FORT LEWIS COLLEGE

# Fundamental Groupoids for Graphs

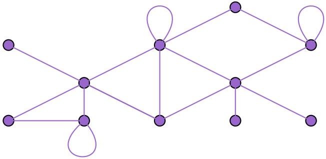
Laura Scull

FMCS, Kananaskis, 2024

# Category of Graphs

Gph is the category with:

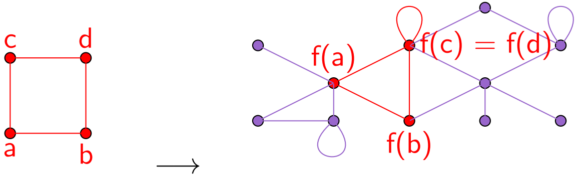
- **Objects** are graphs $G$ with:
  - A set of vertices $V(G)$
  - A set of edges $E(G)$ which are unordered pairs of vertices $\{v, w\}$; notate $v \sim w$
  - We have at most one edge between any two vertices; loops are allowed.



# Graph Homomorphisms

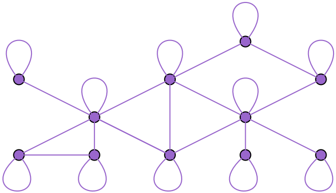
Gph is the category with:

- **Homomorphisms** $f : G \to H$ map vertices to vertices and respect adjacency:
  - $f : V(G) \to V(H)$ a function of sets
  - If $v \sim w \in E(G)$, then $f(v) \sim f(w) \in E(H)$
  - If $v \sim w$ we can map $v, w$ to the same vertex if we have a loop



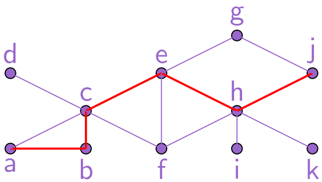
# Reflexive Graphs

Let's consider only reflexive graphs:



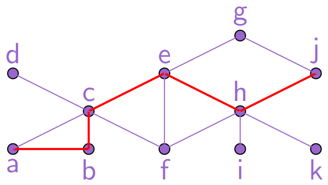
# Reflexive Fundamental Groupoid

Look at (looped) walks: *abbcehhj*



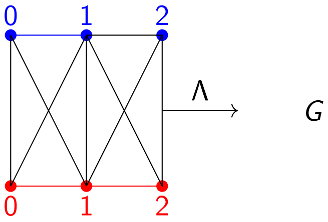
# Prunes of Walks

Remove repeated vertices: $abbcehhj = abcehj$



# $\times$-homotopy of Walks

defined by $\Lambda : I_n \times I_m \to G$

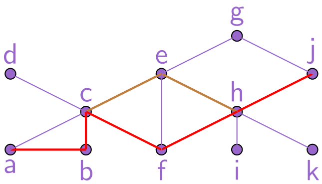



$I_2 \times I_1$

# $\times$-homotopy of Walks

$\times$-homotopy: change one vertex to another connected vertex

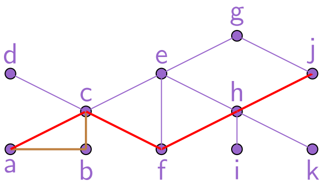
*abcehj = abcfhj*



# Equivalences of Walks

- prunes: remove repeats
- $\times$-homotopy: change one vertex to another connected vertex

*abcfhj* = *accfhj* = *acfjk*

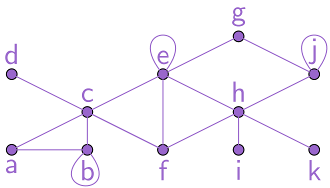


# Reflexive Fundamental Groupoid

For a reflexive graph $G$, we define the **reflexive fundamental groupoid** $\Pi_r(G)$, as follows:

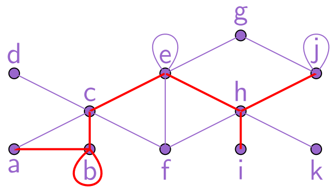
- Objects of $\Pi_r(G)$ are vertices $v$ of the graph $G$.
- An arrow $v_0 \to v_n$ in $\Pi_r(G)$ is given by a walk $v_0 v_1 v_2 v_3 \dots v_n$ where $v_i \sim v_{i+1}$.
- two walks represent the same arrow if they are equvialent under:
  - prunes which remove repetition : $[vv] = [v]$
  - homotopy rel endpoints as map $I_n \to G$, $I_n$ looped ie shifting one vertex to an adjacent vertex
- Composition of morphisms is defined using concatenation of walks.

# Non-reflexive Graphs



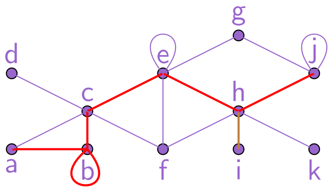
# Fundamental Groupoid

Look at unlooped walks: *abbcehihj*



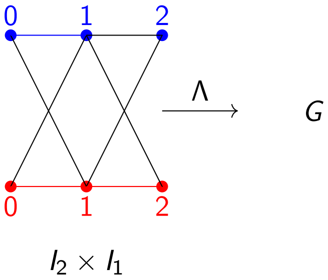
# Prunes of Walks

Remove out-and-back: *abbcehihj* = *abbcehj*



# ×-homotopy of Walks

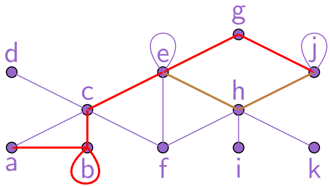
defined by $\Lambda : P_n \times I_m \to G$



# ×-homotopy of Walks

Change one vertex to another, does NOT need to be connected

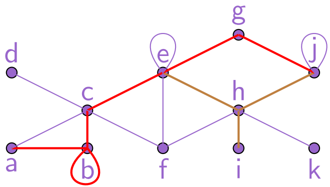
*abbceh* = *abbcegj*



# Equivalences of Walks

- prunes: remove out-and-back
- ×-homotopy: change one vertex to another vertex (doesn't need to be connected)

*abbcehihj* = *abbceh* = *abbcegj*



# Fundamental Groupoid of $G$

For a graph $G$, we define the **fundamental groupoid** $\Pi(G)$ as follows:

- Objects of $\Pi(G)$ are vertices $v$ of the graph $G$.
- An arrow $v_0 \to v_n$ in $\Pi(G)$ is given by a walk $v_0 v_1 v_2 v_3 \dots v_n$ where $v_i \sim v_{i+1}$.

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- two walks represent the same arrow if they are equvialent under:
  - prunes which remove a backtrack : $[vwv] = [v]$
  - homotopy rel endpoints as map $P_n \to G$, $P_n$ unlooped ie shifting one vertex to a (not necessarily adjacent) vertex
- Composition of morphisms is defined using concatenation of walks.

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    ie shifting one vertex to a (not necessarily adjacent) vertex
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- Walks are even or odd

# Walks of Edges

Idea:

- graphs are built out of edges, connected with vertices
- create walks that are sequences of edges

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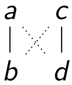
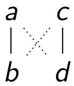
- two consecutive edges are ×-homotopic as maps $K_2 \to G$



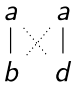
# Walks of Edges

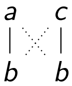
Idea:

- graphs are built out of edges, connected with vertices
- create walks that are sequences of edges
- two consecutive edges are $\times$-homotopic as maps $K_2 \to G$



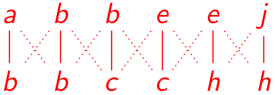

In particular, two edges are adjacent when they share a vertex:

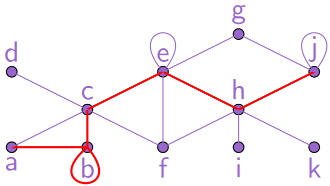


# Walks of Edges

*abbcehj* becomes



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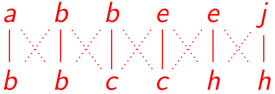
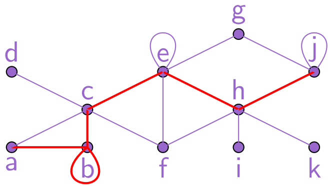



Edges are ordered and cartwheel through the graph

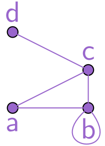
# Edge Graph

Define the edge graph of $G$, denoted, $G^E$, as looped subgraph of exponential $G^{K_2}$:

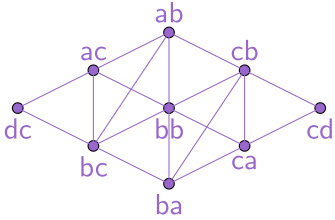
- vertices of $G^E$ are graph homomorphisms $K_2 \to G$
  ie a (directed) edge
- edges of $G^E$ between homomorphisms that are $\times$-homotopic
- $G^E$ is a reflexive graph

# Example of $G^E$

Suppose we have $G$:




Then $G^E$ is the reflexive graph (loops suppressed):



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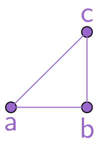


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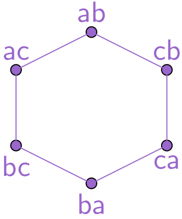
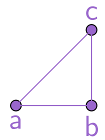


# First Connection

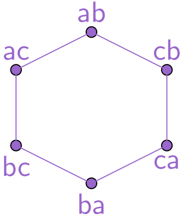
Since $G^E$ is reflexive, we can form the reflexive fundamental groupoid $\Pi_r(G^E)$

Thm $\Pi_r(G^E)$ is equivalent to the even subgroupoid of $\Pi(G)$

$G$:



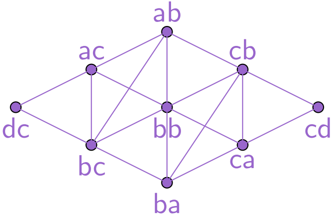

$G^E$:



# $\mathbb{Z}/2$ action on Edges

We have said:

- we have $\mathbb{Z}/2$ action on $G^E$ that flips edge, reversing direction
- we can form an equivariant fundamental groupoid



# Equivariant Fundamental Groupoid

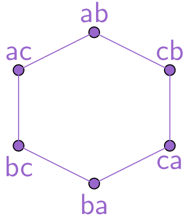
$G$ a reflexive graph with a $\mathbb{Z}/2$ action.
$\Pi_{\mathbb{Z}/2}G$ defined by:

- objects are vertices of $G$
- arrows $x_0 \to x_n$ of two forms:
  - a walk $\alpha$ from $x_0$ to $x_n$ in $\Pi_r(G)$. Denote by $(\alpha, 1)$.
  - a walk $\beta$ from $x_0$ to $\tau x_n$ in $\Pi_r(G)$, plus a 'jump' by the non-zero element $\tau \in \mathbb{Z}/2$. Denote by $(\beta, \tau)$.

Composition via concatenation:

- $(\alpha, 1) * (\beta, 1) = (\alpha * \beta), 1)$
- $(\alpha, 1) * (\beta, \tau) = (\alpha * \beta), \tau)$
- $(\alpha, \tau) * (\beta, 1) = (\alpha * \tau(\beta), \tau)$
- $(\alpha, \tau) * (\beta, \tau) = (\alpha * \tau(\beta), 1)$

# A Loop in $\Pi_{\mathbb{Z}/2} G$




- graph is reflexive, loops suppressed
- equip with antipodal $\mathbb{Z}/2$-action

A loop in $\Pi_{\mathbb{Z}/2} G$: $(ab - cd - ca - ba, \tau)$

# Better Connection

Given $G$ a graph we have:

- $\Pi(G)$ the non-reflexive fundamental groupoid of $G$
- $G^E$ the reflexive edge graph with $\mathbb{Z}/2$ action
- $\Pi_{\mathbb{Z}/2}(G^E)$ the $\mathbb{Z}/2$ reflexive fundamental groupoid of $G^E$

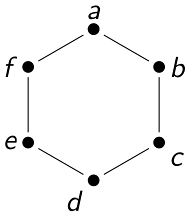
# Better Connection

Given $G$ a graph we have:

- $\Pi(G)$ the non-reflexive fundamental groupoid of $G$
- $G^E$ the reflexive edge graph with $\mathbb{Z}/2$ action
- $\Pi_{\mathbb{Z}/2}(G^E)$ the $\mathbb{Z}/2$ reflexive fundamental groupoid of $G^E$

Thm $\Pi(G)$ is equivalent to $\Pi_{\mathbb{Z}/2}(G^E)$

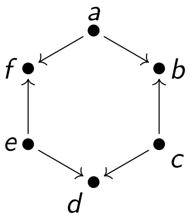
# Example of Even Walk



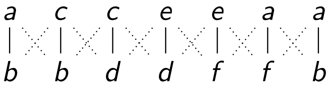

Consider

$$(abcdefa)$$

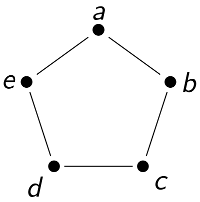
# Example of Even Walk




$$(abcdefa) \to (\alpha, 1):$$

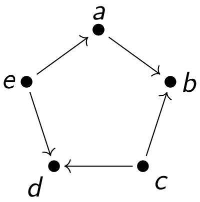


# Example of Odd Walk

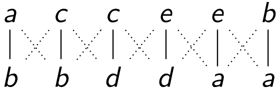



$(abcdea)$

# Example of Odd Walk




$$(abcdea) \to (\alpha, \tau):$$



# Future Questions:

- simplicial complexes
  - Neighbourhood complex
  - Hom complex
- higher homotopy groups??

# Time for a Hike!