Some Categorical Aspects of Moonshine

Jack Jia

Theorem (Mckay, 1978)

$196884 = 196883 + 1$

The Monster

Classification of Finite Simple Groups (1982-2011, by various mathematicians)

Any finite simple group is either:

The Monster

Classification of Finite Simple Groups (1982-2011, by various mathematicians)

Any finite simple group is either:

- an element of one of the 18 infinite families.

The Monster

Classification of Finite Simple Groups (1982-2011, by various mathematicians)

Any finite simple group is either:

- an element of one of the 18 infinite families.
- one of the 26 sporadic groups.

The Monster

Classification of Finite Simple Groups (1982-2011, by various mathematicians)

Any finite simple group is either:

- an element of one of the 18 infinite families.
- one of the 26 sporadic groups.

Among these sporadic groups, the largest is called "the monster" (M), it has order:

The Monster

Classification of Finite Simple Groups (1982-2011, by various mathematicians)

Any finite simple group is either:

- an element of one of the 18 infinite families.
- one of the 26 sporadic groups.

Among these sporadic groups, the largest is called "the monster" (M), it has order:

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

$$\approx 8 \cdot 10^{53}.$$

The Monster

Classification of Finite Simple Groups (1982-2011, by various mathematicians)

Any finite simple group is either:

- an element of one of the 18 infinite families.
- one of the 26 sporadic groups.

Among these sporadic groups, the largest is called "the monster" (M), it has order:

$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

$\approx 8 \cdot 10^{53}$.

Just for comparison, the number of atoms on earth is around $1.33 \cdot 10^{50}$.

Modular Functions

j-invariant

The full modular group $\Gamma := PSL_2(\mathbb{Z}) \circlearrowright \mathcal{H}$, the upper-half plane, by the fractional linear transformation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}.$$

A modular function of weight k is a meromorphic function that is 'invariant' under this action:
$f(M \cdot z) = (cz+d)^k f(z), \quad \forall M \in PSL_2(\mathbb{Z})$.

Modular Functions

j-invariant

The full modular group $\Gamma := PSL_2(\mathbb{Z}) \circlearrowright \mathcal{H}$, the upper-half plane, by the fractional linear transformation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}.$$

A modular function of weight k is a meromorphic function that is 'invariant' under this action:
$f(M \cdot z) = (cz+d)^k f(z), \quad \forall M \in PSL_2(\mathbb{Z})$.
It turns out that every modular function can be written as a rational function of a certain modular function of weight 0: this is the $j-$invaraint.

Modular Functions

j-invariant

The full modular group $\Gamma := PSL_2(\mathbb{Z}) \circlearrowright \mathcal{H}$, the upper-half plane, by the fractional linear transformation:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az+b}{cz+d}.$$

A modular function of weight k is a meromorphic function that is 'invariant' under this action:
$f(M \cdot z) = (cz+d)^k f(z), \quad \forall M \in PSL_2(\mathbb{Z})$.
It turns out that every modular function can be written as a rational function of a certain modular function of weight 0: this is the $j-$invaraint.

Below is the q-expansion ($q := e^{2\pi i \tau}$) of $\tilde{j}$:

$$\tilde{j}(\tau) = q^{-1} + 196884q + 21493760q^2 + ...,$$

Monstrous Moonshine

Comparison of Coefficients

Coefficients of $\tilde{j}$:

- 1, 196884, 21493760, 864299970,...

Degrees of irreducible representations of M:

- 1, 196883, 21296876, 842609326,...

Comparison of Coefficients

Coefficients of $\tilde{j}$:

- 1, 196884, 21493760, 864299970,...

Degrees of irreducible representations of M:

- 1, 196883, 21296876, 842609326,...

We can find the following relations:

- $196884 = 196883 + 1$
- $21493760 = 21296876 + 196883 + 1$
- $864299970 = 842609326 + 21296876 + 2 \cdot 196883 + 2 \cdot 1$

Monstrous Moonshine

Thompson suggested that these equations are really hinting the existence of an infinite-dimensional graded representation:

Thompson's Conjecture (1979)

There exists a somehow 'natural' ($\natural$) graded representation of M: $(\rho^\natural, V^\natural)$, where

$$V^\natural = V_{-1} \oplus V_1 \oplus V_2 \oplus V_3 \oplus ...,$$

such that the normalized j-invariant generates the dimensions of each graded part, namely

$$\tilde{j}(\tau) = \dim(V_{-1})q^{-1} + \sum_{i=1}^{\infty} \dim(V_i)q^i.$$

Monstrous Moonshine

Thompson's Conjecture (1979) (Continued)

More specifically, let $(\rho_0, W_0), (\rho_1, W_1), (\rho_2, W_2)...$ be the irreducible representations of M, ordered by dimension, then we have

- $V_{-1} = W_0$,
- $V_1 = W_1 \oplus W_0$,
- $V_2 = W_2 \oplus W_1 \oplus W_0$,
- $V_3 = W_3 \oplus W_2 \oplus 2W_1 \oplus 2W_0$, etc.

Monstrous Moonshine

Remark

Thompson's conjecture is not exactly the Conway-Norton's Moonshine conjecture, which says for each conjugacy class $[g]$ in M the McKay-Thompson series

$$T_{[g]} := \sum_{i \geqslant -1} \mathrm{Tr}(\rho^{\natural}(g)_{|V_i})q^i$$

is the q-expansion of the normalized Hauptmodul of a subgroup $\Gamma_{[g]}$ of $PSL_2(R)$ commensurable with $PSL_2(Z)$.

Monstrous Moonshine

Remark

Thompson's conjecture is not exactly the Conway-Norton's Moonshine conjecture, which says for each conjugacy class $[g]$ in M the McKay-Thompson series

$$T_{[g]} := \sum_{i \geqslant -1} \mathrm{Tr}(\rho^\natural(g)_{|V_i})q^i$$

is the q-expansion of the normalized Hauptmodul of a subgroup $\Gamma_{[g]}$ of $PSL_2(R)$ commensurable with $PSL_2(Z)$.

In 1979, Atkin, Fong, and Smith proved the existence of $V^\natural$ by checking enough congruences using a computer. But this proof is not very satisfactory as it does not provide an explicit construction of $V^\natural$.
In 1988, Frenkel, Lepowsky, and Meurman constructed the moonshine module $V^\natural$, which is acted on by the monster and has the correct dimensions on its grading.

Monstrous Moonshine

Question: Are these two $V^\natural$'s the same?

Question: Are these two $V^\natural$'s the same?

Here is a summary of Borcherds' proof (1992):

$$\begin{tikzcd}
\begin{array}{c}\text{Moonshine Module } V^\natural \\ \text{(Frenkel-Lepowsky-Meurman, 1988)}\end{array} \arrow[dr, "\textbf{Quant}"] & & \\
& \mathfrak{m} \arrow[d, dashed, "\cong"] & \text{Completely Replicable} \arrow[d] \\
& \mathfrak{l} \arrow[ur, "\text{Twisted Denominator Formula}"] & \text{HauptModul} \\
\begin{array}{c}\text{Automorphic Infinite Product} \\ \text{(Koike-Norton-Zaiger)}\end{array} \arrow[ur, "\text{Generators/Relations}"] & &
\end{tikzcd}$$

Vertex Operator Algebras

The moonshine module $V^\natural$, has the structure of a *Vertex Operator Algebra (VOA)*.

Unfortunately, there is no easy definition of a VOA.

Vertex Operator Algebras

The moonshine module $V^\natural$, has the structure of a *Vertex Operator Algebra (VOA)*.

Unfortunately, there is no easy definition of a VOA.

Vertex operators first appeared in string theory as a device for computing string amplitudes.

In general, given a vector space $\mathcal{V}$ (state-space), a *vertex operator* is an element of the set $\textbf{End}(\mathcal{V})[[z^{\pm 1}]]$, and a VOA is the 'algebra' of these vertex operators.

Vertex Operator Algebras

The moonshine module $V^\natural$, has the structure of a *Vertex Operator Algebra (VOA)*.

Unfortunately, there is no easy definition of a VOA.

Vertex operators first appeared in string theory as a device for computing string amplitudes.

In general, given a vector space $\mathcal{V}$ (state-space), a *vertex operator* is an element of the set $\mathbf{End}(\mathcal{V})[[z^{\pm 1}]]$, and a VOA is the 'algebra' of these vertex operators.

Conceptually, a VOA is the algebra of the symmetries of a 2-D *Conformal Field Theory (CFT)*, a 2-D CFT can be viewed as a functor from **C**, the category of Riemann surfaces (world-sheets) to **Hilb**, the category of Hilbert spaces (state-spaces).

The Moonshine Module $V^\natural$

The Leech lattice Λ_{24} is the unique 24-dimensional even unimodular lattice in which the length of every non-zero vector is at least 2. This lattice also provides the densest sphere packing in dimension 24. (2016, Viazovska et al.)

The Moonshine Module $V^\natural$

The Leech lattice Λ_{24} is the unique 24-dimensional even unimodular lattice in which the length of every non-zero vector is at least 2. This lattice also provides the densest sphere packing in dimension 24. (2016, Viazovska et al.)

The moonshine module $V^\natural$ can be interpreted as the vertex operator algebra of a bosonic string theory in a specific toroidal space-time built out of Λ_{24}.

The Moonshine Module $V^\natural$

The Leech lattice Λ_{24} is the unique 24-dimensional even unimodular lattice in which the length of every non-zero vector is at least 2. This lattice also provides the densest sphere packing in dimension 24. (2016, Viazovska et al.)

The moonshine module $V^\natural$ can be interpreted as the vertex operator algebra of a bosonic string theory in a specific toroidal space-time built out of Λ_{24}.

For a complete treatment of VOAs and early development of bosonic string theory, as well as the connection between string theory and monstrous moonshine, check the original book by F-L-M:

- I. Frenkel, J. Lepowsky, and A. Meurman. *Vertex Operator Algebras and the Monster*. Pure and applied mathematics. Academic Press, Inc., 1988.

Categorical Aspects

Categorical Characterization of VOAs

VOAs form a monoidal category, this category is equivalent to the category of algebras over the holomorphic punctured sphere operad; more precisely, the category of monoidal functors

$$V : \mathrm{2Cob}^0_{\mathrm{conf}} \to \mathrm{Hilb}$$

with some constraints.

Categorical Aspects

Categorical Characterization of VOAs

VOAs form a monoidal category, this category is equivalent to the category of algebras over the holomorphic punctured sphere operad; more precisely, the category of monoidal functors

$$V : 2\mathrm{Cob}^0_{\mathrm{conf}} \to \mathrm{Hilb}$$

with some constraints.

- Y.-Z. Huang, *Geometric Interpretation of Vertex Operator Algebras*, Proc. Natl. Acad. Sci. USA 88 (1991) pp. 9964-9968.

Categorical Aspects

the Functor **Quant**

Borcherds constructed $\mathfrak{m}$ from $V^\natural$ using a functor **Quant**, which takes in a VOA and outputs a Lie algebra. The action of M is then automatically transferred by functoriality.

Categorical Aspects

the Functor **Quant**

Borcherds constructed $\mathfrak{m}$ from $V^\natural$ using a functor **Quant**, which takes in a VOA and outputs a Lie algebra. The action of M is then automatically transferred by functoriality.

At critical dimension (26), this functor satisfies an oscillator-cancellation property (no-ghost theorem) and is naturally isomorphic to the BRST-quantization functor H^1_{BRST} in string theory.

Categorical Aspects

the Functor **Quant**

Borcherds constructed $\mathfrak{m}$ from $V^\natural$ using a functor **Quant**, which takes in a VOA and outputs a Lie algebra. The action of M is then automatically transferred by functoriality.

At critical dimension (26), this functor satisfies an oscillator-cancellation property (no-ghost theorem) and is naturally isomorphic to the BRST-quantization functor H^1_{BRST} in string theory.

- P. Deligne et al. *Quantum Fields and Strings: A Course for Mathematicians: Volume 2*. American Mathematical Society, (1999) pp. 807-1012.

Categorical Aspects

Categorical Characterization of VAs

There is a less rigid notion of VOA: *Vertex Algebra (VA)*. A CFT-type VA that satisfies some conformal conditions is a VOA.

Categorical Characterization of VAs

There is a less rigid notion of VOA: *Vertex Algebra (VA)*. A CFT-type VA that satisfies some conformal conditions is a VOA.

Borcherds gave a categorical characterization of VAs by defining them as the 'singular commutative rings' in a specific category.

Categorical Aspects

Categorical Characterization of VAs

There is a less rigid notion of VOA: *Vertex Algebra (VA)*. A CFT-type VA that satisfies some conformal conditions is a VOA.

Borcherds gave a categorical characterization of VAs by defining them as the 'singular commutative rings' in a specific category.

There are still many open problems in the study of VAs, for example, the study of orbifolds of VAs (fixed subalgebras under finite automorphism groups) is notoriously hard.

Categorical Aspects

Categorical Characterization of VAs

There is a less rigid notion of VOA: *Vertex Algebra (VA)*. A CFT-type VA that satisfies some conformal conditions is a VOA.

Borcherds gave a categorical characterization of VAs by defining them as the 'singular commutative rings' in a specific category.

There are still many open problems in the study of VAs, for example, the study of orbifolds of VAs (fixed subalgebras under finite automorphism groups) is notoriously hard.

- R. E. Borcherds. *Quantum Vertex Algebras*. Advanced Studies in Pure Mathematics 31, (2001) pp. 51-74.

Thank You