The Computational Meaning of Probabilistic Coherence Spaces[1]

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The computational meaning of a denotational semantics

Probabilistic Coherence Spaces | | **Scott's Domains**

types	$\longmapsto$	positively convex modules $P \subseteq (\mathbb{R}^+)^W$ s.t. ...	$\longmapsto$	cpo $(S, \leq)$ s.t. ...
terms M, N	$\longmapsto$	vectors in P $[\![M]\!] = [\![N]\!]$	$\longmapsto$	points in S $[\![M]\!] = [\![N]\!]$

Theorem (pPCF, Danos-Ehrhard)

Let M : `int` be closed and probabilistic,

$$[\![M]\!]_n = \mathrm{Red}^{\infty}_{M,\underline{n}}$$

Theorem (PCF, Plotkin)

Let M : `int` be closed,

$$[\![M]\!] = n \text{ iff } M \xrightarrow{*} \underline{n}$$

Theorem ($p\Lambda$ Ehrhard-Pagani-Tasson)

Let M be closed, untyped and probabilistic,

$$\sum_{d \in |\mathbf{D}_2|} [\![M]\!]_d = \mathrm{Red}^{\infty}_{M,\mathtt{hnf}}$$

Theorem (Λ, Hyland)

Let M be closed and untyped,

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$|\mathcal{A}|$ a set (possibly infinite), called *web*

$\mathrm{P}(\mathcal{A})$ a set of vectors $\subseteq (\mathbb{R}^+)^{|\mathcal{A}|}$ such that

closure: $\mathrm{P}(\mathcal{A})^{\perp\perp} = \mathrm{P}(\mathcal{A})$

- inner product: $v, u \in (\mathbb{R}^+)^{|\mathcal{A}|}$, $\langle u, v\rangle = \sum_{a\in|\mathcal{A}|} u_a v_a$
- polarity: $P \subseteq (\mathbb{R}^+)^{|\mathcal{A}|}$, $P^\perp = \left\{v \in (\mathbb{R}^+)^{|\mathcal{A}|} \,;\, \forall u \in P, \langle v, u\rangle \leq 1\right\}$

bound: $\forall a \in |\mathcal{A}|, \exists v \in \mathrm{P}(\mathcal{A}), v_a \neq 0$

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Example

$\lvert\mathbf{1}\rvert = \{*\}$	$\mathrm{P}(\mathbf{1}) = [0,1]$	$\mathrm{P}(\mathbf{1})^\perp = \mathrm{P}(\mathbf{1})$
$\lvert\mathbf{B}\rvert = \{\mathtt{t}, \mathtt{f}\}$	$\mathrm{P}(\mathbf{B}) = \{(p,q) \,;\, p+q \leq 1\}$	$\mathrm{P}(\mathbf{B})^\perp = \{(p,q) \,;\, p,q \leq 1\}$
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objects: probabilistic coherence spaces $\mathcal{A}, \mathcal{B}, \ldots$

morphisms: functions $f : \mathrm{P}(\mathcal{A}) \mapsto \mathrm{P}(\mathcal{B})$ which are entire, i.e.

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This CCC is the co-Kleisly of the exponential comonad of a category of linear morphisms interpreting Linear Logic: $A \Rightarrow B = !A \multimap B$

Interpretation of simply typed calculi

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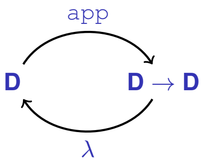

D is given as the minimal solution of

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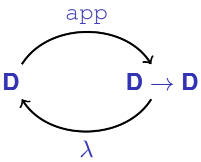

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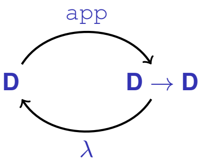

$\mathbf{D}$ is given as the minimal solution of

$$\mathbf{D} = \mathbf{D}^{\mathbb{N}} \to \bot$$

$$\begin{array}{ll}
\mathbf{D}_0 & = (\emptyset, \mathbf{0}) \\
 & \rotatebox{90}{$\leq$} \\
\mathbf{D}_1 & = \mathbf{D}_0^{\mathbb{N}} \to \bot \\
 & \rotatebox{90}{$\leq$} \\
\mathbf{D}_2 & = \mathbf{D}_1^{\mathbb{N}} \to \bot \\
 & \rotatebox{90}{$\leq$} \\
\mathbf{D}_3 & = \mathbf{D}_2^{\mathbb{N}} \to \bot \\
 & \vdots \\
\mathbf{D} & = \bigvee_i \mathbf{D}_i
\end{array}$$

Interpretation of untyped calculi

$x_1, \ldots, x_n \vdash M$ is associated with $[\![M]\!] : \mathrm{P}(\mathbf{D}) \times \ldots \times \mathrm{P}(\mathbf{D}) \mapsto \mathrm{P}(\mathbf{D})$

Remark: $[\![M]\!]_{|\mathbf{D}_0}, [\![M]\!]_{|\mathbf{D}_1}, [\![M]\!]_{|\mathbf{D}_2}, [\![M]\!]_{|\mathbf{D}_3}, \ldots$ are the approximants of $[\![M]\!]$.

The computational meaning of a denotational semantics

Probabilistic Coherence Spaces			Scott's Domains	
types	$\longmapsto$	positively convex modules $P \subseteq (\mathbb{R}^+)^W$ s.t. ...	$\longmapsto$	cpo $(S, \leq)$ s.t. ...
terms M, N	$\longmapsto$	vectors in P $[\![M]\!] = [\![N]\!]$	$\longmapsto$	points in S $[\![M]\!] = [\![N]\!]$

Theorem (pPCF, Danos-Ehrhard)

Let M : $\mathtt{int}$ be closed and probabilistic,

$$[\![M]\!]_n = \mathrm{Red}^{\infty}_{M,\underline{n}}$$

Theorem (PCF, Plotkin)

Let M : $\mathtt{int}$ be closed,

$$[\![M]\!] = n \textit{ iff } M \xrightarrow{*} \underline{n}$$

Theorem ($p\Lambda$ Ehrhard-Pagani-Tasson)

Let M be closed, untyped and probabilistic,

$$\sum_{d \in |\mathbf{D}_2|} [\![M]\!]_d = \mathrm{Red}^{\infty}_{M,\mathtt{hnf}}$$

Theorem (Λ, Hyland)

Let M be closed and untyped,

$$[\![M]\!] \neq \bot \textit{ iff } M \xrightarrow{*} \mathtt{hnf}$$

$$p\Lambda \qquad M, N ::= x \mid \lambda x.M \mid MN \mid M +_p N \qquad \text{with } p \in [0,1]$$

Definition ($\xrightarrow{p}$)

$$\frac{}{(\lambda x.M)N \xrightarrow{1} M\{N/x\}} \qquad \frac{}{M +_p N \xrightarrow{p} M} \qquad \frac{}{M +_p N \xrightarrow{1-p} N}$$

$$\frac{M \xrightarrow{p} M'}{\lambda x.M \xrightarrow{p} \lambda x.M'} \qquad \frac{M \xrightarrow{p} M' \quad M \text{ not a } \lambda}{MN \xrightarrow{p} M'N}$$

Example ($\Delta = \lambda x.xx$, $\Theta = (\lambda yf.f(yyf))(\lambda yf.f(yyf))$, $M = \lambda y.x +_{\frac{1}{3}} \lambda y.y$)

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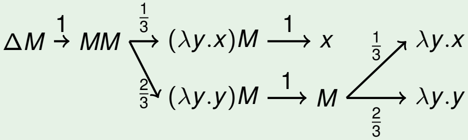

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$$\Theta M \xrightarrow{1}\ \xrightarrow{1} M(\Theta M) \xrightarrow{\frac{1}{3}} (\lambda y.x)(\Theta M) \xrightarrow{1} x$$

$$M(\Theta M) \xrightarrow{\frac{2}{3}} (\lambda y.y)(\Theta M) \xrightarrow{1} \Theta M$$

Definition ($\mathrm{Red}^n \in [0,1]^{p\Lambda \times p\Lambda}$)

$$\mathrm{Red}^1_{M,N} := \begin{cases} p & \text{if } M \xrightarrow{p} N, \\ 1 & \text{if } M \in \mathtt{hnf}, \\ 0 & \text{otherwise.} \end{cases} \qquad \mathrm{Red}^{n+1}_{M,N} := \sum_{L \in \Lambda^+} \mathrm{Red}^1_{M,L}\, \mathrm{Red}^n_{L,N}$$

Definition ($\mathrm{Red}^\infty \in [0,1]^{p\Lambda \times \mathtt{hnf}}$)

Since `hnf` are absorbing states, the following is well-defined:

$$\mathrm{Red}^\infty_{M,H} := \lim_{n \to \infty} \mathrm{Red}^n_{M,H} \qquad \mathrm{Red}^\infty_{M,\mathtt{hnf}} := \sum_{H \in \mathtt{hnf}} \mathrm{Red}^\infty_{M,H}$$

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$$\mathrm{Red}^n_{\Delta M,x} = \begin{cases} \frac{1}{3} & n \geq 3, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Red}^\infty_{\Delta M,x} = \frac{1}{3}$$

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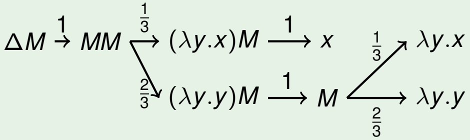

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$$\mathrm{Red}^n_{\Theta M, x} = \frac{1}{3} \sum_{i=0}^{k} \left(\frac{2}{3}\right)^i \quad \text{with } k = \left\lfloor \frac{n}{4} \right\rfloor$$

$$\mathrm{Red}^\infty_{\Theta M, x} = 1$$

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