Investigating Structure in Turing Categories:

How can certain structure, such as range maps, be manifested in a Turing category?

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(Based on research conducted under supervision of Dr. Pieter Hofstra)

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Motivation

Traditional Computation

(i) Based on $\mathbb{N}$: Is done on its recursively enumerable subsets

(ii) Modeled by Turing machines - which correspond to computable functions

(i) **Key feature to abstract:** any computable function has a code $c_f \in \mathbb{N}$ and there exists a universal application $\cdot$ such that $c_f \cdot x = f(x)$

(ii) Has a lot of additional structure
eg. equality, ranges, coproducts etc.

Cartesian Restriction Categories

Let C be a category

A **restriction combinator** $\overline{(-)}$

Sends a map $f : A \to B$ in C to a map $\overline{f} : A \to A$ so that:

[R.1] $f\overline{f} = f$

[R.2] $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ whenever $dom(f) = dom(g)$

[R.3] $\overline{g\overline{f}} = \overline{g}\,\overline{f}$ whenever $dom(f) = dom(g)$

[R.4] $\overline{g}f = f\overline{gf}$ whenever $cod(f) = dom(g)$

A **cartesian restriction category**

Is a category C endowed with a restriction combinator and containing the following:

(i) A restriction-terminal object 1

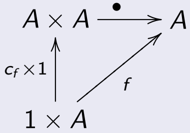
(ii) All partial products

Turing Category C

A Turing category is a cartesian restriction category

that has a Turing object A and map $\bullet : A \times A \to A$ such that

(i) for every $f : A \to A$ there exists a total map, called a code, $c_f : 1 \to A$



(ii) for all $X \in C$ there exists an embedding-retraction pair (m_X, r_X) of X into A

The Main Example

Standard Computation Model

Objects: $1, \mathbb{N}, \mathbb{N}^2, \ldots$
Maps: m-tuples of computable maps $f : \mathbb{N}^n \to \mathbb{N}$
Application: $\bullet : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$
With some enumeration system of all computable maps $\{\phi_0, \phi_1, \ldots\}$,
$n \cdot m = \phi_n(m)$

Restriction: as in Par

$$\overline{f}(x) = \begin{cases} x & \text{if } f(x) \downarrow \\ \uparrow & \text{otherwise.} \end{cases}$$

Embedding-retraction pairs: isomorphisms, for all $n, m > 0, \mathbb{N}^n \cong \mathbb{N}^m$

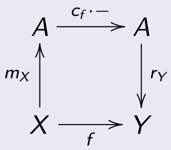
An Arbitrary Map in a Turing Category

Suppose C is a Turing category

For any map $f : X \to Y$

There exists a factorization:

$$f = r_Y(c_f \cdot -)m_X$$



Partial Combinatory Algebras (PCA's)

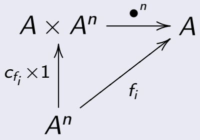
Suppose A is an object in a cartesian restriction category

A PCA $\mathbb{A} = (A, \bullet)$ has

(i) An object A

(ii) A map $\bullet : A \times A \to A$

(iii) Combinatory completeness:
for every polynomial map $f : A^n \to A^m$, for each of the m components, there exists a total map $c_{f_i} : 1 \to A$ such that the following diagram commutes (that is, each component is $\mathbb{A}$ - **computable**)



The $\bullet$ map is never associative - assume expression is bracketed to the left

Alternative PCA Definition

Combinatory completeness in terms of **combinators**

A has elements (**combinators**) k and s such that for all $a, b \in A$:

(i) $k \cdot a \cdot b \cong a$

(ii) $s \cdot a \downarrow$, $s \cdot a \cdot b \downarrow$, and $s \cdot a \cdot b \cdot c \cong a \cdot c \cdot (b \cdot c)$

- Like the algebra version of the lambda calculus abstraction, for eg. $kab = \lambda ab.a$

PCA Examples

(i) Standard computation model

(ii) $(\mathbb{N}(A), \bullet)$, **computation with an oracle** $A \subset \mathbb{N}$
A answers the question "is x in A?"
Computation denoted $n \cdot_A m$.

(iii) **Reflexive Structures**
Suppose A is an object in a Cartesian Closed Category such that A^A is a retract of A, $r : A \to A^A$
Application:
$A \times A \overset{r \times 1}{\to} A^A \times A \overset{ev}{\to} A$
where ev is the evaluation map.

Non-example

Any application that is associative or commutative

Turing Categories based around a PCA

Let C be a category with $\bullet : A \times A \to A$, and suppose $\mathbb{A} = (A, \bullet)$ is a PCA

Turing categories based around A

(i) $\text{Comp}(\mathbb{A})$: $\{1, A, A^2, ...\}$ with $\mathbb{A}$ - computable maps

(ii) $\text{Split}(\text{Comp}(\mathbb{A}))$: formally split all idempotents in $\text{Comp}(\mathbb{A})$

PCA is at the Heart of a Turing Category C

Embedding

Suppose C is a Turing category with Turing object A, then $\mathbb{A}$ is a PCA, and

$\mathrm{Comp}(\mathbb{A}) \hookrightarrow \mathrm{C} \hookrightarrow \mathrm{Split}(\mathrm{Comp}(\mathbb{A}))$

In the standard model, Split(Comp($\mathbb{N}$))

Objects: all recursively enumerable sets
Maps: all functions computable by Turing machines

Range Category

Suppose C is a restriction category

A range combinator $\widehat{(-)}$

Sends a map $f : X \to Y$ in C to a map $\hat{f} : Y \to Y$ so that

[RR.1] $\overline{\hat{f}} = \hat{f}$

[RR.2] $\hat{f} f = f$

[RR.3] $\widehat{\overline{g} f} = \overline{g}\hat{f}$ with codom(f) = dom(g)

[RR.4] $\widehat{g\hat{f}} = \widehat{gf}$ with codom (f) = dom(g)

Often, a fifth axiom is added: **[RR.5]** $f\hat{g} = h\hat{g}$ whenever $fg = hg$

C is called a **range category** when it has a range combinator.
Given a restriction structure $\overline{(-)}$ on C, the range structure $\widehat{(-)}$, if it exists, is unique.

Open Maps

Notation

C - a restriction category
$\mathcal{O}(A)$ - the poset of restriction idempotents of an object A
Given $f : A \to B$, write
$f^* : \mathcal{O}(B) \to \mathcal{O}(A)$ - the "inverse image":
for any $e \in \mathcal{O}(B), f^*(e) = \overline{ef} \leq \overline{f}$.
$ee' = e \wedge e'$.

Given $f : A \to B$, it is **open** when:

There is a poset morphism $\exists_f : \mathcal{O}(A) \to \mathcal{O}(B)$ such that
[O1] $\exists_f(f^*(e')) \leq e'$ for all $e' \in \mathcal{O}(B)$
[O2] $e \wedge f^*(e') \leq f^*(\exists_f(e) \wedge e')$ for all $e \in \mathcal{O}(A), e' \in \mathcal{O}(B)$
[O3] $e' \wedge \exists_f(e) \leq \exists_f(f^*(e') \wedge e)$ for all $e \in \mathcal{O}(A), e' \in \mathcal{O}(B)$

Ranges and Open Maps

In a range category

Every $f : X \to Y$ is open
$\exists_f(e) = \widehat{fe}$, where $e \in \mathcal{O}(X)$
$\exists_f(1) = \hat{f}$

In an arbitrary category

Let C be a restriction category. The subcategory of C on the open maps is a range category

A cartesian range category is a cartesian restriction category with ranges such that $\widehat{f \times g} = \hat{f} \times \hat{g}$

Ranges and Idempotents

Let C be a cartesian restriction category

Lemma

Suppose (m, r) is an embedding-retraction pair of X into A, and $m : X \to A$ is open. Then

(i) $r' := r\hat{m}$ is also a retraction of X, and $\overline{mr'} = mr'$.

(ii) r' is open, and for any $e \in \mathcal{O}(A), \exists_{r'}(e) = r'em = \widehat{r'e}$

Lemma

If **[RR.5]** holds, any split idempotent has the same splitting as a restriction idempotent.

Thus, when m is open, we may assume it is the splitting of a restriction idempotent

A criterion

Let C be a Turing category with Turing object A

Suppose each map $A \to A$ in C is open, and for each X there exists an embedding $m_X : X \to A$ that is open. Then,

(i) C,
(ii) Comp($\mathbb{A}$), and
(iii) Split(Comp($\mathbb{A}$))

are all range categories.

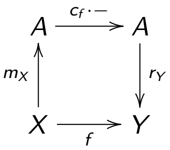
Alternatively,

Suppose each map $A \to A$ in C is open, and assume **[RR.5]** for the open map (range) subcategory of C. Again, (i), (ii), and (iii) are then range categories.

Range of a Map in a Turing Category

In both cases, the range of any map $f : X \to Y$ in C is defined by:

$$\begin{aligned}
\hat{f} &= r_Y\widehat{(c_f\cdot -)}m_X \\
&= r_Y\widehat{\widehat{(c_f\cdot -)}}m_X \\
&= r_Y\widehat{(c_f\cdot -)}\widehat{m_X r_X}m_Y \\
&= r_Y\widehat{(c_f\cdot -)}m_X m_Y
\end{aligned}$$



Range combinators in a PCA

Suppose $\mathbb{A} = (A, \bullet)$ is a PCA in a cartesian range category C, then

(i) $\mathbb{A}$ has (**weak**) **range combinators** whenever for every $a \cdot - : A \to A$ there exists a combinator r_a such that $(r_a \cdot a) \cdot - = \widehat{a \cdot -}$.

(ii) $\mathbb{A}$ has a **strong range combinator** if there exists a combinator r such that $(r \cdot a) \cdot - = \widehat{a \cdot -}$ for every map $a \cdot - : A \to A$

Ranges in a Turing category and the underlying PCA

Proposition

Let $\mathbb{A}$ be a PCA in a cartesian restriction category. Then $\mathbb{A}$ has weak ranges $\Leftrightarrow$ $\mathrm{Comp}(\mathbb{A})$ is a range category. In this case, $\mathrm{Comp}(\mathbb{A}) \hookrightarrow \mathrm{C}$ is a range preserving inclusion.

Corollary

When C is a Turing category with Turing object A, then C is a range category implies that $\mathrm{Comp}(\mathbb{A})$ has ranges. The converse holds whenever **[RR.5]** holds.

Ranges in the standard model

The Standard Model has a strong range combinator

IsInRange (n, x) {
    For(int $i = 0$ to ∞) {
        For(int $j = 0$ to i) {
            Do 1 step of each of each computation $\phi_n(j)$
            If $\phi_n(j)$ halts at this step {
                Test $\phi_n(j) = x$
                If TRUE, return x
            }
        }
    }
}

Further Exploration

- Just described range maps in Turing categories

Other Structure to Look for in a Turing Category

(i) Equality map: $=(a, b)$ is computable, so it is in $(\mathbb{N}, \bullet)$

(ii) Coproducts: $\mathbb{N} + \mathbb{N}$ is in Split(Comp($\mathbb{N}$))

(iii) What else?