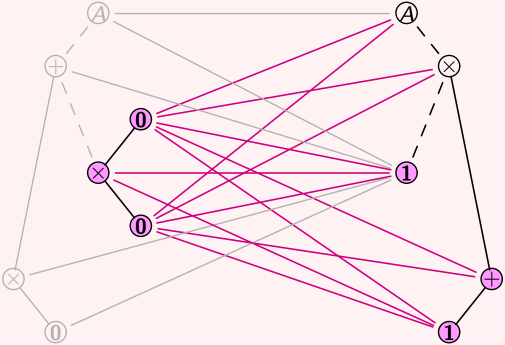
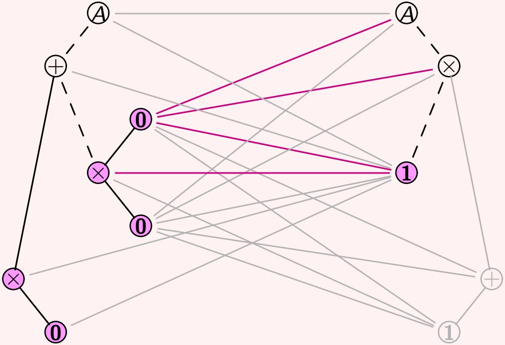


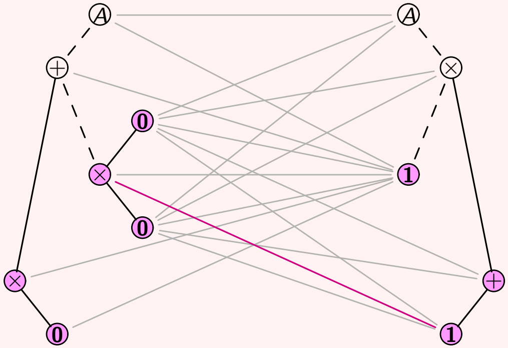
Inductive saturation



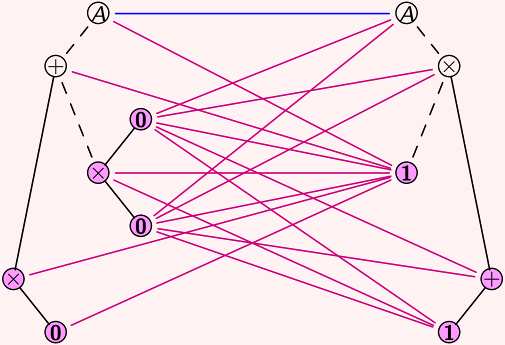
Inductive saturation



Inductive saturation



Inductive saturation



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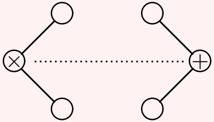

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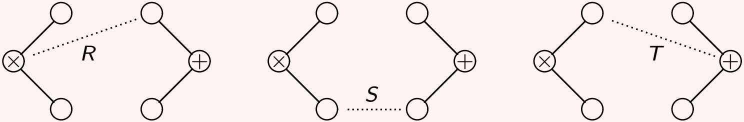
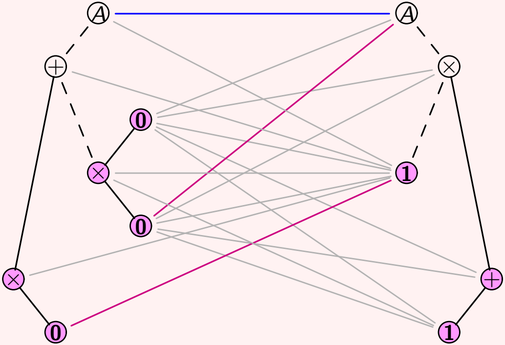
Matching injections and projections

Equivalent nets may factor through different injections or projections, but to allow induction nets must at least have the same domain and codomain.



Idea: 'highest' links, and in particular rooted links, are most significant (downward movement in saturation is unrestricted)

If σR contains a rooted link, so does some $S \Leftrightarrow R$



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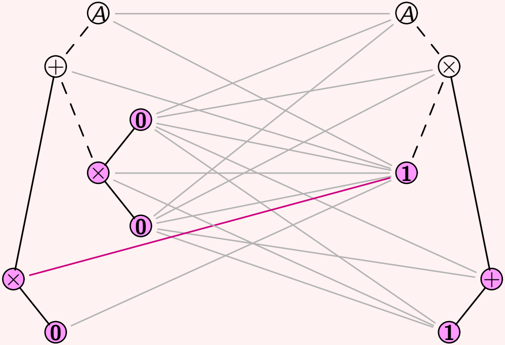

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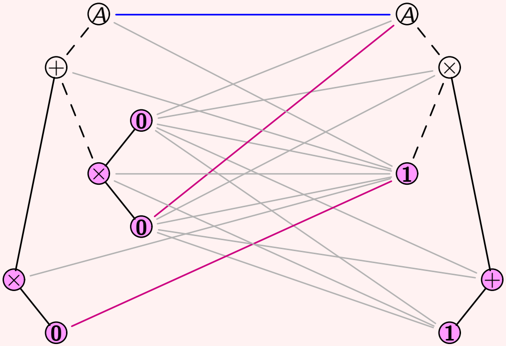

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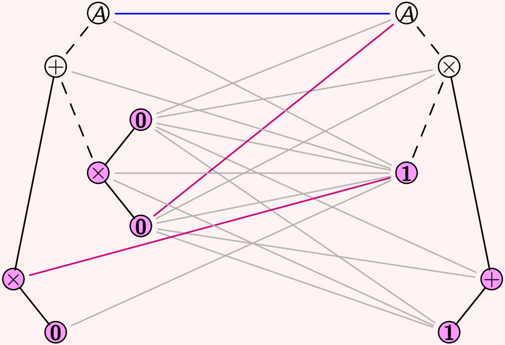

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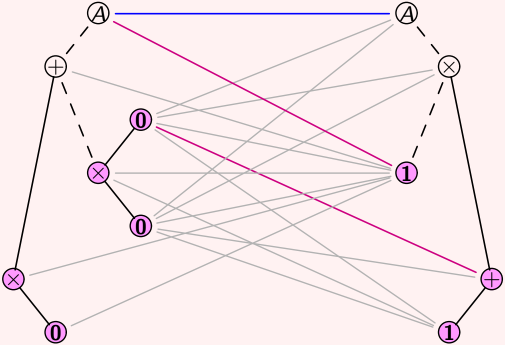

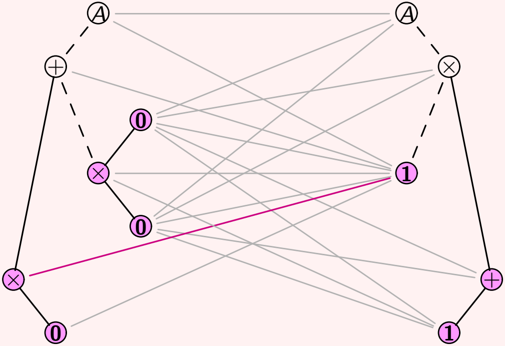
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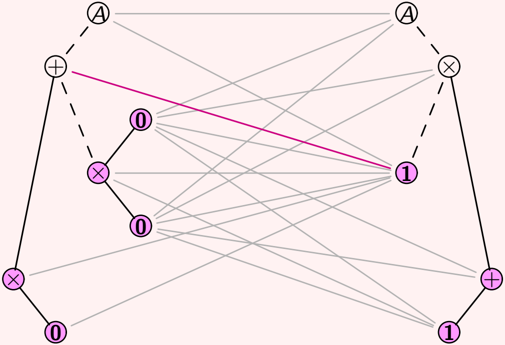
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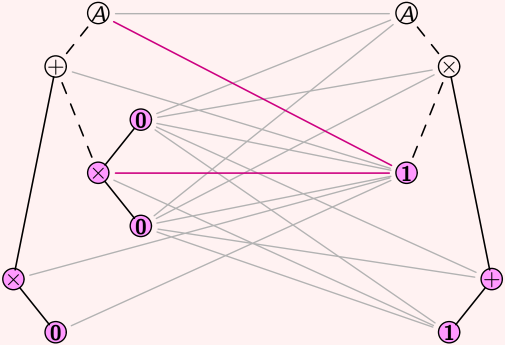

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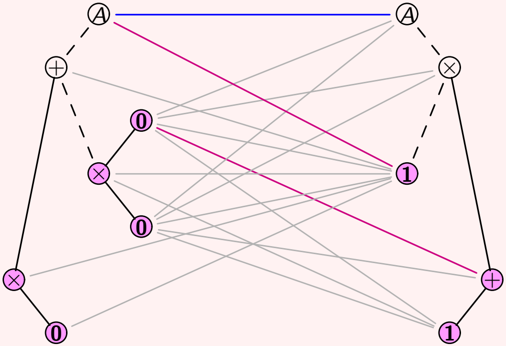

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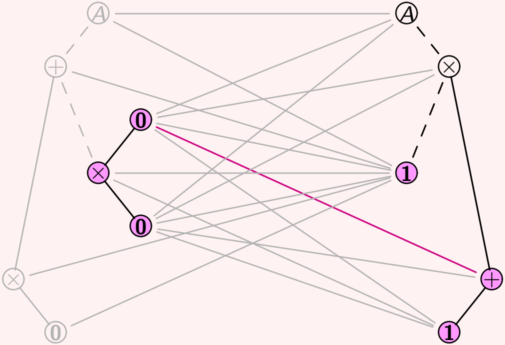
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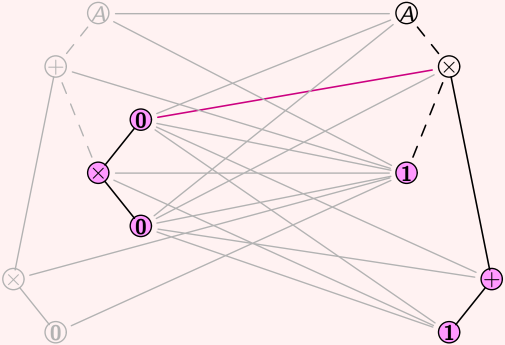

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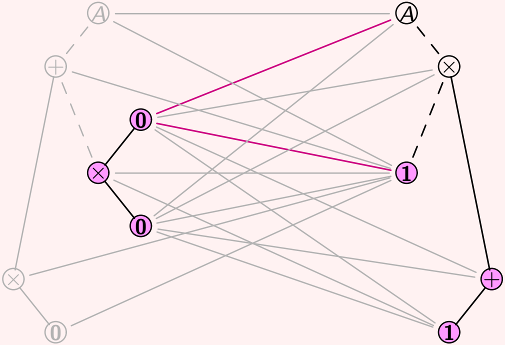

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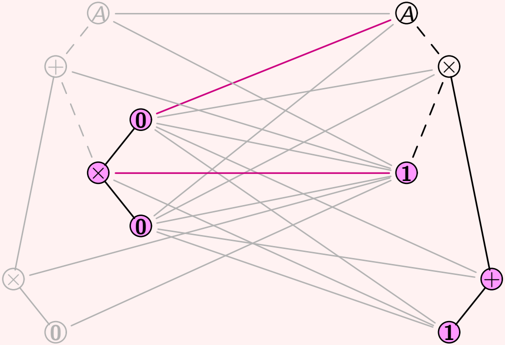

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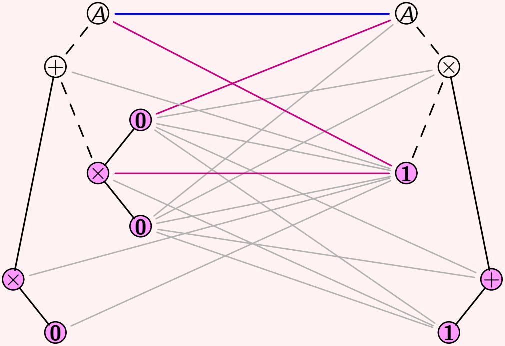

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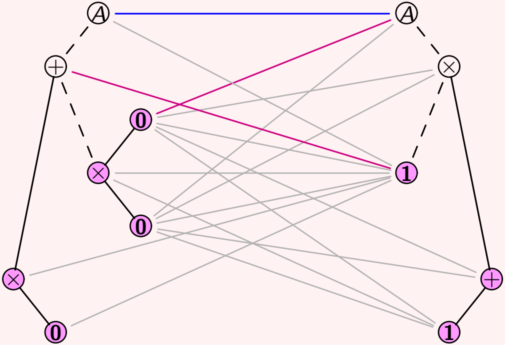

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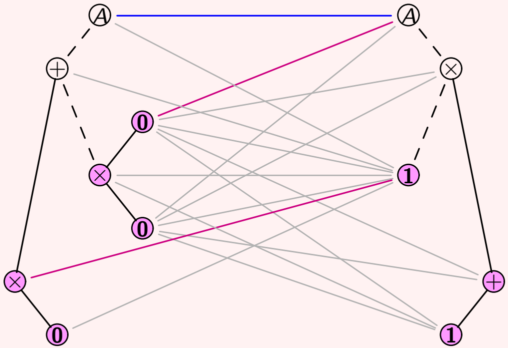

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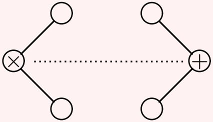

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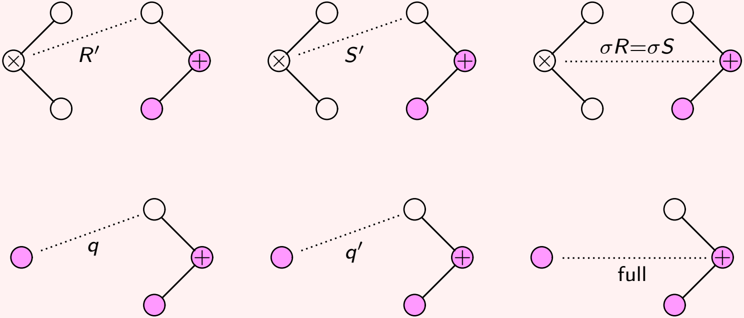
Injections into pointed objects



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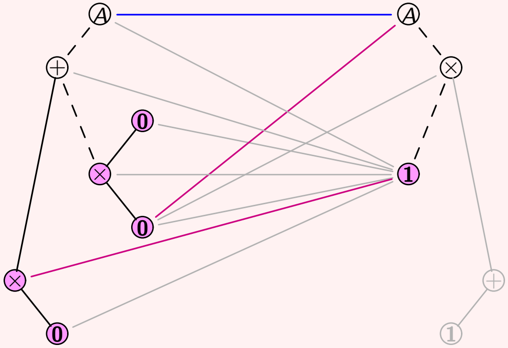

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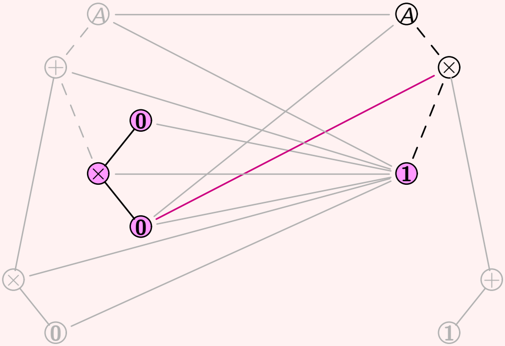

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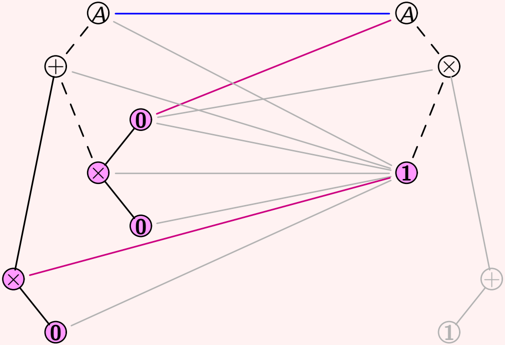

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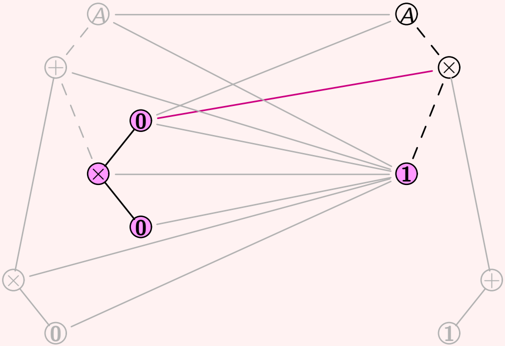

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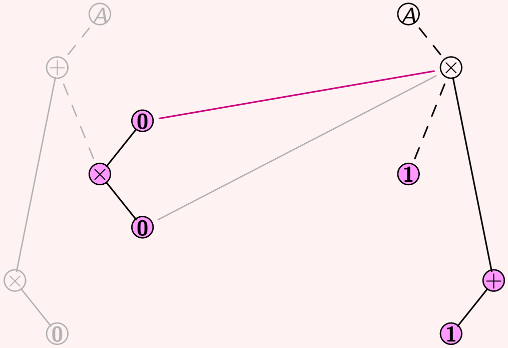

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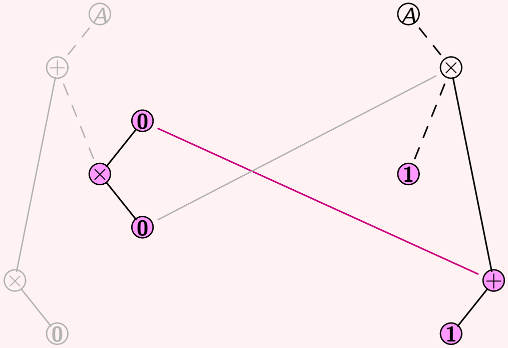

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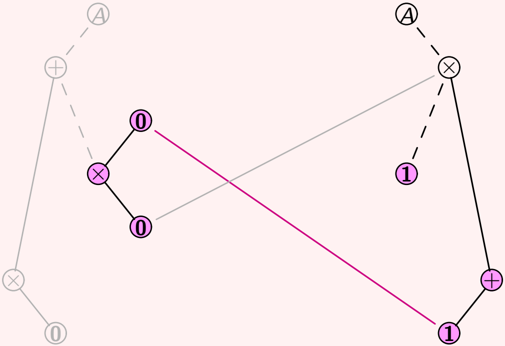

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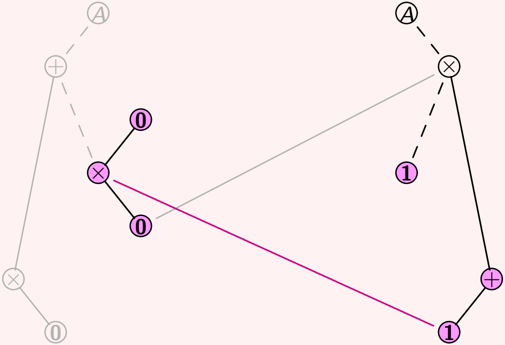

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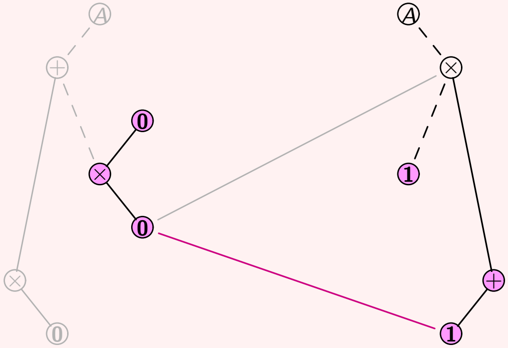

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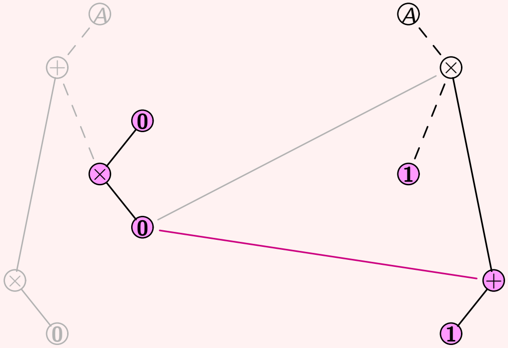

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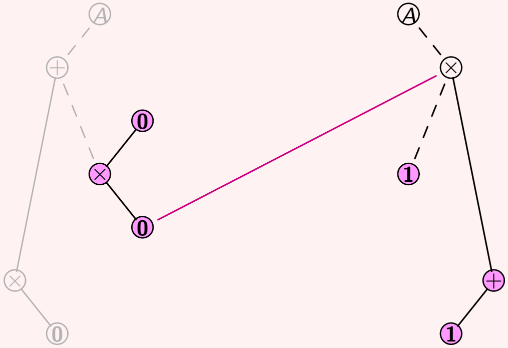

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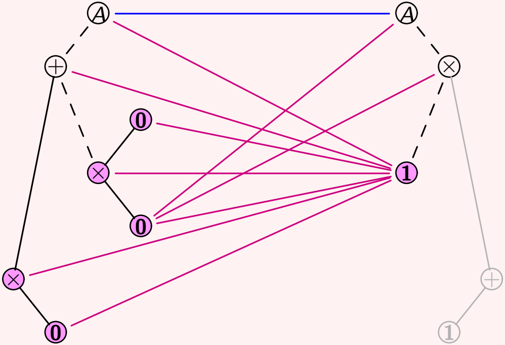

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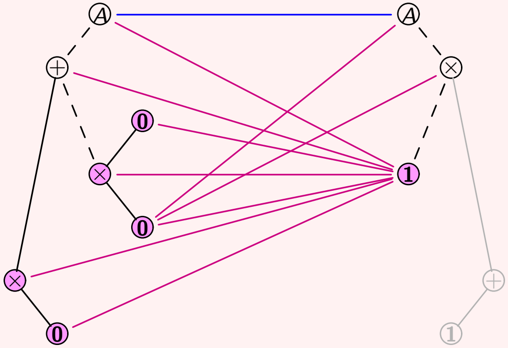

Now the induction hypothesis can be applied

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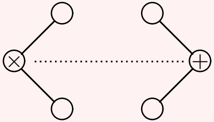

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Questions?