The Free Tangent Structure

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What are Tangent Structures?

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$$\eta : id_{\mathcal{M}} \to T$$
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$$l : T \to T^2$$
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satisfying a set of axioms.

Why the Weil Algebras?

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Weil algebras and Tangent structure
Properties of Weil Algebras
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- Used in Synthetic Differential Geometry to describe structure of infinitesimals

$$D = \{d \in \mathbb{R} | d^2 = 0\}$$

where tangent bundles are then given as

$$TM = M^D$$

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- Even without SDG, they somehow model the correct type of behaviour
- (Soon) Another strong relationship with tangent structure!

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- We can write them in their presentations (not unique)

$$A = k[a_1, ..., a_n]/Q_A$$

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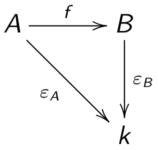

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$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow^{\varepsilon_A} & \downarrow{\scriptstyle \varepsilon_B} \\ & & k \end{array}$$

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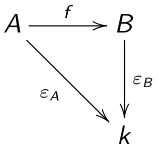

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Remark: We are only interested in k being $\{0, 1\}$, $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{R}$

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Remark: Since k is a zero object, then products can be regarded as pullbacks over k

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(The morphisms we will discuss later)

Weil algebras and graphs

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Recall the presentations

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We can represent such Weil algebras with (finite simple) graphs.

Weil algebras can be represented as graphs...

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Weil Algebra	Presentation	Graph

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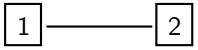
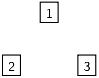
$W^2 = W \times W$	$k[x_1, x_2]/x_1^2, x_2^2, x_1x_2$	
$3W = W \otimes W \otimes W$	$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2$	

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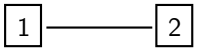
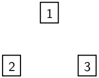
$W^2 = W \times W$	$k[x_1, x_2]/x_1^2, x_2^2, x_1 x_2$	
$3W = W \otimes W \otimes W$	$k[x_1, x_2, x_3]/x_1^2, x_2^2, x_3^2$	

So, to ask exactly which Weil algebras we want is to ask which graphs we want.

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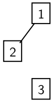

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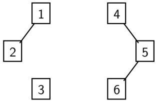

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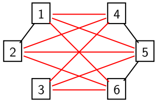

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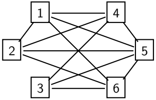

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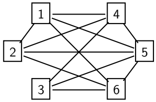

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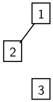

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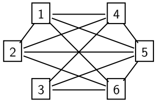

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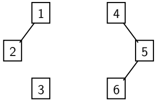

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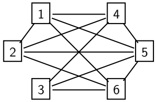

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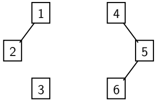

So, we want all the graphs that can be constructed from taking graph joins and disjoint unions of the one-vertex graph...

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- We therefore want all Weil algebras corresponding to cographs
- What about the morphisms? We can take all the morphisms so that $\mathbf{Weil_1}$ is a full subcategory of $\mathbf{Weil}$ (or equivalently of $\mathbf{Alg}/k$)!

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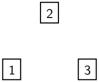

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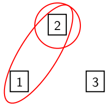

- Take graph for codomain
  - Draw red circles for x_1

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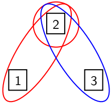
$$x_2 \mapsto y_2 y_3$$

We represent this as follows:



- Take graph for codomain
  - Draw red circles for x_1
  - Draw blue circles for x_2

Almost there...

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Properties of Weil Algebras

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The Free Tangent Structure

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$$F : \mathbf{Weil_1} \to [\mathcal{M}, \mathcal{M}]$$

which "picks out" the tangent structure $\mathbb{T}$!

Behold... the Functor

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Behold... the Functor

Theorem Given a category $\mathcal{M}$ and regarding $\mathbf{Weil_1}$ as monoidal with respect to $\otimes$ and $[\mathcal{M}, \mathcal{M}]$ with respect to $\circ$, to give a tangent structure $\mathbb{T}$ is equivalent to giving a monoidal functor

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$$\begin{array}{ccc} B \otimes (A_1 \times A_2) \otimes C & \xrightarrow{B \otimes \pi_1 \otimes C} & B \otimes A_1 \otimes C \\ {\scriptstyle B \otimes \pi_2 \otimes C} \downarrow & & \downarrow {\scriptstyle B \otimes \varepsilon_1 \otimes C} \\ B \otimes A_2 \otimes C & \xrightarrow[B \otimes \varepsilon_2 \otimes C]{} & B \otimes C \end{array}$$