Adjoints in Double Categories of Quantales and Cauchy Completeness

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Idea

- Lawvere (1973): Cauchy completeness for $\mathcal{V}$-categories introduced metric spaces as categories enriched in $[0, \infty]$
- Paré (2021): Adjoints and Cauchy completeness in double categories concentrating on commutative rings
- N (2024): Generalized and applied to rigs and quantales
- Today: Three double "categories" of quantales

Suplattices (JT)

Sup: The category of complete lattices and $\bigvee$-preserving maps

Example: $\mathrm{Rel}_{I,J}(X) = \{I \multimap J\}$, the set of X-valued relation $I \times J \to X$, for a suplattice X

Sup is a symmetric monoidal closed category since, for all X,

$$\mathbf{Sup}(X, -)\colon \mathbf{Sup} \to \mathbf{Sup}$$

has a left adjoint, denoted by $X \otimes (-)$

- Can show $\mathbf{Sup}(X \otimes Y, Z) \cong \mathbf{Bilinear}(\mathbf{X} \times \mathbf{Y}, \mathbf{Z})$
- Construct $X \otimes Y$ by generators $x \otimes y$ and relations
- Or define $X \otimes Y = \mathbf{Sup}(X, Y^\circ)^\circ$

Quantales

Quant The category of monoids in **Sup**

Objects: quantales Q, $\bigvee$, $\cdot$, e

Morphisms: preserve $\bigvee$, $\cdot$, e

Example: Given $I \overset{r}{-\!\!\!\bullet\!\!\!\to} J \overset{s}{-\!\!\!\bullet\!\!\!\to} K$ on Q, define $(sr)_{ik} = \bigvee_{j \in J} s_{jk} r_{ij}$.

Then $\mathrm{Rel}_I Q = \mathrm{Rel}_{I,I}(Q)$ is a quantale with $e_{ii} = e, e_{ij} = \perp; i \neq j$

Note: $I \xrightarrow{\alpha} J$ induces $\mathrm{Rel}_J Q \xrightarrow{\mathrm{Rel}_\alpha Q} \mathrm{Rel}_I Q$ in **Quant**

Double Categories

A double category $\mathbb{D}$ is a pseudo internal category in CAT

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\bullet} \mathbb{D}_1 \overset{s}{\underset{t}{\substack{\longrightarrow \\ \xleftarrow{\mathrm{id}^\bullet} \\ \longrightarrow}}} \mathbb{D}_0$$

Objects D of $\mathbb{D}_0$, called objects of $\mathbb{D}$

Morphisms $D \xrightarrow{f} E$ of $\mathbb{D}_0$, called horizontal morphisms of $\mathbb{D}$

Objects $D \overset{v}{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;} F$ of $\mathbb{D}_1$, called vertical morphism of $\mathbb{D}$

Morphisms
$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\varphi"] & E \arrow[d, "w", "\bullet" marking] \\ F \arrow[r, "g"'] & G \end{tikzcd}$$
of $\mathbb{D}_1$, called cells of $\mathbb{D}$

A cell is special if f and g are identity morphisms. The vertical morphisms and special cells form a bicategory denoted by $\mathrm{Vert}(\mathbb{D})$.

Example 1: A strict double category of quantales

$\mathbb{Q}\mathrm{uant}_{str}$

Objects: quantales Q,

Horizontal morphisms: quantale homomorphisms $Q \xrightarrow{f} R$

Vertical morphisms: lax maps $Q \overset{v}{\twoheadrightarrow} S$, i.e., v is monotone with $v(q)v(\bar{q}) \leq v(q\bar{q})$ and $e_S \leq v(e_Q)$

Cells:
$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "v"'] \arrow[dr, phantom, "\leq"] & R \arrow[d, "w"] \\ S \arrow[r, "g"'] & T \end{tikzcd}$$
i.e., $g(v(q)) \leq w(f(q))$

Note: Every quantale map $Q \xrightarrow{f} R$ has a lax right adjoint $R \xrightarrow{f^*} Q$, since f preserves $\bigvee$, $f(e_Q) \leq e_R$, $f(f^* r f^* \bar{r}) \leq f(f^* r) f(f^* \bar{r}) \leq r\bar{r}$.

Companions and Conjoints

A companion for $X \xrightarrow{f} Y$ is a vertical morphism $X \xrightarrow[\bullet]{f_*} Y$ and cells

$$
\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\bullet" marking, "\mathrm{id}^\bullet_X"'] \arrow[dr, phantom, "\eta"] & X \arrow[d, "\bullet" marking, "f_*"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\bullet" marking, "f_*"'] \arrow[dr, phantom, "\varepsilon"] & Y \arrow[d, "\bullet" marking, "\mathrm{id}^\bullet_Y"] \\
Y \arrow[r, "\mathrm{id}_Y"'] & Y
\end{tikzcd}
$$

whose horizontal and vertical compositions are identities.

A conjoint for f is a vertical morphism $Y \xrightarrow[\bullet]{f^*} X$ and cells

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\bullet" marking, "\mathrm{id}^\bullet_X"'] \arrow[dr, phantom, "\bar\eta"] & Y \arrow[d, "\bullet" marking, "f^*"] \\
X \arrow[r, "\mathrm{id}_X"'] & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y \arrow[r, "\mathrm{id}_Y"] \arrow[d, "\bullet" marking, "f^*"'] \arrow[dr, phantom, "\bar\varepsilon"] & Y \arrow[d, "\mathrm{id}^\bullet_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

Proposition

If f has a companion and conjoint, then $f_* \dashv f^*$ in $\mathrm{Vert}(\mathbb{D})$.

Cauchy Completeness

Definition

An object Y of $\mathbb{D}$ is Cauchy complete if every left adjoint vertical morphism $v\colon X \xrightarrow{\bullet} Y$ is a companion of some $f\colon X \longrightarrow Y$; and $\mathbb{D}$ is a Cauchy double category, if every object is Cauchy complete.

Proposition

$\mathbb{Q}\mathrm{uant}_{str}$ has companions and conjoints; and is Cauchy.

Proof.

Suppose $v \dashv w$ in $\mathbb{Q}\mathrm{uant}$, where $Q \xrightarrow{\bullet\, v} R$. Since v is lax and preserves $\bigvee$, to see it is a quantale morphism, it suffices to show $v(e_Q) \leq e_R$ and $v(q\bar{q}) \leq v(q)v(\bar{q})$. But, $e_Q \leq w(e_R)$ and

$$q\bar{q} \leq wv(q)wv(\bar{q}) \leq w(v(q)v(\bar{q}))$$

$\square$

Example 2

$\mathbb{Q}\mathrm{uant}$

Objects: quantales Q,

Horizontal morphisms: quantale homomorphisms $Q \xrightarrow{f} R$

Vertical morphisms: (S, Q)-bimodules $Q \overset{M}{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;} S$

Cells:
$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d, "M"'] \arrow[dr, phantom, "\varphi"] & R \arrow[d, "N"] \\ S \arrow[r, "g"'] & T \end{tikzcd}$$
(S, Q)-homomorphsms $M \xrightarrow{\varphi} N$

Problem: $\mathbb{Q}\mathrm{uant}$ is not Cauchy, since one can show that

1. $\mathbb{Q}\mathrm{uant}$ has companions with $f_* \cong R$, for all $Q \xrightarrow{f} R$, but
2. $Q \overset{M}{\longrightarrow\!\!\!\!\!\!\!\bullet\;\;} R$ has a right adjoint iff M is projective as an R-module

Projective R-Modules

Let $Q \xrightarrow{f} \mathrm{Rel}_I R$ be a non-unitary homomorphism, and define

$$M_f = \{\mathbf{x} \in R^I\} | \mathbf{x} f(e) = \mathbf{x}\}$$

where $R^I = \coprod_{i \in I} R = \prod_{i \in I} R$. Then M_f is an (R, Q)-bimodule via

$$(r\mathbf{x})_i = r x_i \qquad \mathbf{x} q = \mathbf{x} f(q)$$

Theorem [N, Lawvere Festschrift]

Suppose $Q \overset{M}{\longrightarrow\!\!\!\!\!\bullet} R$. Then M is projective iff there is a a non-unitary homomorphism $Q \xrightarrow{f} \mathrm{Rel}_I R$ s.t. $M \cong M_f$ as an (R, Q)-bimodule.

Proof.

$(\Leftarrow)$ Define $M_f \underset{\tau}{\overset{\sigma}{\rightleftarrows}} R^I$ by $\sigma(\mathbf{x}) = \mathbf{x}$ and $\tau(\mathbf{x}) = \mathbf{x} f(e) \in M_f$, since $\tau(\mathbf{x}) f(e) = \mathbf{x} f(e) f(e) = \mathbf{x} f(e) = \tau(\mathbf{x})$, and so M_f projective.

$(\Rightarrow)$ See [N] $\square$

Another Horizontal "Category" of Quantales

$Q \xrightarrow{(I,f)} R$; I a set and $Q \xrightarrow{f} \mathrm{Rel}_I R$ a non-unitary homomorphism

Composition: $Q \xrightarrow{(I,f)} R \xrightarrow{(J,g)} S$, denoted $(J \times I, g * f)$ via

$$Q \xrightarrow{f} \mathrm{Rel}_I R \xrightarrow{\mathrm{Rel}(g)} \mathrm{Rel}_I(\mathrm{Rel}_J S) \cong \mathrm{Rel}_{J \times I} S$$

Problem: Associativity fails

$$\Big((K,h) \circ (J,g)\Big) \circ (I,f) = \Big(K \times (J \times I), h * (g * f)\Big)$$

$$(K,h) \circ \Big((J,g) \circ (I,f)\Big) = \Big((K \times J) \times I, (h * g) * f\Big)$$

Solution: Use 2-cells $\begin{tikzcd} Q \arrow[r, bend left, "{(I,f)}"] \arrow[r, bend right, "{(\hat{I},\hat{f})}"'] \arrow[r, phantom, "\downarrow\alpha"] & R \end{tikzcd}$ with $\hat{I} \xrightarrow{\alpha} I$ iso; $\begin{tikzcd} & \mathrm{Rel}_I R \arrow[dd, "\mathrm{Rel}_\alpha Q"] \\ Q \arrow[ur, "f"] \arrow[dr, "\hat{f}"'] & \\ & \mathrm{Rel}_{\hat{I}} R \end{tikzcd}$

Get a bicategory of quantales!

Example 3: A Verity double bicategory of quantales

$\mathbb{Q}\mathrm{uant}_{\mathrm{bicat}}$

Objects: quantales Q,

Horizontal bicategory: $Q \xrightarrow{(I,f)} R$ and
$$\begin{tikzcd} Q \arrow[r, bend left, "{(I,f)}"{name=U}] \arrow[r, bend right, "{(\hat{I},\hat{f})}"'{name=D}] & R \arrow[from=U, to=D, phantom, "\downarrow\alpha"] \end{tikzcd}$$

Vertical bicategory: (S,Q)-bimodules $Q \xrightarrow{M} S$ and
$$\begin{tikzcd} Q \arrow[r, bend left, "M"{name=U}] \arrow[r, bend right, "M'"'{name=D}] & S \arrow[from=U, to=D, phantom, "\downarrow\mu"] \end{tikzcd}$$

Squares:
$$\begin{tikzcd} Q \arrow[r, "{(I,f)}"] \arrow[d, "M"'] \arrow[dr, phantom, "\varphi"] & R \arrow[d, "N"] \\ S \arrow[r, "{(J,g)}"'] & T \end{tikzcd}$$
(S, Q)-homomorphsms $M \xrightarrow{\varphi} \mathrm{Rel}_{I,J}(N)$

Plus actions, and compatibility (Verity '92).

Example 3, cont.

Vertical composition uses:

$$\begin{tikzcd}
\mathrm{Rel}_I Q \arrow[d, "\mathrm{Rel}_{I,J}(M)"'] \arrow[dd, bend left=60, "\mathrm{Rel}_{I,K}(N \otimes_R M)"] \\
\mathrm{Rel}_J R \arrow[d, "\mathrm{Rel}_{J,K}(N)"'] \arrow[r, phantom, "\to"] & {} \\
\mathrm{Rel}_K S
\end{tikzcd}$$

Note: $\mathbb{I}^{\mathrm{op}} \times \mathbb{Q}\mathrm{uant} \xrightarrow{\mathrm{Rel}} \mathbb{Q}\mathrm{uant}$ is a lax double functor, where $\mathbb{I}$ is the double category of sets, bijective functions, and cells

$$\begin{tikzcd}
I \arrow[r, "\alpha"] \arrow[d, "{(I,K)}"'] \arrow[dr, phantom, "\xrightarrow{(\alpha,\beta)}"] & J \arrow[d, "{(J,L)}"] \\
K \arrow[r, "\beta"'] & L
\end{tikzcd}$$

Cauchy Completeness

Lemma

$\mathbb{Q}\mathrm{uant}_{\mathrm{bicat}}$ *has companions.*

Proof.

Given $Q \xrightarrow{(I,f)} R$, show $(I,f)_* = M_f \subseteq R^I$. $\square$

Theorem

$\mathbb{Q}\mathrm{uant}_{\mathrm{bicat}}$ *is Cauchy.*

Proof.

Suppose $Q \xrightarrow{M} R$ has a right adjoint. Then M is projective as a left R-module, and so $M \cong M_f$ as an (R,Q)-bimodule, for some $Q \xrightarrow{(I,f)} R$. Thus, $M \cong (I,f)_*$, as desired. $\square$

Remark

$\mathbb{Q}\mathrm{uant}_{\mathrm{bicat}}$ is the "Kleisli double bicategory" of a graded monad, i.e., a lax double functor $\mathbb{I}^{\mathrm{op}} \times \mathbb{Q}\mathrm{uant} \longrightarrow \mathbb{Q}\mathrm{uant}$ such that

$$(\mathbb{I}^{\mathrm{op}}, \times) \longrightarrow (\mathbf{Lax}(\mathbb{Q}\mathrm{uant}, \mathbb{Q}\mathrm{uant}), \circ)$$

is strong monoidal.

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