Drazin Inverses in Categories
arXiv:2402.18226

JS PL (he/him), joint work with Robin Cockett and Priyaa Varshinee Srinivasan

Dedicated to Phil Scott.

Joint Work with Robin and Priyaa[1]

https://arxiv.org/pdf/2402.18226.pdf

- I gave 3-part series on this story at the CT seminar at Macquarie
- I gave a seminar talk about this at Tallinn to Pawel and Tarmo's groups
- Robin gave a talk about this at CT2024.
- Yet Robin still wanted me to give some tutorials about this at FMCS

[1] Photo from FMCS2023

Drazin Inverses in Ring Theory

In a **ring** R, a **Drazin inverse** of $x \in R$ is a $x^D \in R$ such that:

- **[D.1]** There is a $k \in \mathbb{N}$ such that $x^{k+1}x^D = x^k$
- **[D.2]** $x^D x x^D = x^D$
- **[D.3]** $x^D x = x x^D$

While a Drazin inverse may not always exist, **if a Drazin inverse exists then it is unique**, so we may speak of *the* Drazin inverse.

The “inverse” part in the term “Drazin inverse” is justified since it is a generalization of the usual notion of inverse. In a **ring** R, if $x \in R$ is invertible, then x^{-1} is the Drazin inverse of x.

The term Drazin inverse is named after Michael P. Drazin (1929 – still alive!), who originally introduced the concept of Drazin inverses in rings under the name “pseudo-inverse”

Michael P. Drazin **Pseudo-inverses in associative rings and semigroups.** (1958)

Drazin Inverses in Ring Theory

In a **semigroup** R, a **Drazin inverse** of $x \in R$ is a $x^D \in R$ such that:

- **[D.1]** There is a $k \in \mathbb{N}$ such that $x^{k+1}x^D = x^k$
- **[D.2]** $x^D x x^D = x^D$
- **[D.3]** $x^D x = x x^D$

While a Drazin inverse may not always exist, **if a Drazin inverse exists then it is unique**, so we may speak of *the* Drazin inverse.

The "inverse" part in the term "Drazin inverse" is justified since it is a generalization of the usual notion of invertise. In a **monoid** R, if $x \in R$ is invertible, then x^{-1} is the Drazin inverse of x.

The term Drazin inverse is named after Michael P. Drazin (1929 – still alive!), who originally introduced the concept of Drazin inverses in rings under the name "pseudo-inverse"

Michael P. Drazin **Pseudo-inverses in associative rings and semigroups.** (1958)

Drazin Inverses all around

Drazin inverses have an extensive literature and active area of research:

- Studied in-depth in ring theory and semigroup theory
- Connected to strong π-regularity
- Connected to Fitting's results (Fitting's Lemma or Fitting's Decomposition result)
- Studied in matrix theory since every complex square matrix has a Drazin inverse. So the Drazin inverse has many application and is a very useful tool for computations

But what about Drazin inverses in category theory?

For any object A in a category $\mathbb{X}$, the homset $\mathbb{X}(A, A)$ of endomorphisms of type A is a monoid with respect to composition. As such, we may consider Drazin inverses in $\mathbb{X}(A, A)$, or in other words, we may talk about Drazin inverses of endomorphisms in an arbitrary category.

However not much has been done with Drazin inverses in category theory!

To the best of our knowledge, the only discussion of Drazin inverses in category theory appears in a section of a paper by Puystjens and Robinson:

R. Puystjens & D. W. Robinson **Generalized Inverses of Morphisms with Kernels**. (1987)

where they provide an existence property of Drazin inverses in an additive/Abelian category.

At some point after we wrote this paper:

R. Cockett & J.-S. P. Lemay **Moore-Penrose Dagger Categories**. (2023)

Robin became fascinated by Drazin inverses! We realized lots can be said about Drazin inverses using a categorical point of view.

Quick summary of our paper

The purpose of our paper was to develop Drazin inverses from a categorical perspective. We both review the ring/semigroup theory stuff, and also provide novel results.

- Drazin inverses in a category
- Consider Drazin categories and many examples
- How Drazin inverses behave well with well-known categorical constructions
- A 2-categorical perspective on Drazin inverses (rank!)
- Relating Drazin inverses to idempotent splitting
- Relate Drazin inverse to eventual image duality:

T. Leinster **The Eventual Image**. (2022)

- Drazin inverses in additive/Abelian categories, recapturing Fitting's results
- Generalize the notion of Drazin inverses to that of **Drazin opposing pairs**.

We'll go through some of this today. For more details, please go see our paper:

`https://arxiv.org/pdf/2402.18226.pdf`

Warning, it is a long paper: Robin claimed this was suppose to be a 20 page paper, which I never believed!

Drazin Inverses in a Category

Definition

In a category $\mathbb{X}$, a **Drazin inverse** of $x : A \to A$ is an endomorphism $x^D : A \to A$ such that[a]:

- **[D.1]** There is a $k \in \mathbb{N}$ such that $x^{k+1}x^D = x^k$
- **[D.2]** $x^D x x^D = x^D$
- **[D.3]** $x^D x = x x^D$

If $x : A \to A$ has a Drazin inverse $x^D : A \to A$, we say that x is **Drazin**, and call the least k such that $x^{k+1}x^D = x^k$ the **Drazin index of** x, which we denote by $\mathrm{ind}(x) = k$.

[a] Composition is written in *diagrammatic order* – not that it will matter much...

Drazin Inverses are unique!

Proposition

In a category $\mathbb{X}$, if $x : A \to A$ has a Drazin inverse, then it is unique.

Proof.

Suppose that $x : A \to A$ has two possible Drazin inverses $y : A \to A$ and $z : A \to A$.

So explicitly, there is a $k \in \mathbb{N}$ such that $x^k xy = x^k$, and also that $yxy = y$ and $xy = yx$, and there is a $k' \in \mathbb{N}$ such that $x^{k'} xz = x^{k'}$, and also that $xz = zx$ and $zxz = z$.

Now set $j = \max(k, k')$. Then we can compute that:

$$x^{j+1}y = x^j \qquad y = x^j y^{j+1} \qquad zx^{j+1} = x^j \qquad z = z^{j+1}x^j$$

Then we compute that:

$$y = x^j y^{j+1} = zx^{j+1}y^{j+1} = zxx^j y^{j+1} = zxy = z^{j+1}xx^j y = z^{j+1}x^{j+1}y = z^{j+1}x^j = z$$

So $y = z$, and we conclude that the Drazin inverse is unique. □

From now on we may speak of *the* Drazin inverse of an endomorphism x (if it exists of course) and denote it by x^D.

Drazin Categories **NEW**

Definition

A **Drazin category** is a category such that every endomorphism has a Drazin inverse.

Since Drazin inverses are unique, being Drazin is a property of a category rather than a structure.

It is always possible to construct a Drazin category from any category by considering the full subcategory determined by the objects whose every endomorphism is Drazin.

Definition

In a category $\mathbb{X}$, an object A is a **Drazin object** if every endomorphism $x : A \to A$ is Drazin. Let $\mathsf{D}(\mathbb{X})$ be the full subcategory of Drazin objects of $\mathbb{X}$.

Lemma

For any category $\mathbb{X}$, $\mathsf{D}(\mathbb{X})$ is a Drazin category. Moreover, $\mathbb{X}$ is Drazin if and only if $\mathsf{D}(\mathbb{X}) = \mathbb{X}$.

Drazin Inverse Example: Matrices

Example

Let F be field and $\mathsf{MAT}(F)$ be the category of matrices over k, that is, the category whose objects are natural numbers $n \in \mathbb{N}$ and where a map $A : n \to m$ is an $n \times m$ F-matrix. Composition given by matrix multiplication and the identity on n is the n-dimensional identity matrix. Endomorphisms in $\mathsf{MAT}(F)$ correspond precisely to square matrices: so an endomorphism $A : n \to n$ is an $n \times n$ square matrix A. Then $\mathsf{MAT}(F)$ is a Drazin category.

So for an $n \times n$ matrix A, to compute its Drazin inverse we first write it in the form:

$$A = P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1}$$

for some invertible $n \times n$ matrix P, an invertible $m \times m$ matrix C (where $m \leq n$), and a nilpotent $n - m \times n - m$ matrix N (that is, $N^k = 0$ for some $k \in \mathbb{N}$). Then the Drazin inverse of A is the $n \times n$ matrix A^D defined as follows:

$$A^D = P \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

The Drazin index of A corresponds precisely to the **index** of A, which is the least $k \in \mathbb{N}$ such that $\mathsf{rank}(A^{k+1}) = \mathsf{rank}(A^k)$.

Drazin inverses of **complex** matrices are well studied and have many applications.

S. Campbell & C. Meyer. **Generalized inverses of linear transformations.**

Drazin Inverse Example: Modules

Example

Let R be a ring and let R-MOD be the category of (left) R-modules and R-linear morphisms between them. In general, R-MOD is not Drazin...

For example when $R = \mathbb{Z}$, the $\mathbb{Z}$-linear endomorphism $f : \mathbb{Z} \to \mathbb{Z}$ defined $f(x) = 2x$ does not have a Drazin inverse. Why?

If f had a Drazin inverse $f^D : \mathbb{Z} \to \mathbb{Z}$, by $\mathbb{Z}$-linearity, it must be of the form $f^D(x) = nx$ for some $n \in \mathbb{Z}$. Then by **[D.1]** we would have that for some $k \in \mathbb{N}$, $2^{k+1}nx = 2^k x$ for all $x \in \mathbb{Z}$. This would imply that $2n = 1$, but since $n \in \mathbb{Z}$, this is a contradiction.

So $\mathbb{Z}$-MOD is not Drazin.

Drazin Inverse Example: Modules

Example

While R-MOD may not always be Drazin, there are various characterizations of Drazin R-linear endomorphism!

Z. Weng **Class of drazin inverses in rings**. (2017)

In particular, an R-linear endomorphism $f : M \to M$ is Drazin if and only if

$$M = \mathrm{im}(f^k) \oplus \ker(f^k)$$

for some $k \geq 1$ and this decomposition is sometimes called **Fitting's decomposition**. In this case, f becomes an isomorphism on $\mathrm{im}(f^k)$, and its Drazin inverse is the inverse on this component.

Then an R-module M is said to satisfy **Fitting's Lemma** if every endomorphism has a Fitting's decomposition (or equivalently if R-MOD(M, M) is **strongly π-regular**).

E. P. Armendariz, J. W. Fisher, and R. L. Snider. **On injective and surjective endo-morphisms of finitely generated modules**. (1978)

As such, the Drazin objects in R-MOD are precisely the R-modules which satisfy Fitting's Lemma.

Drazin Inverse Example: Modules – Fitting Interlude

Hans Fitting was a German mathematician who died in 1938 – unexpectedly – at the young age of thirty-one. His results (written in German) are now so fundamental that they are simply referred to as "Fitting's Lemma" and "Fitting's decomposition theorem".

H. Fitting. **Die theorie der automorphismenringe abelscher gruppen und ihr analogon bei nicht kommutativen gruppen.** (1933)

Fitting's Decomposition Theorem says that for every endomorphism of a **finite length** R-module gives a Fitting Decomposition. **Fitting's Lemma** says that every endomorphism of an *indecomposible* finite length module is either an isomorphism or a nilpotent.

Example

This implies that every R-linear endomorphism of a finite length R-module is Drazin, and thus finite length R-modules are Drazin. So the full subcategory of finite length R-modules is Drazin.

REMARK: While every finite length R-module is Drazin, there are modules which do not have finite length which are Drazin. Ex. $\mathbb{Q}$ seen as a $\mathbb{Z}$-module is Drazin but not of finite length!

Drazin Inverse Example: Finite Sets

Example

Let FinSET be the category of finite sets and functions between them. FinSET is Drazin.

When X is a finite set, one way of understanding the Drazin inverse of a function $f : X \to X$, is to consider the inclusion of subsets:

$$X \supseteq \mathrm{im}(f) \supseteq \mathrm{im}(f^2) \supseteq \ldots \supseteq \mathrm{im}(f^k) = \mathrm{im}(f^{k+1}) = \ldots$$

which must eventually stabilize after at most $k \leq |X|$ steps. Then f becomes an isomorphism on $\mathrm{im}(f^k)$. Then the Drazin inverse of f is:

$$f^D(x) = f|^{-1}_{\mathrm{im}(f^k)}(f^k(x))$$

Lemma

Every finite set enriched category is Drazin.

Drazin Inverse Example: Sets

Example

Let SET be the category of sets and functions between them.

SET is not Drazin. For example the successor function $s : \mathbb{N} \to \mathbb{N}$, $s(n) = n + 1$, does not have a Drazin inverse. Why?

Suppose that s had a Drazin inverse $s^D : \mathbb{N} \to \mathbb{N}$. By **[D.3]**, we would have that:

$$s^D(n) = s^D(s^n(0)) = s^n(s^D(0)) = s^D(0) + n$$

So $s^D(n) = s^D(0) + n$. Now if $\mathrm{ind}(s) = k$, by **[D.1]** we would have that:

$$k = s^k(0) = s^D(s^{k+1}(0)) = s^D(k+1) = s^D(0) + k + 1$$

This implies that $0 = s^D(0) + 1$ – which is a contradiction since $s^D(0) \in \mathbb{N}$.

But we may still ask what are the Drazin objects are...

Lemma

A set X is Drazin in SET *if and only if X is a finite set. Therefore* $\mathrm{D}(\mathrm{SET}) = \mathrm{FinSET}$.

Properties of Drazin Inverses

Now let's look at some properties of Drazin inverses.

WARNING about composition

Unfortunately, Drazin inverses do not necessarily play well with composition.

Even if x and y are Drazin, xy may not be Drazin...

And even if xy is Drazin, we might not have that $(xy)^D$ is equal to $y^D x^D$...

Strongly π-Regular

Definition

In a category $\mathbb{X}$, $x : A \to A$ is **strongly π-regular** if there exists endomorphisms $y : A \to A$ and $z : A \to A$, and $p, q \in \mathbb{N}$ such that $yx^{p+1} = x^p$ and $x^{q+1}z = x^q$.

Lemma

In a category $\mathbb{X}$, $x : A \to A$ is Drazin if and only if it x is strongly π-regular.

Proof.

For $\Rightarrow$ set $y = z = x^D$. For $\Leftarrow$, set $k = \max(p, q)$ and $x^D := x^k z^{k+1} = y^{k+1} x^k$. □

Strongly π-regular rings have been studied in-depth:

E. P. Armendariz, J. W. Fisher, and R. L. Snider. **On injective and surjective endo-morphisms of finitely generated modules**. (1978)

P. Ara. **Strongly π-regular rings have stable range one.** (1996)

G. Azumaya. **Strongly π-regular rings.** (1954)

M. F. Dischinger. **Sur les anneaux fortement π-regulier.** (1979)

W. K. Nicholson. **Strongly clean rings and Fitting's Lemma.** (1999)

Isomorphisms and Drazin Index of 0

Lemma

In a category $\mathbb{X}$*,* $x : A \to A$ *is Drazin with* $\mathrm{ind}(x) = 0$ *if and only if* x *is an isomorphism. Explicitly:*

- *If* x *is an isomorphism, then it is Drazin where* $x^D = x^{-1}$ *and* $\mathrm{ind}(x) = 0$*;*
- *If* x *is Drazin and* $\mathrm{ind}(x) = 0$*, then* x *is an isomorphism where* $x^{-1} = x^D$*.*

In particular, the identity $1_A : A \to A$ *is Drazin and its own Drazin inverse,* $1_A^D = 1_A$*.*

Group Inverses and Drazin Index ≤ 1

Definition

In a category $\mathbb{X}$, a **group inverse** of $x : A \to A$ is an endomorphism $x^D : A \to A$ such that the following equalities hold:

- **[G.1]** $xx^Dx = x$;
- **[G.2]** $x^Dxx^D = x^D$;
- **[G.3]** $x^Dx = xx^D$.

Lemma

In a category $\mathbb{X}$, $x : A \to A$ is Drazin with $\mathrm{ind}(x) \leq 1$ if and only if x has a group inverse. Explicitly:

- *If x has a group inverse x^D, then x is Drazin with the group inverse x^D being its Drazin inverse and $\mathrm{ind}(x) \leq 1$;*
- *If x is Drazin and $\mathrm{ind}(x) \leq 1$, then its Drazin inverse x^D is its group inverse.*

Drazin Inverse of a Drazin Inverse

Lemma

In a category $\mathbb{X}$, let $x : A \to A$ be Drazin. Then:

- x^D *is Drazin where* $x^{DD} := xx^Dx$ *and* $\mathrm{ind}(x^D) \leq 1$;
- x^{DD} *is Drazin where* $x^{DDD} = x^D$;
- *If* $\mathrm{ind}(x) \leq 1$, *then* $x^{DD} = x$.

Iteration

Lemma

In a category $\mathbb{X}$, if $x : A \to A$ is Drazin, then x^n is Drazin where $(x^n)^D = (x^D)^n$.

Lemma

In a category $\mathbb{X}$, $x : A \to A$ is Drazin if and only if there is a $k \in \mathbb{N}$ such that $x^{k+1} : A \to A$ is Drazin.

Absolute

We now turn our attention to other properties of Drazin inverse that have a more categorical flavour.... We first observe that Drazin inverses are *absolute*, that is, every functor preserves Drazin inverses on the nose.

Proposition

Let $\mathsf{F} : \mathbb{X} \to \mathbb{Y}$ *be a functor and let* $x : A \to A$ *be Drazin in* $\mathbb{X}$*. Then* $\mathsf{F}(x) : \mathsf{F}(A) \to \mathsf{F}(A)$ *is Drazin in* $\mathbb{Y}$ *where* $\mathsf{F}(x)^D = \mathsf{F}(x^D)$ *and* $\mathsf{ind}(\mathsf{F}(x)) \leq \mathsf{ind}(x)$*.*

Corollary

If a category $\mathbb{X}$ *is equivalent to a category* $\mathbb{Y}$ *which is Drazin, then* $\mathbb{X}$ *is Drazin.*

Example

So for a field k, let k-FVEC be the category of finite dimensional k-vector spaces and k-linear maps between them. Since k-FVEC is equivalent to MAT(k), we get that k-FVEC is Drazin. (Another way to see this is that finite dimensional vector spaces are have finite length).

Commuting Squares

Proposition

In a category $\mathbb{X}$, let $x : A \to A$ and $y : B \to B$ be Drazin. If the diagram on the left commutes, then the diagram on the right commutes:

$$\begin{array}{ccc} A & \xrightarrow{x} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[y]{} & B \end{array} \quad \Rightarrow \quad \begin{array}{ccc} A & \xrightarrow{x^D} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[y^D]{} & B \end{array}$$

This is quite useful for constructing new Drazin categories:

- (Co)Slice categories;
- (Co)algebras of endofuctions;
- Chu Construction

Drazin Inverses and Idempotents

There is a deep connection between Drazin inverses and idempotents!

Lemma

Let $x : A \to A$ *be Drazin. Define the map* $e_x := x^D x : A \to A$ *(or equivalently by* **[D.3]** *as* $e_x = x x^D$*). Then* e_x *is an idempotent.*

Lemma

An idempotent $e : A \to A$ *is Drazin, its own Drazin inverse,* $e^D = e$*, and* $\mathrm{ind}(e) \leq 1$*. Moreover,* $\mathrm{ind}(e) = 0$ *if and only if* $e = 1_A$*.*

Drazin Inverses and Idempotent Splitting

Another way of understanding Drazin inverses is as isomorphisms in the **idempotent splitting**.

For a category $\mathbb{X}$, let $\mathsf{Split}(\mathbb{X})$ be its idempotent splitting. Recall:

- Objects are pairs (A, e) consisting of an object A and an idempotent $e : A \to A$
- A $f : (A, e) \to (B, e')$ is a map $f : A \to B$ such that $efe' = f$ (or equivalently $ef = f = fe'$).
- Composition same as in $\mathbb{X}$
- Identity maps are $1_{(A,e)} := e : (A, e) \to (A, e)$.

Theorem

$x : A \to A$ is Drazin in $\mathbb{X}$ if and only if there is an idempotent $e : A \to A$ such that for some $k \in \mathbb{N}$, $x^{k+1} : (A, e) \to (A, e)$ is an isomorphism in $\mathsf{Split}(\mathbb{X})$.

Proof.

For the $\Rightarrow$ take the idempotent e_x. Then $x^{k+1} : (A, e_x) \to (A, e_x)$ is an isomorphism in $\mathsf{Split}(\mathbb{X})$.

The $\Leftarrow$ direction requires more work. Briefly, if $x^{k+1} : (A, e) \to (A, e)$ is an isomorphism in $\mathsf{Split}(\mathbb{X})$ with inverse $v : (A, e) \to (A, e)$. Then the Drazin inverse of x is $x^D := vx^k = x^k v$. □

Quick Word about Eventual Image Duality and Drazin Inverses

We also look at the relationship between Drazin inverses and Leinster's eventual image duality.

Briefly, an endomorphism $x : A \to A$ has an Eventual Image Duality if the diagram:

$$\ldots \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} \ldots$$

has both a limit and colimit, which are canonically isomorphic.

T. Leinster **The Eventual Image**. (2022)

Definition

$x : A \to A$ is **Drazin split** if it is Drazin and the induced idempotent $e_x := xx^D : A \to A$ splits.

Lemma

An endomorphism that is Drazin split has eventual image duality.

Expressive Rank

For a square matrix A (over a field), an intuitive way of finding its Drazin inverse is to iterate A until the rank does not change (which is always guaranteed to happen):

$$\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^{k+2}) = \dots$$

When this happens, one can reverse any later iterations and thus build a Drazin inverse. The same principle holds true for linear endomorphisms on a finite-dimensional vector space or endomorphisms on a finite set.

We'd like to make this procedure rigorous categorically.

This involves making precise what is meant by "rank". In linear algebra, the rank of a matrix or a linear transformation is the dimension of its image space. So we wish to generalize this in a category by associating every map to a natural number which represents its rank.

We express this in terms of a *colax* functor into a 2-category which we call Rank.

Rank

To help with notation, for $n, m \in \mathbb{N}$ we denote $n \wedge m = \min(n, m)$.

Rank is the 2-category defined as follows:

[0-cells]: $n \in \mathbb{N}$;

[1-cells]: $m : n_1 \to n_2$ where $m \leq n_1 \wedge n_2$, the identity on n is $n : n \to n$, and composition of $m_1 : n_1 \to n_2$ and $m_2 : n_2 \to n_3$ is $m_1 \wedge m_2 : n_1 \to n_3$;

[2-cells]: $m_1 \Rightarrow m_2$ if and only if $m_1 \leq m_2$.

For a category $\mathbb{X}$, by a colax functor rank : $\mathbb{X} \to$ Rank, we mean a mapping which associates objects of $\mathbb{X}$ to 0-cells, maps of $\mathbb{X}$ to 1-cells, so $\text{rank}(f) = m : \text{rank}(A) = n_1 \to \text{rank}(B) = n_2$ where $m \leq n_1$ and $m \leq n_2$. We also ask that a colax functor preserves identities, $\text{rank}(1_A) = \text{rank}(A)$, while for composition we only require that $\text{rank}(fg) \leq \text{rank}(f) \wedge \text{rank}(g)$.

For objects, we think of $\text{rank}(A)$ as the dimension of A.

For maps we think of $\text{rank}(f)$ as the rank of f.

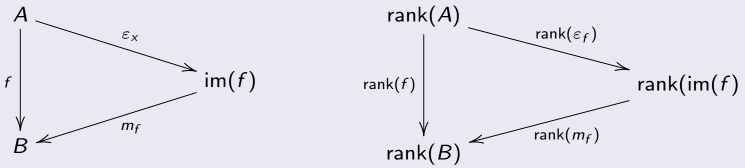
Note that for every map $f : A \to B$, by definition we have that $\text{rank}(f) \leq \text{rank}(A)$ and $\text{rank}(f) \leq \text{rank}(B)$.

Expressive Rank

Definition

A category $\mathbb{X}$ is said to have **expressive rank** if:

- **[ER.1]** $\mathbb{X}$ comes equipped with a colax functor $\text{rank} : \mathbb{X} \to \text{Rank}$;
- **[ER.2]** $\mathbb{X}$ has a factorization system $(\mathcal{E}, \mathcal{M})$ which **expresses rank**:



  then $\text{rank}(f) = \text{rank}(\varepsilon_f) = \text{rank}(m_f) = \text{rank}(\text{im}(f))$.
- **[ER.3]** rank reflects isomorphisms, that is, if $f : A \to B$ and $\text{rank}(f) = \text{rank}(A) = \text{rank}(B)$, then f is an isomorphism.

In a category $\mathbb{X}$ with expressive rank, we call $\text{rank}(A)$ the **dimension** of an object A, and we call $\text{rank}(A)$ the **rank** of a map f.

Expressive Rank Examples

Example

For a field k, MAT(k) has expressive rank, where the factorization system is the usual surjection-injection factorization system, $\text{rank}(n) = n$, and for a matrix A, $\text{rank}(A)$ is the usual rank of the matrix.

Example

For a field k, k-FVEC also has expressive rank, where the factorization system is the usual surjection-injection factorization system, for a finite-dimensional vector space V, $\text{rank}(V) = \dim(V)$, and for a linear transformation f, $\text{rank}(f) = \dim(\text{im}(f))$.

Example

FinSet has expressive rank, where the factorization system is the usual surjection-injection factorization system, for a finite set X, $\text{rank}(X) = |X|$, and for a function f, $\text{rank}(f) = |\text{im}(f)|$.

Example

The category of finite length modules over a ring also has expressive rank.

Expressive Rank

Theorem

A category which has an expressive rank is Drazin.

Expressive Rank

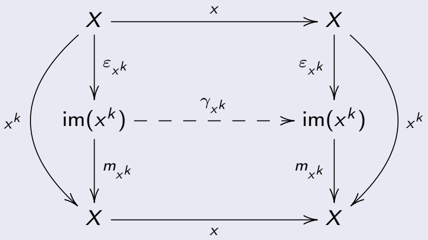
Proof.

For any endomorphism $x : A \to A$ we get a descending chain of inequalities:

$$\mathrm{rank}(A) = \mathrm{rank}(x^0) \geq \mathrm{rank}(x^1) \geq \mathrm{rank}(x^2) \geq \dots$$

Let $r = \min_{n \in \mathbb{N}}(\mathrm{rank}(x^n))$ and set k to be the least natural number such that $\mathrm{rank}(x^k) = r$. Once the sequence hits this rank, all subsequent ranks are equal, so $r = \mathrm{rank}(x^k) = \mathrm{rank}(x^{k+1}) = \dots$.

Then we show that the unique map $\gamma_{x^k} : \mathrm{im}(x^k) \to \mathrm{im}(x^k)$ induced by the factorization system which makes the following diagram commute:



is an isomorphism using that rank reflects isomorphisms. Then define $x^D :\ = \varepsilon_{x^k}(\gamma_{x^k}^{-1})^{k+1} m_{x^k}$. □

Today's Story

Now we are going to discuss Drazin inverses in *additive categories*. We will discuss:

- Revisit π-regularity
- Core-Nilpotent Decomposition
- Kernel-Cokernel Coincidence
- Generalizing Matrix Example in finite biproduct setting
- Generalizing Image-Kernel Decomposition in Abelian category setting

What Robin means by an additive category

In our paper, and in this talk, by additive category we mean a category $\mathbb{X}$ which is enriched over Abelian groups, that is, every homset $\mathbb{X}(A, B)$ is an Abelian group (written additively) and composition is a group homomorphism.

Sums, Negatives, and Scalar Multiplication

- Similarly to the case of the usual inverse, the Drazin inverse does not necessarily behave well with sums. Indeed, the sum of Drazin endomorphisms x and y is not necessarily Drazin, and even if $x + y$ was Drazin, then the Drazin inverse $(x + y)^D$ is not necessarily the sum of the Drazin inverses $x^D + y^D$.

- If $x : A \to A$ is Drazin, then $-x : A \to A$ is Drazin where $(-x)^D = -x^D$ and $\mathrm{ind}(-x) = \mathrm{ind}(x)$.

- If we happen to be in a setting where we can scalar multiply maps by rationals $\frac{p}{q} \in \mathbb{Q}$, then if x is Drazin, then so is $\frac{p}{q}x$ where $(\frac{p}{q}x)^D = \frac{q}{p}x^D$ if $\frac{p}{q} \neq 0$.

- In particular, in such a setting, for $m \geq 1$, we would have that if x is Drazin, then $mx = x + x + \ldots + x$ is also Drazin where $(mx)^D = \frac{1}{m}x^D$.

- Even more generally, if R is a ring, then in a category $\mathbb{X}$ which is enriched over R-modules, for any unit $u \in R$, if x is Drazin then ux is Drazin where $(ux)^D = u^{-1}x^D$.

Revisiting Strongly π-Regular

Definition

In a category $\mathbb{X}$, $x : A \to A$ is **strongly π-regular** if there exists endomorphisms $y : A \to A$ and $z : A \to A$, and $p, q \in \mathbb{N}$ such that $yx^{p+1} = x^p$ and $x^{q+1}z = x^q$.

Lemma

In a category $\mathbb{X}$, $x : A \to A$ is Drazin if and only if it x is strongly π-regular.

Strongly π-regular rings have been studied in-depth:

- E. P. Armendariz, J. W. Fisher, and R. L. Snider. **On injective and surjective endo-morphisms of finitely generated modules**. (1978)
- P. Ara. **Strongly π-regular rings have stable range one.** (1996)
- G. Azumaya. **Strongly π-regular rings.** (1954)
- M. F. Dischinger. **Sur les anneaux fortement π-regulier.** (1979)
- W. K. Nicholson. **Strongly clean rings and Fitting's Lemma.** (1999)

Revisiting Strongly π-Regular

Notice the definition of strongly π-regular can be separated in two:

Definition

In a category $\mathbb{X}$, $x : A \to A$ is said to be:

- **right π-regular** if there is an endomorphism $x^R : A \to A$ and a $k \in \mathbb{N}$ such that $x^{k+1}x^R = x^k$. The $k \in \mathbb{N}$ is a called a **right π-index**.
- **left π-regular** if there is an endomorphism $x^L : A \to A$ and a $k \in \mathbb{N}$ such that $x^L x^{k+1} = x^k$. The $k \in \mathbb{N}$ is a called a **left π-index**.

- Clearly, an endomorphism which is strongly π-regular is equivalent to being both right π-regular and left π-regular.
- So being both right π-regular and left π-regular is equivalent to being Drazin.
- However in general, being right π-regular is not equivalent to being left π-regular. So being right/left π-regular is not equivalent to being strong π-regular/Drazin.

Dischinger's Result

C. R. Acad. Sc. Paris, t. 283 (11 octobre 1976) Série A — 571

ALGÈBRE. — *Sur les anneaux fortement π-réguliers.*

Note (*) de M. **Friedrich Dischinger**, présentée par M. Jean Leray.

On montre qu'un anneau π-régulier à droite est nécessairement π-régulier à gauche. En vertu de ce résultat, il est possible de remplacer les notions de π-régularité à droite ou à gauche par la notion de π-régularité forte. D'autre part, on introduit la notion nouvelle de π-régularité complète, et on donne des conditions suffisantes pour la coïncidence de ces deux notions.

1. Soit A un anneau associatif et unitaire, mais non nécessairement commutatif. Un élément a de A a été appelé π-régulier à droite (resp. à gauche) ([1]) s'il existe un $n \geqq 1$ et un $b \in$ A tel que $a^{n+1}\,b = a^n$ (resp. $ba^{n+1} = a^n$) et on dit que a est fortement π-régulier s'il est π-régulier à droite et à gauche. Soit u un élément de A inversible à droite; alors, u est un exemple d'un élément π-régulier à droite, et pour que u soit fortement π-régulier, il faut et il suffit que u soit inversible à gauche.

On dit que l'anneau A est π-régulier à droite (resp. à gauche, resp. fortement π-régulier) si tout élément de A est π-régulier à droite (resp. à gauche, resp. fortement π-régulier). Si A est π-régulier à droite, on voit que tout élément u de A inversible à droite est inversible à gauche. Plus généralement : tout élément a d'un anneau π-régulier à droite A est-il π-régulier à gauche ? Des réponses partielles à cette question étaient connues : [*voir* ([1]), ([2]) et ([3])]. Nous allons donner la réponse générale, qui est affirmative. Ainsi, les trois propriétés d'un anneau définies ci-dessus coïncident :

THÉORÈME 1. — *Tout anneau* A *π-régulier à droite est π-régulier à gauche.*

La démonstration donnée ici, qui est beaucoup plus simple que la démonstration donnée d'abord par l'auteur, est due à H. Zöschinger.

Theorem

In a ring R, for any $a \in R$, a is right π-regular if and only if a left π-regular.

Dischinger's Result

Unfortunately the same is not true for an arbitrary category (in fact Dischinger even remarks that his proof does not extend to semigroups).

In our paper, we show that Dischinger's result generalizes to the setting of an additive category.

Definition

In a category $\mathbb{X}$, an object $A \in \mathbb{X}$ is said to be **right (resp. left) π-regular** if every endomorphism of type $A \to A$ is right (resp. left) π-regular. Similarly, a category $\mathbb{X}$ is said to be **right (resp. left) π-regular** if every endomorphism in $\mathbb{X}$ is right (resp. left) π-regular.

Theorem

In an additive category $\mathbb{X}$, an object A is right π-regular if and only if A is left π-regular. Therefore, an additive category $\mathbb{X}$ is right π-regular if and only if $\mathbb{X}$ is left π-regular.

The proof follows the same steps as Dischinger, however, we fill in some gaps and provide some of the details which were omitted.

Corollary

An additive category $\mathbb{X}$ is Drazin if and only if $\mathbb{X}$ is right (or left) π-regular.

Nilpotents and Zero Morphisms

- In an additive category, an endomorphism $n : A \to A$ is said to be **nilpotent** if there is a $k \in \mathbb{N}$ such that $n^k = 0$, and the smallest such k is called the nilpotent index of n.
- Nilpotent endomorphisms are precisely the Drazin endomorphisms whose Drazin inverse is 0.
- In this case, the Drazin index and the nilpotent index coincide.

Lemma

In any additive category $\mathbb{X}$, $n : A \to A$ is nilpotent if and only if n is Drazin with $n^D = 0$. Explicitly:

- *If n is nilpotent with nilpotent index k, then n is Drazin where $n^D = 0$ and $\mathrm{ind}(n) = k$;*
- *If n is Drazin with Drazin inverse $n^D = 0$, then n is nilpotent with nilpotent index $\mathrm{ind}(n)$.*

In particular, the zero morphism $0 : A \to A$ is Drazin and its own Drazin inverse, $0^D = 0$.

Core-Nilpotent Decomposition

For matrices, an important concept in relation to the Drazin inverse is the notion of the *core-nilpotent decomposition*:

- The *core* of a matrix is defined as the Drazin inverse of its Drazin inverse
- While its *nilpotent part* is the matrix minus its core.

Definition

In any additive category $\mathbb{X}$, for a Drazin endomorphism $x : A \to A$,

- The **core** of x is the endomorphism $c_x : A \to A$ defined by $c_x = x^{DD} = xx^D x$.
- The **nilpotent part** of a Drazin endomorphism is $n_x = x - c_x : A \to A$.

The pair (c_x, n_x) is called a core-nilpotent decomposition.

We wish to prove that having a core-nilpotent decomposition is equivalent to being Drazin. To justify this claim, it is useful to have a definition of a core-nilpotent decomposition which is independent of x being Drazin.

Core-Nilpotent Decomposition

Definition

In an additive category $\mathbb{X}$, a **core-nilpotent decomposition** of $x : A \to A$ is a pair (c, n) of endomorphisms $c : A \to A$ and $n : A \to A$ such that:

- **[CND.1]** c is Drazin with $\text{ind}(c) \leq 1$ (so c has a group inverse);
- **[CND.2]** n is nilpotent with nilpotent index $k \in \mathbb{N}$ (so that $n^k = 0$);
- **[CND.3]** $cn = 0 = nc$;
- **[CND.4]** $x = c + n$.

Theorem

In an additive category $\mathbb{X}$, $x : A \to A$ is Drazin if and only if x has a core-nilpotent decomposition. Explicitly:

- *If x is Drazin then (c_x, n_x) is a core-nilpotent decomposition;*
- *If x has a core-nilpotent decomposition (c, n), then x is Drazin with $x^D := c^D$.*

Moreover, if x is Drazin, then (c_x, n_x) is its unique core-nilpotent decomposition.

Generalizing the matrices approach

Recall that for an $n \times n$ matrix A, to compute its Drazin inverse we first write it in the form:

$$A = P \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} P^{-1}$$

for some invertible $n \times n$ matrix P, an invertible $m \times m$ matrix C (where $m \leq n$), and a nilpotent $n - m \times n - m$ matrix N (that is, $N^k = 0$ for some $k \in \mathbb{N}$).

Then the Drazin inverse of A is the $n \times n$ matrix A^D defined as follows:

$$A^D = P \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

We wish to generalization this approach, and we can do so in a setting with **finite biproducts**!

Finite Biproducts and Matrices

We now work in an additive category with finite biproducts: we denote the biproduct by $\oplus$.

Recall the matrix representation for maps between biproducts. Indeed, recall that a map of type $F : A_1 \oplus \ldots \oplus A_n \to B_1 \oplus \ldots \oplus B_m$ is uniquely determined by a family of maps $f_{i,j} : A_i \to B_j$. As such, F can be represented as an $n \times m$ matrix:

$$F := \begin{bmatrix} f_{1,1} & f_{1,2} & \ldots & f_{1,m} \\ f_{2,1} & f_{2,2} & \ldots & f_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n,1} & f_{n,2} & \ldots & f_{n,m} \end{bmatrix}$$

Moreover, composition corresponds to matrix multiplication, and identities correspond to the identity matrix.

Fitting Decomposition

Definition

In an additive category $\mathbb{X}$ with finite biproducts, a **Fitting decomposition** of $x : A \to A$ is a triple (p, α, η) consisting of an isomorphism $p : A \to I \oplus K$, an isomorphism $\alpha : I \to I$, and a nilpotent endomorphism $\eta : K \to K$ such that the following equality holds:

$$x = p \begin{bmatrix} \alpha & 0 \\ 0 & \eta \end{bmatrix} p^{-1}$$

We will show that a map Fitting decomposition is Drazin. To get the if and only if statement, we need to discuss **idempotent complement splitting**.

Drazin Split

Lemma

Let $x : A \to A$ be Drazin. Define the map $e_x := x^D x : A \to A$ (or equivalently by **[D.3]** *as $e_x = xx^D$). Then e_x is an idempotent.*

Definition

$x : A \to A$ is **Drazin split** if it is Drazin and the induced idempotent $e_x := xx^D : A \to A$ splits.

Drazin Complement Split

- In an additive category, the **complement** of an idempotent $e : A \to A$ is the endomorphism $e^c : A \to A$ defined as $e^c := 1_A - e$.

- The complement of an idempotent is again an idempotent, so we may consider when it splits. In an additive category, we say that an idempotent e is **complement-split** if e^c is split.

Definition

In an additive category $\mathbb{X}$, $x : A \to A$ is **Drazin complement-split** if x is Drazin and its induced impotent $e_x : A \to A$ (i.e. $e_x = xx^D = x^D x$) is complement-split.

Definition

In an additive category $\mathbb{X}$, $x : A \to A$ is **Drazin decomposable** if x is both Drazin split and Drazin complement-split.

Drazin Decomposable and Fitting Decomposition

Theorem

In an additive category $\mathbb{X}$ with finite biproducts, $x : A \to A$ is Drazin decomposable if and only if x has a Fitting's decomposition. Explicitly:

- *If x is Drazin decomposable, where e_x splits via $r : A \to I$ and $s : I \to A$, and e_x^c splits via $r^c : A \to K$ and $s^c : K \to A$, then define the maps $p : A \to I \oplus K$, $\alpha : I \to I$, and $\eta : K \to K$ as follows:*

$$p = \begin{bmatrix} r & r^c \end{bmatrix} \qquad \alpha = sxr \qquad \eta = s^c n_x r^c$$

  Then (p, α, η) is a Fitting's decomposition of x.

- *If x has a Fitting's decomposition (p, α, η), then x is Drazin complement-split with Drazin inverse defined as follows:*

$$x^D = p \begin{bmatrix} \alpha^{-1} & 0 \\ 0 & 0 \end{bmatrix} p^{-1}$$

  and moreover, writing $p : A \to I \oplus K$ and $p^{-1} : I \oplus K \to I$ in matrix form:

$$p = \begin{bmatrix} r & r^c \end{bmatrix} \qquad p^{-1} = \begin{bmatrix} s \\ s^c \end{bmatrix}$$

  the induced idempotent $e_x : A \to A$ splits via $r : A \to I$ and $s : I \to A$, and $e_x^c : A \to A$ splits via $r^c : A \to K$ and $s^c : K \to A$.

Fitting Decomposition and Core-Nilpotent Decomposition

Corollary

In an additive category $\mathbb{X}$ with finite biproducts, if $x : A \to A$ has a Fitting's decomposition (p, α, η), and is therefore Drazin, then the core and nilpotent-part of x are determined by:

$$c_x := p \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} p^{-1} \qquad\qquad n_x := p \begin{bmatrix} 0 & 0 \\ 0 & \eta \end{bmatrix} p^{-1}$$

Kernel-Cokernel Coincidence

Robinson and Puystjens provide conditions for being Drazin in an additive category in terms of kernels and cokernels, and also a formula for the Drazin inverse in this setting.

R. Puystjens & D. W. Robinson **Generalized Inverses of Morphisms with Kernels**. (1987)

THEOREM 2. *Let $\phi: X \to X$ be a morphism of an additive category $\mathscr{C}$, and let $i \geqslant 0$ be an integer. If $\kappa: K \to X$ is a kernel of $\phi^i: X \to X$, then ϕ has a Drazin inverse ϕ^D in $\mathscr{C}$ and $i \geqslant$ Drazin index of ϕ if and only if ϕ^i has a cokernel $\lambda: X \to L$, $\kappa\lambda: K \to L$ is invertible, and $\phi^{i+1} + \lambda(\kappa\lambda)^{-1}\kappa: X \to X$ is invertible. In this case, $\gamma = \lambda(\kappa\lambda)^{-1}: X \to K$ is a cokernel of ϕ^i, $\phi\phi^D + \gamma\kappa = 1_X$, and*

$$\phi^D = \phi^i \left(\phi^{i+1} + \gamma\kappa\right)^{-1} = \left(\phi^{i+1} + \gamma\kappa\right)^{-1}\phi^i.$$

We revisited their result and showed how their setup is in fact equivalent to being Drazin complement-split.

Kernel-Cokernel Coincidence

Theorem

In an additive category $\mathbb{X}$, $x : A \to A$ is Drazin complement-split if and only if there is a $k \in \mathbb{N}$ such that x^{k+1} has a kernel and cokernel:

$$\ker(x^{k+1}) \overset{\kappa}{\rightarrowtail} A \underset{0}{\overset{x^{k+1}}{\rightrightarrows}} A \overset{\lambda}{\twoheadrightarrow} \mathrm{coker}(x^{k+1})$$

such that $\kappa\lambda$ is an isomorphism and $x^{k+1} : (A, e^c_{\lambda,\kappa}) \to (A, e^c_{\lambda,\kappa})$ is an isomorphism in $\mathsf{Split}(\mathbb{X})$*, where $e^c_{\lambda,\kappa}$ is the complement of the idempotent $e_{\lambda,\kappa} = \lambda(\kappa\lambda)^{-1}\kappa$.*

Corollary

In an additive category $\mathbb{X}$, if $x : A \to A$ is Drazin complement-split and $\mathrm{ind}(x) = k$*, then the following equality holds:*

$$x^D = x^k(x^{k+1} + e^c_x)^{-1} = (x^{k+1} + e^c_x)^{-1}x^k$$

Kernel-Cokernel Coincidence – a remark

Rather than asking that x^{k+1} is an isomorphism in the idempotent splitting, Robinson and Puystjens ask that $x^{k+1} + e_{\lambda,\kappa}$ be an actual isomorphism. However, the following lemma shows that these statements are equivalent.

Lemma

In an additive category $\mathbb{X}$, if $f : (A, e) \to (A, e)$ is a map in $\mathrm{Split}(\mathbb{X})$, then $f : (A, e) \to (A, e)$ is an isomorphism in $\mathrm{Split}(\mathbb{X})$ if and only if $f + e^c$ is an isomorphism in $\mathbb{X}$.

Generalizing the module approach

Recall that for an R-linear endomorphism $f : M \to M$, f is Drazin if and only if

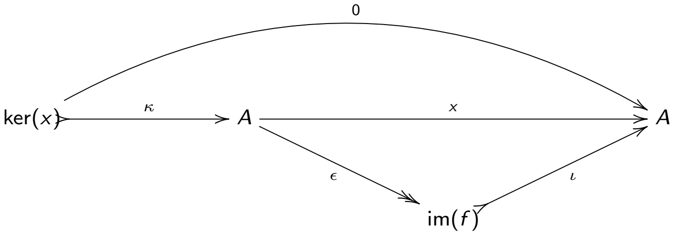
$$M = \mathrm{im}(f^k) \oplus \ker(f^k)$$

for some $k \geq 1$, which we call a **Fitting Image-Kernel Decomposition**.

We generalize this sort of decomposition in an Abelian category and show that it is indeed equivalent to being Drazin.

Image-Kernel Decomposition

In an Abelian category $\mathbb{X}$, denote the kernel and image of an endomorphism $x : A \to A$, as follows:



where recall that κ and ι are monic, and ϵ is epic. Now define ψ as the canonical map:

$$\psi = \begin{bmatrix} \iota \\ \kappa \end{bmatrix} : \mathrm{im}(x) \oplus \ker(x) \to A$$

Definition

In an Abelian category $\mathbb{X}$, a map $x : A \to A$ has an **image-kernel decomposition** in case the map ψ as defined above is an isomorphism.

Theorem

In an Abelian category $\mathbb{X}$, $x : A \to A$ is Drazin if and only if there is some $k \in \mathbb{N}$ such that x^{k+1} has an image-kernel decomposition.

Let's take a quick look at the proof

For the $\Rightarrow$ direction: our objective is to show that $\psi : \mathrm{im}(x^{k+1}) \oplus \ker(x^{k+1}) \to A$ is an isomorphism. To construct its inverse, we need to first construct maps $A \to \mathrm{im}(x^{k+1})$ and $A \to \ker(x^{k+1})$.

- First note that by idempotency and **[D.3]**, we get that $e_x = e_x^{k+1} = (x^D)^{k+1}x^{k+1}$. So by using the universal property of the image, there is a monic map $\mathrm{im}(e_x) \to \mathrm{im}(x^{k+1})$, which then allows us to build a map $\phi_1 : A \to \mathrm{im}(x^{k+1})$ such that $\phi_1 \iota = e_x$.

- On the other hand, we have that $e_x^c x^{k+1} = 0$. So by the universal property of the kernel, let $\phi_2 : A \to \ker(x^{k+1})$ be the unique map such that $\phi_2 \kappa^\circ_{x^{k+1}} = e_x^c$.

Then define ϕ as:

$$\phi = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix} : A \to \mathrm{im}(x^{k+1}) \oplus \ker(x^{k+1})$$

which we show is indeed the inverse of ψ (which requires a bit of work!)

Let's take a quick look at the proof

For the $\Leftarrow$ direction: To show that x is Drazin, we will show that ψ is part of a Fitting decomposition of x.

So let $\phi_1 : A \to \text{im}(x^{k+1})$ and $\phi_2 : A \to \ker(x^{k+1})$ be the components of $\psi^{-1} : A \to \text{im}(x^{k+1}) \oplus \ker(x^{k+1})$.

- Define $\alpha : \text{im}(x^{k+1}) \to \text{im}(x^{k+1})$ to be the composite $\alpha := \iota x \phi_1$. To show that α is an isomorphism, since we are in Abelian category, we showed that α is monic and epic.
- By the universal property of the kernel, there exists a unique map $\eta : \ker(x^{k+1}) \to \ker(x^{k+1})$ such that $\eta\kappa = \kappa x$. Which we show is nilpotent.

Then we also computed out that:

$$\psi^{-1} \begin{bmatrix} \alpha & 0 \\ 0 & \eta \end{bmatrix} \psi = x$$

So we conclude that $(\psi^{-1}, \alpha, \eta)$ is a Fitting's decomposition of x. So we get that x is Drazin.

Drazin Inverses of Opposing Pairs of Maps

Arriving with categorical eyes to the subject of Drazin inverses it is natural to want to have a Drazin inverse of an arbitrary map.

However, to have a Drazin inverse of a map $f : A \to B$, one really needs an *opposing* map $g : B \to A$ to allow for the iteration which is at the heart of the notion of a Drazin inverse.

Drazin Inverses of Opposing Pairs of Maps

In a category $\mathbb{X}$, we denote a pair of maps of dual type $f : A \to B$ and $g : B \to A$ by $(f, g) : A \rightleftarrows B$ and refer to it as an **opposing pair**.

Definition

In a category $\mathbb{X}$, a **Drazin inverse** of $(f, g) : A \rightleftarrows B$ is an opposing pair $(g^{\frac{D}{f}}, f^{\frac{D}{g}}) : A \rightleftarrows B$ satisfying the following properties:

- **[DV.1]** There is a $k \in \mathbb{N}$ such that $(fg)^k ff^{\frac{D}{g}} = (fg)^k$ and $(gf)^k gg^{\frac{D}{f}} = (gf)^k$.
- **[DV.2]** $f^{\frac{D}{g}} ff^{\frac{D}{g}} = f^{\frac{D}{g}}$ and $g^{\frac{D}{f}} gg^{\frac{D}{f}} = g^{\frac{D}{f}}$;
- **[DV.3]** $ff^{\frac{D}{g}} = g^{\frac{D}{f}} g$ and $f^{\frac{D}{g}} f = gg^{\frac{D}{f}}$.

The map $f^{\frac{D}{g}} : B \to A$ is called the Drazin inverse of f over g, while the map $g^{\frac{D}{f}} : A \to B$ is called the Drazin inverse of g over f.

Drazin Opposing Pairs

So how do Drazin inverses of opposing pair relate to the usual Drazin inverse?

Lemma

In a category $\mathbb{X}$*, for an opposing pair* $(f, g) : A \rightleftarrows B$ *,* $fg : A \to A$ *is Drazin if and only if* $gf : B \to B$ *is Drazin. Explicitly,*

- *If* fg *is Drazin, then* gf *is Drazin where* $(gf)^D := g(fg)^D(fg)^D f$*;*
- *If* gf *is Drazin, then* fg *is Drazin where* $(fg)^D := f(gf)^D(gf)^D g$*.*

Definition

In a category $\mathbb{X}$, a **Drazin opposing pair** is an opposing pair $(f, g) : A \rightleftarrows B$ such that fg or gf is Drazin.

Drazin Opposing Pairs

So how do Drazin inverses of opposing pair relate to the usual Drazin inverse?

Theorem

In a category $\mathbb{X}$, $(f, g) : A \rightleftarrows B$ *has a Drazin inverse if and only if* (f, g) *is a Drazin opposing pair. Explicitly:*

- *If* (f, g) *has a Drazin inverse* $(g^{\frac{D}{f}}, f^{\frac{D}{g}})$ *then* (f, g) *is a Drazin opposing pair where*

  $(fg)^D : = g^{\frac{D}{f}} f^{\frac{D}{g}}$ *with and* $(gf)^D : = f^{\frac{D}{g}} g^{\frac{D}{f}}$

- *If* (f, g) *is a Drazin opposing pair then* (f, g) *has a Drazin inverse* $(g^{\frac{D}{f}}, f^{\frac{D}{g}})$ *where* $f^{\frac{D}{g}} := g(fg)^D = (gf)^D g$ *and* $g^{\frac{D}{f}} := f(gf)^D = (fg)^D f$

Definition

In a category $\mathbb{X}$, an opposing pair $(f, g) : A \rightleftarrows B$ **is Drazin** if (f, g) is a Drazin opposing pair or equivalently if (f, g) has a Drazin inverse.

Lemma

In a category $\mathbb{X}$, $x : A \to A$ *is Drazin if and only if* $(x, 1_A) : A \rightleftarrows A$ *is Drazin, or equivalently if* $(1_A, x) : A \rightleftarrows A$ *is Drazin. Explicitly:*

- *If* x *is Drazin then* $(x, 1_A)$ *has a Drazin inverse with* $x^{\frac{D}{1_A}} = x^D$ *and* $1^{\frac{D}{x}} = xx^D$;
- *If* $(x, 1_A)$ *is Drazin, then* x *is Drazin where* $x^D = x^{\frac{D}{1_A}} 1^{\frac{D}{x}} = 1^{\frac{D}{x}} x^{\frac{D}{1_A}}$.

Corollary

A category $\mathbb{X}$ *is Drazin if and only if every opposing pair in* $\mathbb{X}$ *is Drazin.*

Drazin Inverses in Dagger Categories

An important source of a pair of opposing maps in a dagger category is the pair of a map and its adjoint.

A **dagger category** is a category $\mathbb{X}$ equipped with a functor $\dagger : \mathbb{X}^{op} \to \mathbb{X}$ which is the identity on objects and involutive. Explicitly, this means that for every map $f : A \to B$, there is a map of dual type $f^\dagger : B \to A$ called the **adjoint** of f such that:

$$(fg)^\dagger = g^\dagger f^\dagger \qquad 1_A^\dagger = 1_A \qquad f^{\dagger\dagger} = f$$

So for any map $f : A \to B$, we get the opposing pair $(f, f^\dagger) : A \rightleftarrows B$, which we call an **adjoint opposing pair**.

Lemma

In a dagger category $\mathbb{X}$, if $(f, f^\dagger) : A \rightleftarrows B$ has a Drazin inverse then $f^{\frac{D}{f^\dagger}} = ((f^\dagger)^{\frac{D}{f}})^\dagger$ and so $(f, f^\dagger)^D = ((f^\dagger)^{\frac{D}{f}}, (f^{\frac{D}{f^\dagger}})^\dagger)$ is an adjoint opposing pair.

Moore-Penrose Inverses

We use to Drazin opposing pairs to give a new perspective on *Moore-Penrose inverses*.
In a dagger category $\mathbb{X}$, a **Moore-Penrose inverse** of a map $f : A \to B$ is a map of dual type $f^\circ : B \to A$ such that:

[MP.1] $ff^\circ f = f$

[MP.2] $f^\circ ff^\circ = f^\circ$

[MP.3] $(ff^\circ)^\dagger = ff^\circ$

[MP.4] $(f^\circ f)^\dagger = f^\circ f$

Like Drazin inverses, Moore-Penrose inverses are unique.

R. Cockett & J.-S. P. Lemay **Moore-Penrose Dagger Categories**. (2023)

Proposition

In a dagger category $\mathbb{X}$, $f : A \to B$ has a Moore-Penrose inverse if and only if $(f, f^\dagger) : A \rightleftarrows B$ is **a** *Drazin inverse. Explicitly:*

- *If f has a Moore-Penrose inverse, then $(f, f^\dagger)$ is the Drazin inverse of $(f^{\circ\dagger}, f^\circ)$.*
- *If $(f, f^\dagger)$ is a Drazin inverse of an adjoint opposing pair $(g, g^\dagger)$, then f has a Moore-Penrose inverse with $f^\circ = g^\dagger fg^\dagger$.*

More about Drazin Inverses to come? Future work?

https://arxiv.org/pdf/2402.18226.pdf

HOPE YOU ENJOYED MY TALK!

THANK YOU FOR LISTENING!

MERCI!

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https://sites.google.com/view/jspl-personal-webpage/