McKay for Reflection Groups and Semiorthogonal Decompositions

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Foundational Methods in Computer Science
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Outline

- Classical McKay Correspondence
- Derived equivalence and generalizing McKay
- Semi-orthogonal decompositions
- Reflection groups example
- Further work on reflection groups

Classical McKay Correspondence

McKay (1980): Finite subgroups $G \leq \mathrm{SL}(2, \mathbb{C})$

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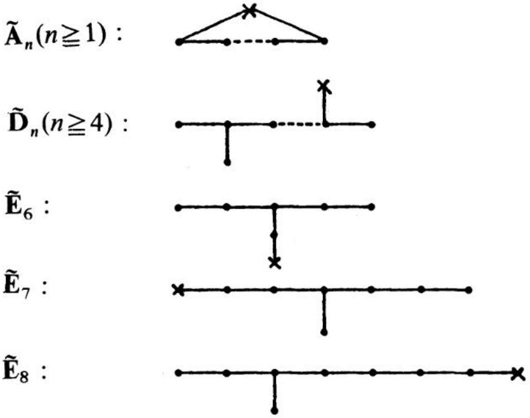
Visualization of the connection: ADE Dynkin diagrams



Example: A_n

Natural representation:

$$\rho_{\mathsf{nat}} : \mathbb{Z}/(n+1) \ \leq \ \mathsf{SL}(2,\mathbb{C})$$

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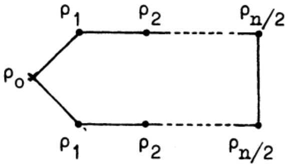

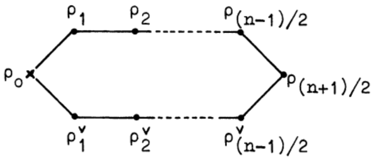

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- The equivalence comes from the universal closed subscheme of G-Hilb($\mathbb{C}^2$) as a moduli space.

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- Polishchuk–Van den Bergh Conjecture: $G \leq \mathrm{GL}(2, \mathbb{C})$ reflection group
  $D^b_G(\mathbb{C}^2)$ has SOD in bijection with irred. rep.s of G

Semi-Orthogonal Decompositions

SOD with $E_1, \ldots, E_m$ exceptional sequence:

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- When G is a reflection group (my case of interest) it is conjectured the exceptional objects should be in bijection with the non-trivial irreducible representations of G.

Some reflection groups of interest

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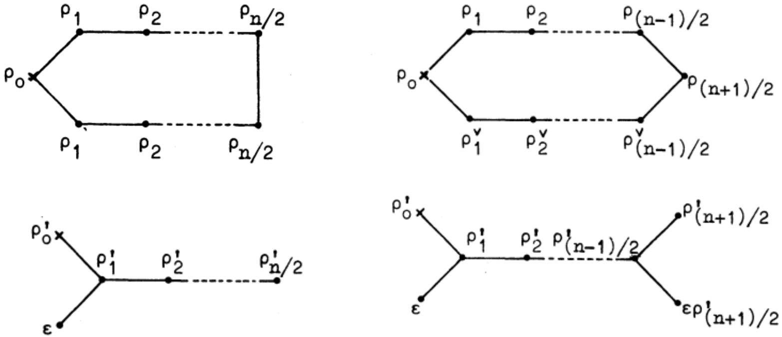
- Classical McKay groups in SL$(2,\mathbb{C})$ are all index 2 subgroups of reflection groups in $U(2,\mathbb{C})$.
- Representation theory closely related to intersections with SL$(2,\mathbb{C})$
- A'_n and A_n example:



A'_n case

- Potter (2018): Explicit description of SOD for A'_n.
- Let $G \leq \mathrm{GL}(2, \mathbb{C})$, $H := G \cap \mathrm{SL}(2, \mathbb{C})$, $A := G/H \simeq \mathbb{Z}/(2)$, $Y := H\text{-Hilb}(\mathbb{C}^2)$, M the minimal res. of $\mathbb{C}^2/G$ (itself)

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  - $D^b_A(Y) \cong \langle D([Y/A]^{\mathrm{can}}, D(\tilde{D}_1), \ldots, D(\tilde{D}_r), \text{intersections of } \tilde{D}_i \rangle$ D_i are components of branch locus of A acting on H-Hilb($\mathbb{C}^2$).

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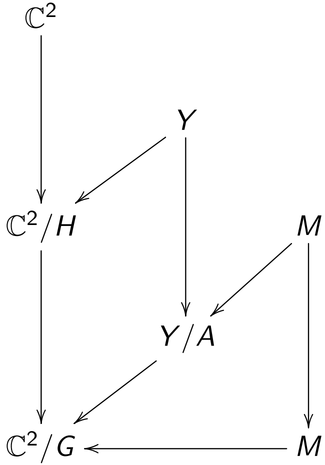
  - If Y/A is smooth... (toric charts for Y)
  - $D([Y/A]^{\mathrm{can}} \cong \langle D(M), E_1, \ldots, E_s\rangle$ where the E_i are divisors to be blown down in Y/A to get M.



$\mathbb{C}^2$
Y
$\mathbb{C}^2/H$
M
Y/A
$\mathbb{C}^2/G$
M

Results in A'_n case

- n even: $D_G(\mathbb{C}^2) \cong \langle D(M), E_1, \ldots, E_{\frac{n}{2}}, D(\tilde{D}) \rangle$
- n odd: $D_G(\mathbb{C}^2) \cong \langle D(M), E_1, \ldots, E_{\frac{n+1}{2}}, D(\tilde{D}_1), D(\tilde{D}_2) \rangle$
- Capellan (2024) confirms matching of semi-orthogonal decomposition with representations.

Work in progress: other reflection groups

- Same strategy for SOD's as Potter.
- However, these singularities are not toric
- Main tool: explicit computations in H-Hilb($\mathbb{C}^2$) using Ito and Nakamura's work Example: $G = G_{12}$

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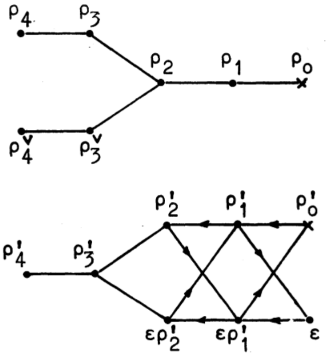

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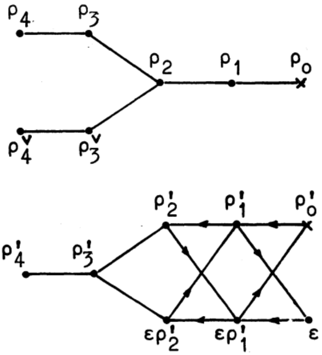

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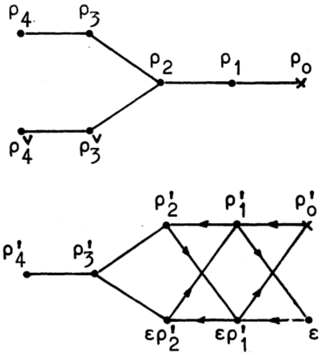

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Interesting difference: Branch locus of Y/A comes from both $\mathbb{C}^2/G$ branch locus and a fixed $\mathbb{P}^1$ in the exceptional locus of Y

Thank you!