Medial Linearly Distributive Categories

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Fox's Theorem

Definition

Consider a symmetric monoidal category, or SMC, $(\mathcal{X}, \oslash, I, \alpha, \rho, \gamma)$.

- A *cocommutative comonoid* is a triple $\langle A, \Delta_A, t_A \rangle$ of an object A in $\mathcal{X}$ equipped with two morphisms, the *diagonal* $\Delta_A : A \to A \oslash A$ and the *counit* $t_A : A \to I$ such that:

$$\Delta_A; (1_A \oslash \Delta_A) = \Delta_A; (\Delta_A \oslash 1_A); \alpha_{A,A,A}$$

$$\Delta_A; (1_A \oslash t_A) = \rho_A \qquad \Delta_A; \gamma_{A,A} = \Delta_A$$

- A *comonoid morphism* $f : \langle A, \Delta_A, t_A \rangle \to \langle B, \Delta_B, t_B \rangle$ is a morphism $f : A \to B$ in $\mathcal{X}$ such that

$$f; \Delta_B = \Delta_A; (f \oslash f) \qquad f; t_B = t_A$$

Fox's Theorem (cont'd)

Let $\mathrm{C}(\mathcal{X})$ denote the category of cocommutative comonoids and comonoid morphisms.

Proposition

$\mathrm{C}(\mathcal{X})$ *is a cartesian category.*

Theorem (Fox [6])

The functor $\mathrm{C}(-)\colon \underline{\mathsf{SMON}} \to \underline{\mathsf{CART}}$ *is right adjoint to the inclusion.*

Corollary

A SMC $\mathcal{X}$ *is cartesian if and only if it is isomorphic to its category of cocommutative comonoids* $\mathrm{C}(\mathcal{X})$*.*

Cartesian LDCs

Definition (Cockett, Seely [4])

A *symmetric linearly distributive category*, or SLDC, $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ consists of:

- a category $(\mathbb{X}, ;, 1_A)$,
- a symmetric *tensor* monoidal structure $(\mathbb{X}, \otimes, \top)$,
- a symmetric *par* monoidal structure $(\mathbb{X}, \oplus, \bot)$, and
- left and right *linear distributivity* natural transformations

$$\delta^R : (A \oplus B) \otimes C \to A \oplus (B \otimes C) \qquad \delta^L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

A *cartesian linearly distributive category* is a SLDC such that the tensor $\otimes$ is the product ($\top$ is the terminal object) and $\oplus$ is the coproduct ($\bot$ is the initial object).

Cartesian LDCs (cont'd)

Example

1. Cartesian $*$-autonomous category $\iff$ Boolean algebra
   $\to$ Joyal's paradox: a cartesian closed category $+$ involutive negation $\implies$ any two arrows $A \to B$ identified [7, Thm 12]
2. Distributive lattice
   $\to$ A distributive category is a cartesian LDC if and only if it is a preorder [4, Prop 5.1].
3. Category with all finite biproducts
   $\to$ Rel, with the disjoint union
   $\to$ SupLat, with the cartesian product
   $\to$ Ab, with the direct sum
   $\to$ Compact closed category with products (or coproducts) [9]

Motivation

Motivation: Is there a Fox-like theorem for cartesian LDCs?

- A SLDC is cartesian if and only it is isomorphic to its own LDC of what?
- Does the construction work for all SLDC?
- Does this determine a functor which is adjoint to the inclusion?

Characterizing cartesian LDCs

By Fox's theorem and its dual statement, we get:

Proposition

A SLDC $\mathbb{X}$ is cartesian if and only if there are natural transformations

$$\Delta_A : A \to A \otimes A \qquad t_A : A \to \top \qquad \nabla_A : A \oplus A \to A \qquad s_A : \bot \to A$$

such that, $\forall A, B \in \mathbb{X}$,

- $\langle A, \Delta_A, t_A \rangle$ *determines a $\otimes$-cocommutative comonoid,*
- $\langle A, \nabla_A, s_A \rangle$ *determines a $\oplus$-commutative monoid, and*

$$\Delta_{A\otimes B} = (\Delta_A \otimes \Delta_B); s^{\otimes}_{A,A,B,B} \qquad t_{A\otimes B} = (t_A \otimes t_B); {u^R_{\otimes\top}}^{-1}$$

$$\nabla_{A\oplus B} = s^{\oplus}_{A,B,A,B}; (\nabla_A \oplus \nabla_B) \qquad s_{A\oplus B} = {u^R_{\oplus\bot}}^{-1}; (s_A \oplus s_B)$$

$$\Delta_\top = u^R_{\otimes\top} \qquad t_\top = 1_\top \qquad \nabla_\bot = u^R_{\oplus\bot} \qquad s_\bot = 1_\bot$$

Characterizing cartesian LDCs

If we consider any SLDC $\mathbb{X}$ and try forming the category of such quintuples $\langle A, \Delta_A, t_A, \nabla_A, s_A \rangle$, we quickly realize this does not define a LDC:

$$\nabla_{A \otimes B} : (A \otimes B) \oplus (A \otimes B) \xrightarrow{?} (A \oplus A) \otimes (B \oplus B) \xrightarrow{\nabla_A \otimes \nabla_B} A \otimes B$$

$$\Delta_{A \oplus B} : A \oplus B \xrightarrow{\Delta_A \oplus \Delta_B} (A \otimes A) \oplus (B \otimes B) \xrightarrow{?} (A \oplus B) \otimes (A \oplus B)$$

$$s_{A \otimes B} : \bot \xrightarrow{?} \bot \otimes \bot \xrightarrow{s_A \otimes s_B} A \otimes B \qquad t_{A \oplus B} : A \oplus B \xrightarrow{t_A \oplus t_B} \top \oplus \top \xrightarrow{?} \top$$

The above “unknown” arrows may or may not exist in any given LDCs. They do exist in all cartesian LDCs by the universal properties of products and coproducts.

$\Longrightarrow$ We need a SLDC $\mathbb{X}$ which has arrows $\forall A, B, C, D,$

$$(A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$$

$$\bot \to \bot \otimes \bot \qquad \top \oplus \top \to \top$$

Medial rule

$(A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$ is known as the **medial rule**.

It has appeared alongside switch (linear distributivity) in different systems of logic, especially within **deep inference** (introduced by A. Guglielmi):
$\to$ it allows the systems to become local (the rules, in particular contraction, can be given in their atomic state)

Medial rule has been considered in a local system for classical logic [2], for intuitionistic logic [13] and for linear logic [11].

The medial rule has also been studied in the categorical semantics for classical logic and in defining the concept of “Boolean category”:

- $*$-autonomous categories with finitary medial and the absorption law (Lamarche [10])
- B3-category (Strassburger [12])

Defining medial LDCs

Definition

A symmetric *medial LDC* is a SLDC $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ equipped with morphisms (called the *nullary medial* and *comedial* maps)

$$\nabla_\top : \top \oplus \top \to \top \qquad \Delta_\bot : \bot \to \bot \otimes \bot$$

and a *medial* natural transformation

$$\mu_{A,B,C,D} : (A \otimes B) \oplus (C \otimes D) \to (A \oplus C) \otimes (B \oplus D)$$

such that

- $\nabla_\top$ equips $\top$ with a commutative semigroup structure,
- $\Delta_\bot$ equips $\bot$ with a cocommutative semigroup structure,
- the *medial maps* interact coherently with γ, α, δ, and
- the *absorption laws* holds.

Mix LDCs

Definition (Cockett, Seely [3])

A LDC $\mathbb{X}$ is *mix* if there is a morphism $m : \bot \to \top$ such that $\forall A, B \in \mathbb{X}$, the two induced maps $A \otimes B \to A \oplus B$ are equal:

$$A \otimes B \cong (A \oplus \bot) \otimes B \xrightarrow{\delta^R} A \oplus (\bot \otimes B) \xrightarrow{1 \oplus (m \otimes 1)} A \oplus (\top \otimes B) \cong A \oplus B$$

and

$$A \otimes B \cong A \otimes (\bot \oplus B) \xrightarrow{\delta^L} (A \otimes \bot) \oplus B \xrightarrow{(1 \otimes m) \oplus 1} (A \otimes \top) \oplus B \cong A \oplus B$$

Proposition

A medial LDC is mix.

Duoidal categories

LDCs are not the only category with two monoidal structures:

Definition (Aguiar, Mahajan [1])

A *duoidal category* $(\mathcal{X}, \star, J, \diamond, I)$ is category $\mathcal{X}$ with two monoidal structures $(\mathcal{X}, \star, J)$ and $(\mathcal{X}, \diamond, I)$ equipped with morphisms

$$\Delta_I : I \to I \star I \qquad \nabla_J : J \diamond J \to J \qquad \iota : I \to J$$

and an *interchange* natural transformation

$$\zeta_{A,B,C,D} : (A \star B) \diamond (C \star D) \to (A \diamond C) \star (B \diamond D)$$

satisfying some coherence conditions.

$\to$ A medial LDC is a duoidal category with monoidal structures and further equipped with linear distributivities.

Defining bimonoids

Let $\mathbb{X}$ be a symmetric medial linearly distributive category.

Definition (Aguiar, Mahajan [1])

A bicommutative *bimonoid* in $\mathbb{X}$ is a quintuple $\langle A, \Delta_A, t_A, \nabla_A, s_A \rangle$ consisting of an object A and four morphisms

$$\Delta_A : A \to A \otimes A \qquad t_A : A \to \top \qquad \nabla_A : A \oplus A \to A \qquad s_A : \bot \to A$$

such that $\langle A, \Delta_A, t_A \rangle$ is a cocommutative $\otimes$-comonoid, $\langle A, \nabla_A, s_A \rangle$ is a commutative $\oplus$-monoid, and satisfying coherence conditions

$$\nabla_A ; \Delta_A = (\Delta_A \oplus \Delta_A) ; \mu_{A,A,A,A} ; (\nabla_A \otimes \nabla_A) \qquad s_A ; t_A = m$$

$$\nabla_A ; t_A = (t_A \oplus t_A) ; \nabla_\top \qquad s_A ; \Delta_A = \Delta_\bot ; (s_A \otimes s_A)$$

A *bimonoid morphism* is an arrow $f : A \to B$ that is a $\otimes$-comonoid morphism and $\oplus$-monoid morphism.

Defining bimonoids (cont'd)

Proposition

$\langle \top, u^R_{\otimes\top}, 1_\top, \nabla_\top, m\rangle$ and $\langle \bot, \Delta_\bot, m, u^R_{\oplus\bot}, 1_\bot\rangle$ are bimonoids.

Given two bicommutative bimonoids $\langle A, \Delta_A, t_A, \nabla_A, s_A\rangle$ and $\langle B, \Delta_B, t_B, \nabla_B, s_B\rangle$ in $\mathbb{X}$, then $\langle A\otimes B, \Delta_{A\otimes B}, t_{A\otimes B}, \nabla_{A\otimes B}, s_{A\otimes B}\rangle$ defined by

$$\Delta_{A\otimes B} = (\Delta_A \otimes \Delta_B); s^{\otimes}_{A,A,B,B} \qquad t_{A\otimes B} = (t_A \otimes t_B); {u^R_{\otimes\top}}^{-1}$$
$$\nabla_{A\otimes B} = \mu_{A,B,A,B}; (\nabla_A \otimes \nabla_B) \qquad s_{A\otimes B} = \Delta_\bot; (s_A \otimes s_B)\ ,$$

and $\langle A\oplus B, \Delta_{A\oplus B}, t_{A\oplus B}, \nabla_{A\oplus B}, s_{A\oplus B}\rangle$ defined by

$$\Delta_{A\oplus B} = (\Delta_A \oplus \Delta_B); \mu_{A,A,B,B} \qquad t_{A\oplus B} = (t_A \oplus t_B); \nabla_\top$$
$$\nabla_{A\oplus B} = s^{\oplus}_{A,B,A,B}; (\nabla_A \oplus \nabla_B) \qquad s_{A\oplus B} = {u^R_{\oplus\bot}}^{-1}; (s_A \oplus s_B)\ ,$$

are bicommutative bimonoids.

Cartesian LDC of bimonoids

Definition

Define $\mathrm{Bim}(\mathbb{X})$ to be the category of bicommutative bimonoids and bimonoid morphisms in $\mathbb{X}$.

Theorem

$\mathrm{Bim}(\mathbb{X})$ is a cartesian linearly distributive category.

Examples of medial LDCs

Example

1. $*$-autonomous categories with finitary medial and the absorption law [10]
2. Symmetric monoidal categories, viewed as compact LDCs
   $\rightarrow$ medial maps are given by associativities and symmetries:
   $\alpha_{A,B,C\oslash D}; (1_A \oslash \alpha^{-1}_{B,C,D}); (1_A \oslash (\gamma_{B,C} \oslash 1_D)); (1_A \oslash \alpha_{C,B,D}); \alpha^{-1}_{A,C,B\oslash D}$
   $= \alpha_{A,B,C\oslash D}; (1_A \oslash \gamma_{B,C\oslash D}); (1_A \oslash \alpha_{C,D,B}); (1_A \oslash (1_C \oslash \gamma_{D,C})); \alpha^{-1}_{A,C,B\oslash D}$
   $= s^{\oslash}_{A,B,C,D} : (A \oslash B) \oslash (C \oslash D) \rightarrow (A \oslash C) \oslash (B \oslash D)$
3. Cartesian linearly distributive categories
   $\rightarrow$ medial maps given by universal properties of (co)products:
   $[\amalg^0_{A,C} \times \amalg^0_{B,D}, \amalg^1_{A,C} \times \amalg^1_{B,D}] : (A \times B) + (C \times D) \rightarrow (A + C) \times (B + D)$

note: $\amalg^0_{X,Y} : X \rightarrow X + Y$ and $\amalg^1_{X,Y} : Y \rightarrow X + Y$ denote injections, while $[f, g] : X + Y \rightarrow Z$ denotes unique map given by coproduct

Examples of medial LDCs (cont'd)

Recall the categories of coherent spaces COH (Girard [8]) and hypercoherences HCohL (Ehrhard [5]), models of linear logic and full classical linear logic respectively.

These were generalized (by Lamarche) as follows:

Definition ([10])

Let Q denote a LD-poset.

- A *Q-coherence* $A = (|A|, \rho_A)$ consisting of a poset $(|A|, \sqsubseteq)$ and a symmetric monotone function $\rho_A \colon |A| \times |A| \to Q$.
- A *Q-coherence map* $f \colon A \to B$ is a relation $f \colon |A| \mathrel{\multimap\!\!\!\to} |B|$ which is
  - down-closed in the source: $(a, b) \in f \wedge a' \sqsubseteq a \implies (a', b) \in f$,
  - up-closed in the target: $(a, b) \in f \wedge b \sqsubseteq b' \implies (a, b') \in f$,
  - $(a, b) \in f \wedge (a', b') \in f \implies \rho_A(a, a') \leq \rho_B(b, b')$.

Examples of medial LDCs (cont'd)

Definition ([10])

Define Q-Coh to be the LDC of Q-coherences with

$$A \otimes B = (|A| \times |B|, \rho_{A \otimes B}), \quad \rho_{A \otimes B}((a,b),(a',b')) = \rho_A(a,a') \otimes \rho_B(b,b')$$

$$A \otimes B = (|A| \times |B|, \rho_{A \oplus B}), \quad \rho_{A \oplus B}((a,b),(a',b')) = \rho_A(a,a') \oplus \rho_B(b,b')$$

Theorem

Q-Coh is a medial LDC if and only if Q is a medial LD-poset, with medial maps are relations defined by

$$(a,b,c,d)\mu_{A,B,C,D}(a',c',b',d') \iff a \sqsubseteq a' \wedge b \sqsubseteq b' \wedge c \sqsubseteq c' \wedge d \sqsubseteq d'$$

Example

4. Q-Coh for a medial LD-poset Q
   $\rightarrow$ All distributive lattices are examples of medial LD-posets.

Further Work

- Complete the Fox theorem for medial LDCs
  - Define medial linear functors and linear natural transformations: 2-cat MLDC
  - Determine that Bim(-) extends to a functor SMLDC $\rightarrow$ CLDC
  - Prove Bim(-) is right adjoint to inclusion
- Develop examples further
  - Find more examples of medial LDCs $\mathbb{X}$
  - What is $\mathrm{Bim}(\mathbb{X})$, in particular what is $\mathrm{Bim}(Q\text{-Coh})$?
- Develop a sequent calculus for MLL+medial
  - Is there a version of cut elimination?

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