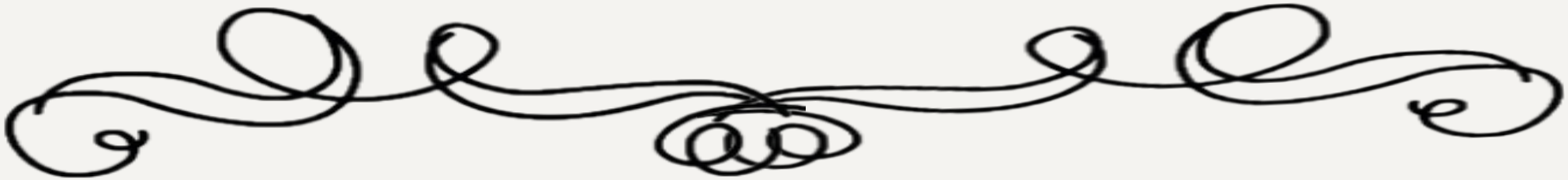
Semantics of CaMPL - Showing the Power Set Functor is Monoidal



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Background/Spoilers

- Categorical Message Passing Language (CaMPL) is a concurrent functional programming language which has its semantics defined in category theory.
- Semantics of non-deterministic programs is given by a sup-lattice enriched category which is created by a change of base from the original semantics.
- Geoff Crutwell's PhD thesis showed that we can define a change of base functor if there is a monoidal functor between the "base" categories.
    - He also describes how monoidal functors can be applied "monoidally" and that applying functors monoidally is "functorial."
- Power set functor gives a monoidal functor that takes sets to sup-lattices.

Change of Base

- We want to define the semantics of non-deterministic programs as a sup-lattice enriched version of the deterministic semantics.
- We also want to ensure the monoidal structures in the semantics, which define some of the other programming features in the language, are preserved.
- So we need a functor that takes the deterministic semantics which are set-enriched to the non-deterministic semantics which are sup-lattice enriched and preserves the monoidal structures in the semantics.

Change of Base

According to results from Geoff Cruttwell's PhD thesis, a change of base functor exists if there is a monoidal functor between the monoidal "base" categories.

- Theorem 5.7.1. Suppose that $N : (V, \bullet, I, \sigma_1) \to (W, \otimes, J, \sigma_2)$ is a braided monoidal functor between braided monoidal categories. Then the change of base 2-functor N_* takes monoidal V-categories to monoidal W-categories.

This means that if we can define a monoidal "N functor" of our own from **Set** to **SupLat**, we also get a change of base functor from the deterministic semantics to the non-deterministic semantics.

Monoidal Functors

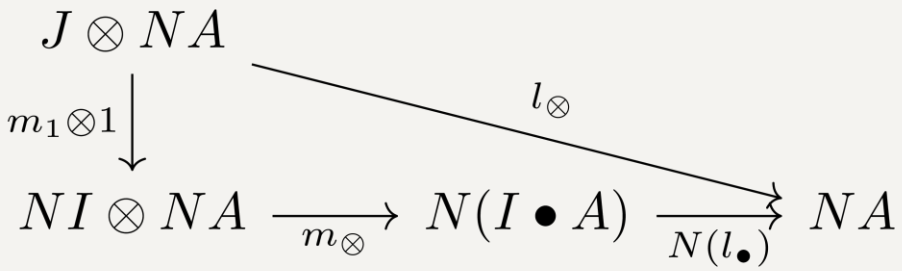
A monoidal functor between monoidal categories $(V, \bullet, I)$ and $(W, \otimes, J)$, is a functor $N : V \to W$, together with natural transformations, called comparison maps:

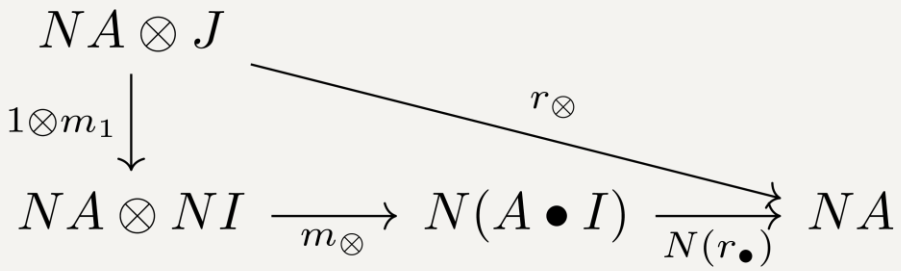
- $m_{\otimes AB} : NA \otimes NB \to N(A \bullet B)$
- $m_1 : J \to NI$

For which the following coherence diagrams commute $\forall A, B, C \in V$.

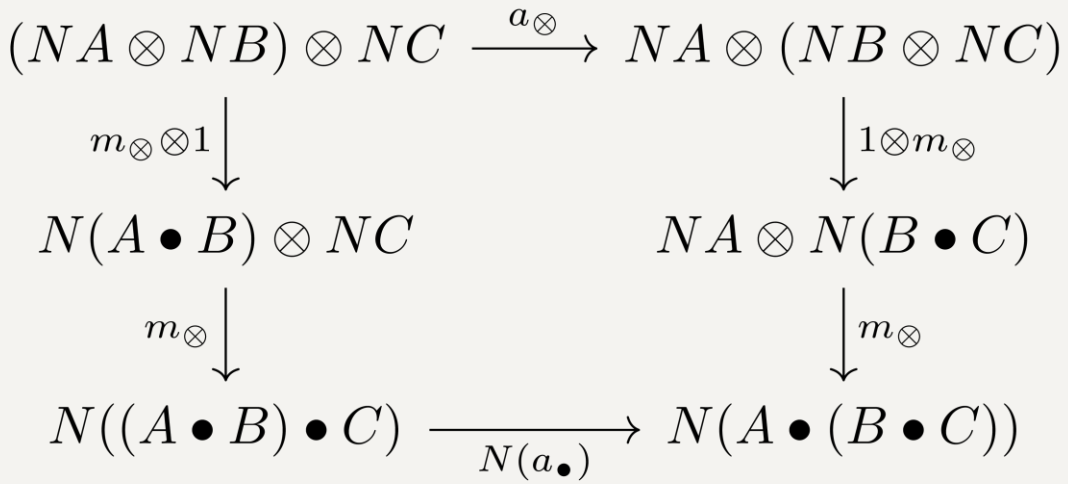
Monoidal Functors

- Left and right unitors:





- Associativity:



What is a Sup-Lattice?

- A **sup-lattice** $\mathcal{L} = (L, \bot, \vee)$ has carrier L, bottom element $\bot \in L$, and join operation $\vee$.
- A partial ordering is given by $\vee$ which can be expressed as $x \leq y \iff x \vee y = y$ and $\bot \leq x$.
- Additionally, sup is a unique map from elements to their **least upper bound** (supremum) element if it exists, so $x \leq sup(x, y)$, $y \leq sup(x, y)$ and $x \leq z,\ y \leq z \implies sup(x, y) \leq z$.
- We can also express the notion of arbitrary sups as joins: $x_i \leq \bigvee_{i \in I} x_i$ and $x_i \leq z,\ \forall i \in I \implies \bigvee_{i \in I} x_i \leq z$

Power Set Functor

- Let $\mathscr{P}$: **Set** $\to$ **Set** denote the power set functor. We can write $\mathscr{P} = \mathcal{PU}$ where $\mathcal{P}$: **Set** $\to$ **SupLat** lifts a set to a sup-lattice where the carrier is its power set and $\bot := \emptyset, \vee := \cup$ and $\mathcal{U}$: **SupLat** $\to$ **Set** is the underlying functor that maps a sup-lattice back to the set that is its carrier.
- These functors form an adjunction $(\eta, \epsilon) : \mathcal{P} \dashv \mathcal{U} :$ **Set** $\to$ **SupLat** with $\eta : 1_{\textbf{Set}} \Rightarrow \mathcal{PU}$ which sends elements to singletons.
  An abstract component map is $\eta_X : X \to \mathcal{PU}(X)\ ;\ x \mapsto \{x\}$.
  The power set is formed by the sup-lattice with bottom element $\emptyset$ and the singletons are joined with $\cup$.

SupLat is a Monoidal Category

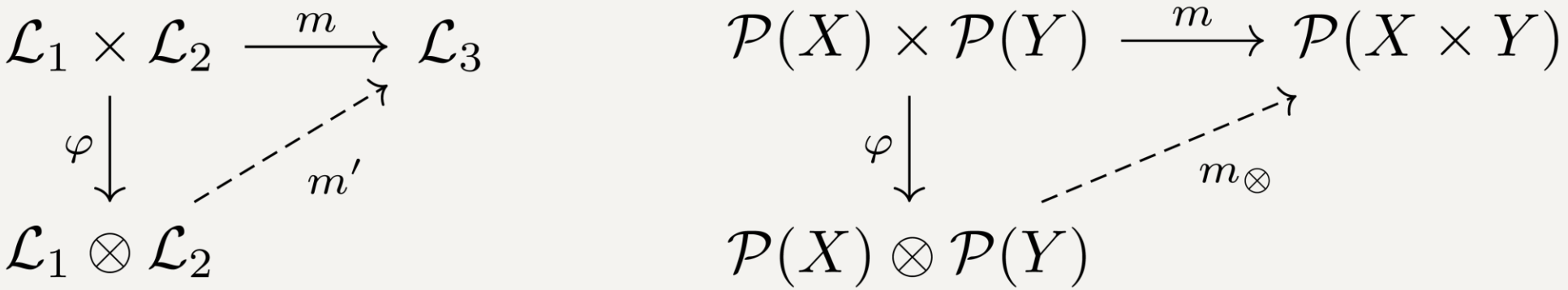
- **SupLat** is the category of sup-lattices where the objects are sup-lattices and the morphisms are sup-preserving maps.
- A sup-preserving map ensures the sup for a set of elements is mapped to the sup for the mappings of those elements: $f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)$
- The unit sup-lattice I has carrier $\{\top, \bot\}$.
- The tensor of sup-lattices, $\mathcal{L}_1 \otimes \mathcal{L}_2$, creates a sup-lattice with carrier $\{\bigvee(x_1 \otimes x_2) \mid x_1 \in L_1, x_2 \in L_2\}$, where $\bot \otimes x = \bot = x \otimes \bot$.
  Maps out of the tensor are sup-preserving and bi-sup-preserving:
  $(\bigvee_{i \in I} x_i) \otimes y = \bigvee_{i \in I} (x_i \otimes y) \quad x \otimes (\bigvee_{j \in J} y_j) = \bigvee_{j \in J} (x \otimes y_j)$

Power Set Functor is a Monoidal Functor

- We want to show that $\mathcal{P}$: **Set** $\to$ **SupLat** is a monoidal functor; that is, the monoidal structure in **Set** $(\times, *, a_\times, l_\times, r_\times)$ is lifted to the monoidal structure in **SupLat** $(\otimes, I, a_\otimes, l_\otimes, r_\otimes)$.
- To show that $\mathcal{P}$ is a monoidal functor, we must define comparison maps and show that the necessary coherence diagrams commute.
- $m_1 : I \to \mathcal{P}(*)$ is given by $\top \mapsto *, \bot \mapsto \emptyset$.
- $m_\otimes : \mathcal{P}(X) \otimes \mathcal{P}(Y) \to \mathcal{P}(X \times Y)$ is given by the universal property of the tensor of sup-lattices.

Power Set Functor is a Monoidal Functor

- The diagram on the left is the universal property of the tensor which says, if we have a bi-sup-preserving map m, there exists the map m':



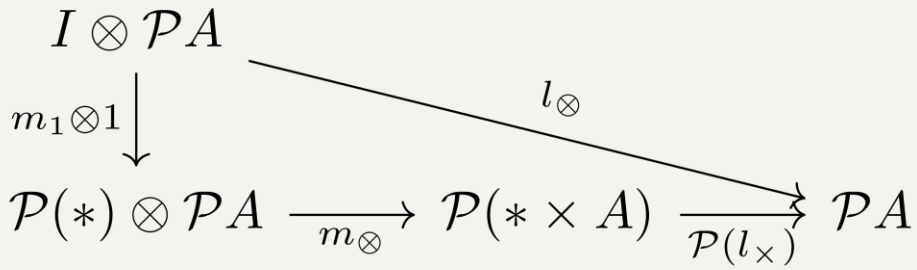
- If we set $\mathcal{L}_3 := \mathcal{P}(X \times Y)$ and show the map m exists, the m' map is the $m_\otimes$ map we want, as shown in the diagram on the right.

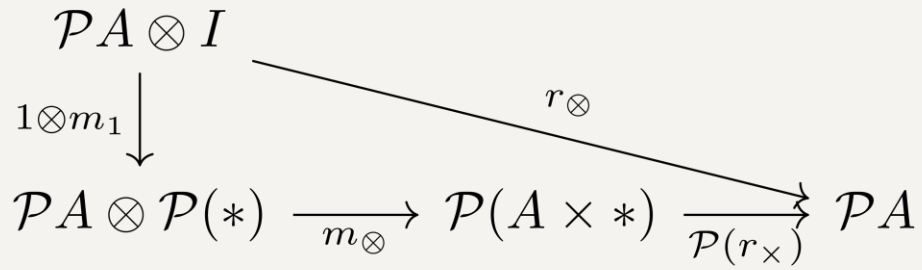
Power Set Functor is a Monoidal Functor

- Suppose (U, V) is an element of the carrier of $\mathcal{P}(X) \times \mathcal{P}(Y)$.
- Define $m : (U, V) \mapsto \{(u, v) \mid u \in U, v \in V\}$.
- Since m is a map out of the product of sup-lattices, it must be bi-sup-preserving, so we want to show that $m(\bigvee_i U_i, V) = \bigvee_i m(U_i, V)$ as well as the symmetric case.
- By definition, $m(\bigvee_i U_i, V) \mapsto \{(u, v) \mid u \in \bigvee_i U_i, v \in V\}$, and $\bigvee_i m(U_i, V) \mapsto \bigvee_i \{(u, v) \mid u \in U_i, v \in V\}$.
- These are equivalent as arbitrary elements from each are included in the other, so we have defined our m map and therefore $m_\otimes$.

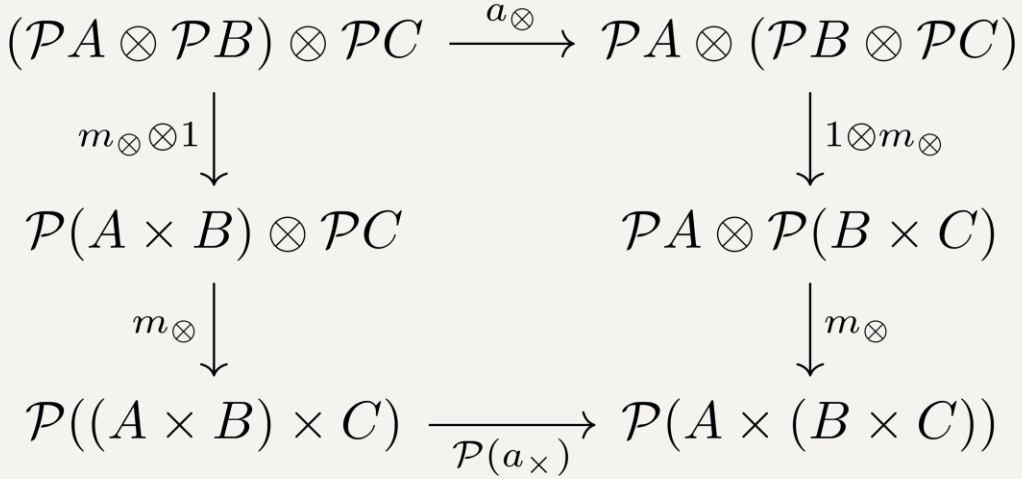
Power Set Functor is a Monoidal Functor

- Left and right unitors:





- Associativity:



Change of Base

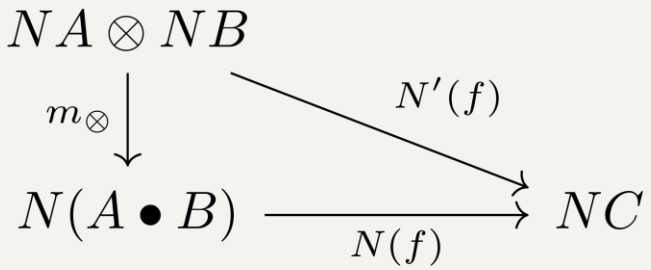
Again, we use results from Geoff's PhD thesis to define the category produced by a change of base.

- Proposition 4.2.1. Let N be as defined, and X a V-category. Then the following defines a W-category N_*X:
  - N_*X has the same objects as those of X,
  - hom-objects are defined by $(N_*X)(x, y) := N(X(x, y))$,
  - composition is N applied monoidally to the composition in X,
  - identities are N applied monoidally to the identities in X.

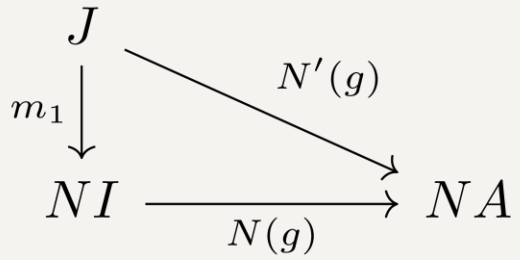
Applying Monoidal Functors Monoidally

There are two ways to apply a monoidal functor N "monoidally":

- Given an arrow in V,
  $f : A \bullet B \to C$
  we apply N monoidally to get an arrow in W,
  $N'(f) : NA \otimes NB \to NC$
  defined as the composite:



- Given an arrow in V,
  $g : I \to A$
  we apply N monoidally to get an arrow in W,
  $N'(g) : J \to NA$
  defined as the composite:



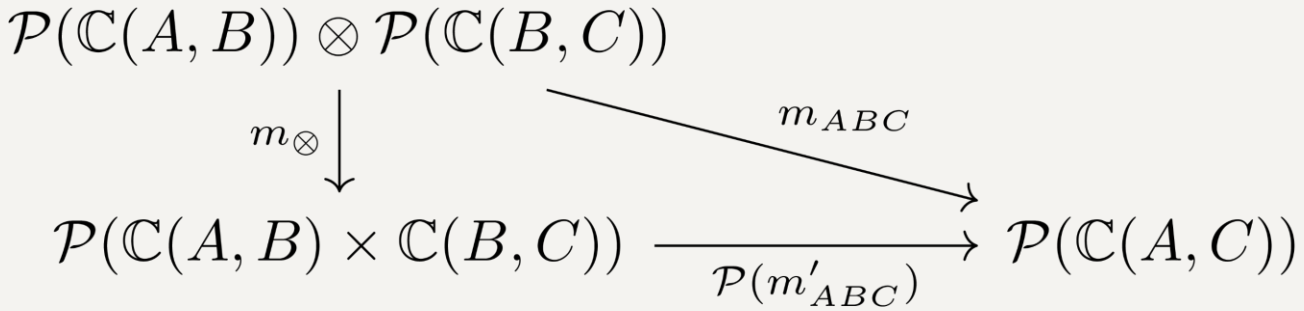
Sup-Lattice Enrichment

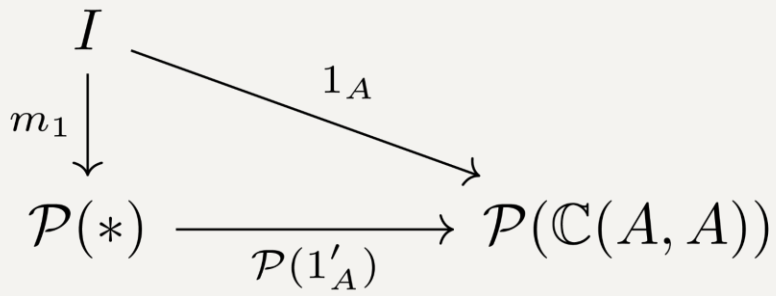
Let $\mathbb{C}$ be the set-enriched deterministic semantics category, then define $\mathcal{P}_*\mathbb{C}$ as the sup-lattice enriched non-deterministic semantics category produced by the change of base:

- Objects : Obj $(\mathbb{C})$
- Hom-objects : $(\mathcal{P}_*\mathbb{C})(A, B) := \mathcal{P}(\mathbb{C}(A, B))$
- Identity : $\mathcal{P}$ applied monoidally to the identities in $\mathbb{C}$ to produce: $1_A : I \to \mathcal{P}(\mathbb{C}(A, A))$
- Composition : $\mathcal{P}$ applied monoidally to the composition in $\mathbb{C}$ to produce: $m_{ABC} : \mathcal{P}(\mathbb{C}(A, B)) \otimes \mathcal{P}(\mathbb{C}(B, C)) \to \mathcal{P}(\mathbb{C}(A, C))$

Sup-Lattice Enrichment

- Composite diagrams for composition and identity maps:





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