What kind of linearly distributive categories do polynomial functors form?

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Linearly distributive categories

Robin Cockett, and Robert Seely. *Weakly distributive categories* (1997)

Linearly distributive categories

Linearly distributive categories (LDCs)[1]**:**

$$(\mathbb{X}, \otimes, \top, a_\otimes, u^L_\otimes, u^R_\otimes) \qquad (\mathbb{X}, \oplus, \bot, a_\oplus, u^L_\oplus, u^R_\oplus)$$

linked by linear distributors:

$$\partial^L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$

$$A \times (B + C) \simeq (A \times B) + (A \times C)$$

$$\partial^R : (B \oplus C) \otimes A \to B \oplus (C \otimes A)$$

Intuition: At a restaurant the waiter, A, can choose to address either person at the table, B or C. Once assigned to B, A cannot choose C.

The distributor is not an equality or isomorphism in general!

Monoidal categories: LDCs in which $\otimes = \oplus; \top = \bot$

[1] Cockett and Seely (1997) "Weakly distributive categories"

Symmetry and $\otimes$-symmetry

A symmetric LDC is an LDC in which $\otimes$ and the $\oplus$ products are symmetric and the diagram commutes.

$$\begin{array}{ccccc}
(A \oplus B) \otimes C & \xrightarrow{\sigma_\otimes} & C \otimes (A \oplus B) & \xrightarrow{\sigma_\oplus} & C \otimes (B \oplus A) \\
\partial^R \downarrow & & & & \downarrow \partial^L \\
A \oplus (B \otimes C) & \xleftarrow[\sigma_\oplus]{} & A \oplus (C \otimes B) & \xleftarrow[\sigma_\otimes]{} & A \oplus (B \otimes C)
\end{array}$$

A $\otimes$-symmetric LDC is an LDC in which only the $\otimes$ product is symmetric.

Spectrum of LDCs

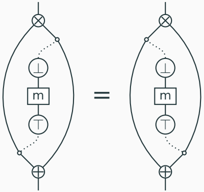
LDC
$(\mathbb{X}, \otimes, \top, \oplus, \bot)$

Monoidal category
$(\mathbb{X}, \otimes, I)$

Mix LDCs

Mix category [2]: LDC with $m : \bot \longrightarrow \top$ called the **mix map** with

$$\mathrm{indep}_{A,B} : A \otimes B \longrightarrow A \oplus B :=$$



$$(1 \oplus (u^L_\oplus)^{-1})(1 \otimes (\mathrm{m} \oplus 1))\delta^L(u^R_\otimes \oplus 1)$$

The indep[3] must be natural in A and B.

[2] Richard Blute, Robin Cockett, and Robert Seely (2000). "Feedback for linearly distributive categories: traces and fixpoints."

[3] Also referred to as the mixor

Spectrum of LDCs

Mix category

$\mathsf{m} : \bot \longrightarrow \top$

$\mathsf{mx} : A \otimes B \longrightarrow A \oplus B$

LDC

$(\mathbb{X}, \otimes, \top, \oplus, \bot)$

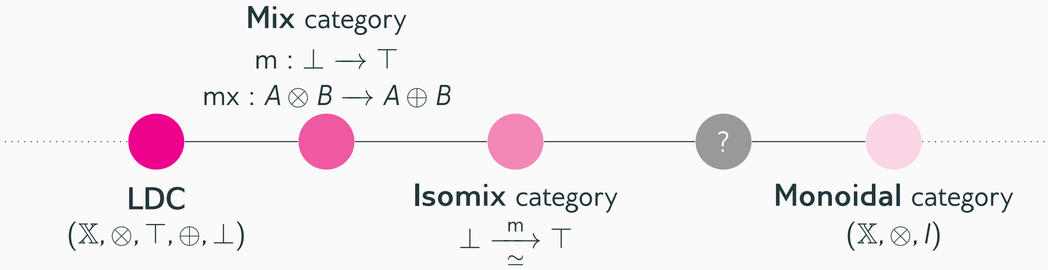
Monoidal category

$(\mathbb{X}, \otimes, I)$

Isomix LDCs

It is an **isomix** category if m is an isomorphism.

m being an isomorphism does not make the indep an isomorphism.



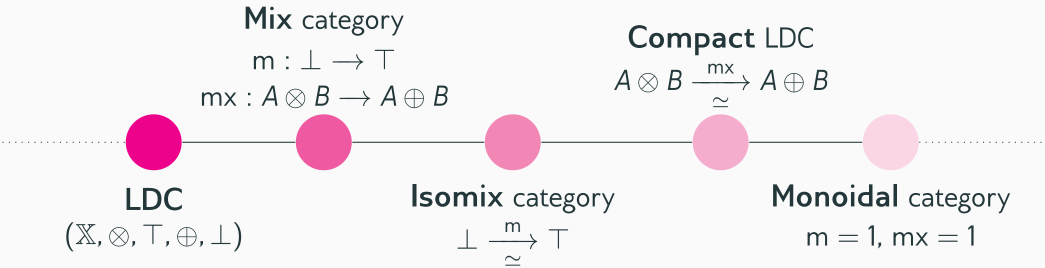
Compact LDCs

A **compact LDC** is an LDC in which every indep map is an isomorphism.

$$\mathsf{indep}_{A,B} : A \otimes B \xrightarrow{\simeq} A \oplus B$$

Compact LDCs $(\mathbb{X}, \otimes, \top, \oplus, \bot)$ are linearly equivalent to the underlying monoidal categories $(\mathbb{X}, \otimes, \top)$ and $(\mathbb{X}, \oplus, \bot)$.

Spectrum of LDCs



Typical examples of LDCs

Every monoidal category is an LDC.

A bounded distributive lattice regarded as a category is an LDC.

All $*$-autonomous categories are LDCs:
Ehrhard's finiteness spaces, Girard's coherence spaces, Chu spaces.

Can we have a simple yet a structurally rich example of LDCs?

Yes!!

This talk is about ...

Poly the category of polynomial functors and transformations as an example of isomix LDCs.

We will find non-trivial examples of various structures of LDCs in Poly.

We will also discover new properties of LDCs by studying Poly.

Poly seems to be a golden goose for LDCs!!!

Acknowledgement

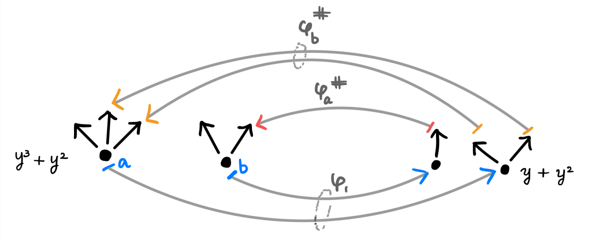
Thanks to Harrison Grodin and Reed Mullanix for their conjecture, that normal duoidal categories are linear distributive categories, which spawned this work.

Recap: Polynomial functors and natural transformations

A **polynomial functor** is a functor that is isomorphic to a coproduct of representables.

$$p \cong \sum_{P:p(1)} y^{p[P]} : \mathsf{Set} \longrightarrow \mathsf{Set}$$

A schematic of a polynomial functor map, $\varphi : (y^3 + y^2) \longrightarrow (y + y^2)$.



Recap: Tensor product in Poly

The tensor product $\otimes$ is given by the Day convolution of Cartesian product $\times$ in Set.

$$p \otimes q = \sum_{P:p(1)} y^{p[P]} \otimes \sum_{Q:q(1)} y^{q[Q]} := \sum_{(P,Q):p(1)\times q(1)} y^{p[P]\times q[Q]}$$

Tensor product is symmetric.

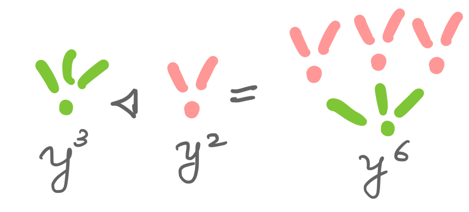
Recap: Tri product in Poly

The substitution product $\triangleleft$ is given by functor composition.

$$p \triangleleft q = \sum_{P:p(1)} y^{p[P]} \triangleleft \sum_{Q:q(1)} y^{q[Q]} := \sum_{P:p(1)} \left(\sum_{Q:q(1)} y^{q[Q]} \right)^{p[P]} = \sum_{P:p(1)} \sum_{f:p[P] \longrightarrow q(1)} \prod_{d:p[P]} \prod_{e:q[f(d)]} y$$

Read $p \triangleleft q$ as q then p.

The tri product $\triangleleft$ is non-symmetric.



Part I: The category Poly is a $\otimes$-symmetric isomix

Duoidal categories

A duoidal category[4] is a category $\mathbb{X}$ with two monoidal structures $(\mathbb{X}, \otimes, \top)$ and $(\mathbb{X}, \triangleleft, \bot)$ along with a natural transformation:

$$\mathrm{duo} : \; (A \triangleleft B) \otimes (C \triangleleft D) \longrightarrow (A \otimes C) \triangleleft (B \otimes D)$$

called the interchange law, and morphisms:

$$e_\top : \; \top \longrightarrow \top \triangleleft \top \qquad e_\bot : \; \bot \otimes \bot \longrightarrow \bot$$

such that the functors $\triangleleft$ and $\bot$ are $\otimes$-lax monoidal, and the assosciativity and unitor natural isomorphisms of $(\triangleleft, \bot)$ are $\otimes$-monoidal natural transformations.

[4]Marcelo Aguiar and Swapneel Arvind Mahajan. Monoidal functors, species and Hopf algebras (2010)

Normal duoidal category

In a duoidal category, we have a map $k : \top \to \bot$

$$\top \xrightarrow{\cong} \top \otimes \top \xrightarrow{\cong} (\top \triangleleft \bot) \otimes (\bot \triangleleft \top) \xrightarrow{\text{duo}} (\top \otimes \bot) \triangleleft (\top \otimes \bot) \xrightarrow{\cong} \bot \triangleleft \bot \xrightarrow{\cong} \bot$$

A duoidal category is normal if the above composite is an isomorphism.

Bilax duoidal functor

Bilax duoidal functor $F : (\mathbb{X}, \otimes, \top, \triangleleft, \bot) \longrightarrow (\mathbb{Y}, \otimes, \top, \triangleleft, \bot)$ consists of a functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ that is

$\otimes$-lax monoidal and $\triangleleft$-colax monoidal satisfying,

$$\begin{array}{ccccc}
F((A \triangleleft B) \otimes (C \triangleleft D)) & \xleftarrow{m_\otimes} & F(A \triangleleft B) \otimes F(C \triangleleft D) & \xrightarrow{n_\triangleleft \otimes n_\triangleleft} & (F(A) \triangleleft F(B)) \otimes (F(C) \triangleleft F(D)) \\
{\scriptstyle F(\mathsf{duo})}\downarrow & & & & \downarrow{\scriptstyle \mathsf{duo}} \\
F((A \otimes C) \triangleleft (B \otimes D)) & \xrightarrow[n_\triangleleft]{} & F(A \otimes C) \triangleleft F(B \otimes D) & \xleftarrow[m_\otimes \triangleleft m_\otimes]{} & (F(A) \otimes F(C)) \triangleleft (F(B) \otimes F(D))
\end{array}$$

and 3 more coherences (two for unit laxors the duoidal map and one for the unit laxors with the k map.

Normal duoidal is also isomix

normalDuo category of normal duoidal categories and bilax duoidal functors

Isomix category of isomix LDCs and isomix functors

Theorem: There is a faithful functor from normalDuo to Isomix.

Proof sketch: Consider $\triangleleft$ to be $\oplus$. Define the left distributor as follows:

$$A \otimes (B \oplus C) \xrightarrow{\cong} (A \oplus I) \otimes (B \oplus C) \xrightarrow{\text{duo}} (A \otimes B) \oplus (I \otimes C) \xrightarrow{\cong} (A \otimes B) \oplus C$$

Lemma: The category $(\text{Poly}, \otimes, \triangleleft, y)$ is normal duoidal hence an isomix LDC. Additionally, Poly is $\otimes$-symmetric.

Attention: Since in this tutorial we will mostly be in a $\otimes$-symmetric setting, we will use $\triangleleft$ instead of $\oplus$.

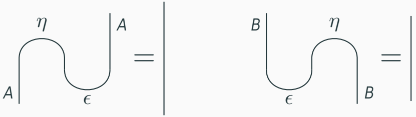
Part II: Meet the linear duals and biclosed LDCs

What is a linear dual?

In an LDC, an object B **is left dual to** A if there exist[5]:

$$\eta : \top \longrightarrow A \triangleleft B \qquad \epsilon : B \otimes A \longrightarrow \bot$$

such that:



A symmetric $*$-**autonomous category** is an LDC in which every object has a chosen dual object.

[5]Robin Cockett, Jurgen Koslowski and Robert Seely. Introduction to Linear bicategories (1999)

Biclosed LDCs

An LDC is:

⊗-closed if for all $A : \mathbb{X}$, the functor $- \otimes A : \mathbb{X} \longrightarrow \mathbb{X}$ has a right adjoint.

$$\mathbb{X}(A \otimes B, C) \cong \mathbb{X}(B, A \multimap C) \qquad \text{ev} : A \otimes (A \multimap B) \longrightarrow B$$

◁-coclosed if for all $A : \mathbb{X}$, the functor $- \triangleleft A : \mathbb{X} \longrightarrow \mathbb{X}$ has a left adjoint.

$$\mathbb{X}(A/C, B) \cong \mathbb{X}(A, B \triangleleft C) \qquad \text{coev} : A \longrightarrow (A/B) \triangleleft B$$

biclosed if $\mathbb{X}$ is both ⊗-closed and ◁-coclosed.

Poly is biclosed

The category Poly is a biclosed isomix LDC where, for any two polynomials p, q:

$$p \multimap q := [p, q] = \prod_{P:p(1)} \sum_{Q:q(1)} \prod_{d:q[Q]} \sum_{d':p[P]} y$$

$$p/q := \begin{bmatrix} q \\ p \end{bmatrix} = \sum_{P:p(1)} y^{q \triangleleft p[P]} = \sum_{P:p(1)} \prod_{Q:q(1)} \prod_{d:q[Q] \longrightarrow p[P]} y$$

Double closure maps (1)

For any object $A : \mathbb{X}$ in a biclosed isomix LDC, there is a sort of a "double closure" map:

$$A \xrightarrow{\text{coev}} \left(\begin{bmatrix} [A,y] \\ y \end{bmatrix} \triangleleft [A,y] \right) \otimes A \xrightarrow{\partial^R} \begin{bmatrix} [A,y] \\ y \end{bmatrix} \triangleleft ([A,y]) \otimes A) \xrightarrow{\text{ev}} \begin{bmatrix} [A,y] \\ y \end{bmatrix}$$

$$\Phi_A : A \longrightarrow \begin{bmatrix} [A,y] \\ y \end{bmatrix}$$

Theorem: If A is right dual to $[A, y]$ with

$$\epsilon = \text{ev} : A \otimes [A, y] \longrightarrow y$$

then Φ_A has a retraction χ_A,

$$A \xrightarrow{\Phi_A} \begin{bmatrix} [A,y] \\ y \end{bmatrix} \xrightarrow{\chi_A} A \qquad \Phi_A ; \chi_A = 1_A$$

For the other direction, we need an extra condition: coev $;\, \partial^R ;\, \chi_A ;\,$ ev $= Id_{[A,y]}$.

Double closure maps (2)

For any object $A : \mathbb{X}$ in a biclosed isomix LDC there is a sort of a "double closure" map:

$$\left[\begin{bmatrix}A\\y\end{bmatrix}, y\right] \xrightarrow{\text{coev}} \left[\begin{bmatrix}A\\y\end{bmatrix}, y\right] \otimes \left(\begin{bmatrix}A\\y\end{bmatrix} \triangleleft A\right) \xrightarrow{\partial^L} \left(\left[\begin{bmatrix}A\\y\end{bmatrix}, y\right] \otimes \begin{bmatrix}A\\y\end{bmatrix}\right) \triangleleft A \xrightarrow{\text{ev}} A$$

$$\Psi_A : \left[\begin{bmatrix}A\\y\end{bmatrix}, y\right] \to A$$

Theorem: If A is left dual to $\begin{bmatrix}A\\y\end{bmatrix}$ with

$$\eta = \text{coev} : y \longrightarrow \begin{bmatrix}A\\y\end{bmatrix} \triangleleft A$$

then Ψ_A has a section Ω_a:

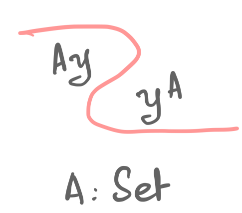
$$A \xrightarrow{\Omega_A} \left[\begin{bmatrix}A\\y\end{bmatrix}, y\right] \xrightarrow{\Psi_A} A \qquad \Omega_A ; \Psi_A = 1_A$$

Double closure maps in Poly

Theorem: A polynomial $p = y^A$ for A : Set if and only if Φ_p is the identity:

$$\mathrm{id}_p = \Phi_p : p \to \begin{bmatrix} [p,y] \\ y \end{bmatrix}$$

Corollary: For any set A, Ay is left dual to y^A with $\eta = \mathrm{coev}$ and $\epsilon = \mathrm{ev}$.



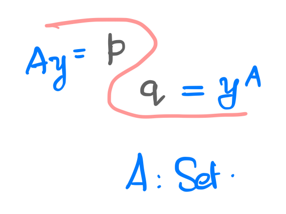
Theorem: A polynomial $q = Ay$ for A : Set if and only if Ψ_q is the identity.

$$\mathrm{id}_q = \Psi_q : \left[\begin{bmatrix} q \\ y \end{bmatrix}, y \right] \to q$$

We get the same corollary again!

What are the linear duals in Poly?

Theorem: If a polynomial p is left dual to q, then $p = Ay$ and $q = y^A$ for some A : Set.



Thus, the only polynomials with duals are linear polynomials and representables!

Theorem: If $q \dashv\!\!| \; p$ and $p \dashv\!\!| \; q$, then $p = q = y$.

Any polynomial which is both a left and a right dual of the same polynomial is trivial.

Part III: The core of mix LDC

The core of a mix LDC

The **core**[6] of a mix LDC is the full subcategory determined by objects A for which the natural transformation is also an isomorphism:

$$\mathrm{indep}_{A,-} : A \otimes - \longrightarrow A \triangleleft - \qquad \mathrm{indep}_{-,A} : - \otimes A \longrightarrow - \triangleleft A$$

The core of a mix category is closed to $\otimes$ and $\oplus$.

The core of an isomix LDC contains the monoidal units $\top$ and $\bot$.

The core of an isomix LDC is linearly equivalent to its underlying monoidal categories.

[6] Richard Blute, Robin Cockett, Robert Seely, "Feedback for linearly dis- tributive categories: traces and fixpoints" (2000)

Left and right core of mix LDCs

For a mix ($\otimes$-symmetric) LDC $(\mathbb{X}, \otimes, \top, \triangleleft, \bot)$,

Left core: Full subcategory of objects $A : \mathbb{X}$ such that

$$\mathsf{indep}_{A,-} : A \otimes - \longrightarrow A \triangleleft -$$

Right core: Full subcategory of objects $B : \mathbb{X}$ such that

$$\mathsf{indep}_{-,B} : - \otimes B \longrightarrow - \triangleleft B$$

Lemma: For an isomix LDC, the unit object is both in the left and the right cores.

In an isomix LDC, the left and the right core are compact LDCs.

Opposing cores

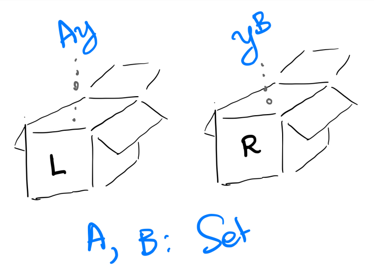
Opposing cores: A mix LDC is said to have opposing cores if

$$\star : \mathsf{core}_{\mathsf{r}}(\mathbb{X})^{\mathsf{op}} \xrightarrow{\cong} \mathsf{core}_{\ell}(\mathbb{X})$$

Examples: Compact closed categories and Poly

Lemma: In Poly, a polynomial p in the left core if and only if $p \cong Ay$ for some A : Set.

Lemma: In Poly, a polynomial q in the right core if and only if $q \cong y^B$ for some B : Set.



Opposing cores of Poly

Corollary: The category Poly has opposing cores.

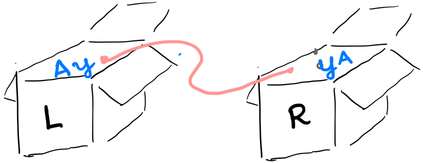
Proof: For any A : Set,

$$\left(y^A\right)^\star := Ay$$

For any A, B : Set, and map $\varphi : y^B \to y^A$,

$$\varphi^\star : Ay \to By; \quad (\varphi^\star)_1 := \varphi^\sharp$$

Corollary: In Poly, a polynomial p is left dual to q if and only if p is from the left core and q is from the right core.



Part IV: Linear monoids and linear comonoids

Robin Cockett, Jurgen Koslowski and Robert Seely.

Introduction to Linear bicategories (1999)

Priyaa Varshinee Srinivasan. (PhD Thesis)

Dagger linear logic for categorical quantum mechanics (2021)

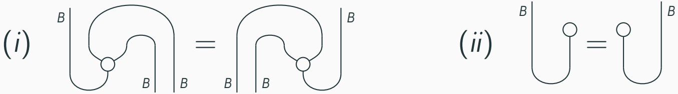
Linear monoids in LDCs

In an LDC, a **linear monoid**[7], $A \overset{\circ}{\dashv\!\!\mid} B$, contains a:

- a $\otimes$-monoid $(A, \curlyvee : A \otimes A \longrightarrow A, \, \circ : \top \longrightarrow A)$

- cyclic duals, $A \dashv\!\!\mid B$ and $B \dashv\!\!\mid A$

together producing a $\triangleleft$-comonoid $(B, \curlywedge : B \longrightarrow B \triangleleft B, \, \circ : B \longrightarrow \bot)$



[7] Robin Cockett, Jurgen Koslowski and Robert Seely.Introduction to Linear bicategories (1999)

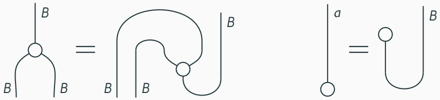
Left and right linear monoids in LDCs

In an LDC, a left linear monoid, $A \overset{\otimes \triangleleft}{\dashv\!\!\mid} B$, contains a:

- a $\otimes$-monoid $(A, \curlyvee : A \otimes A \longrightarrow A, \, \mathsf{Y} : \top \longrightarrow A)$ on the left dual, and

- a dual, $A \dashv\!\!\mid B$.

The $\otimes$-monoid structure on A induces a $\triangleleft$-comonoid structure on B via the duality:



In an LDC, a right linear monoid, $A \overset{\triangleleft \otimes}{\dashv\!\!\mid} B$, contains a:

- a $\otimes$-monoid structure on the right dual B, and

- a dual, $A \dashv\!\!\mid B$.

The $\otimes$-monoid structure on B induces a $\triangleleft$-comonoid structure on A via the duality.

Left and right linear monoids in Poly

The functor $A \mapsto Ay$ is strong monoidal $(\mathsf{Set}, 1, \times) \longrightarrow (\mathsf{Poly}, y, \otimes)$

The functor $A \mapsto y^A$ is strong monoidal $(\mathsf{Set}^{\mathsf{op}}, 1, \times) \longrightarrow (\mathsf{Poly}, y, \otimes)$.

For any monoid $(M, *, u)$ in Set, $My \overset{\otimes \; \triangleleft}{\longrightarrow\!\!\Vdash} y^M$ is a left linear monoid because:

- $(\mathrm{coev}, \mathrm{ev}) : My \dashv\!\!\mid y^M$,

- The functor $M \mapsto My$ is (strong) monoidal preserves monoids.

For any set A, $My \overset{\triangleleft \; \otimes}{\longrightarrow\!\!\Vdash} y^M$ is a right linear monoid because:

- $(\mathrm{coev}, \mathrm{ev}) : My \dashv\!\!\mid y^M$,

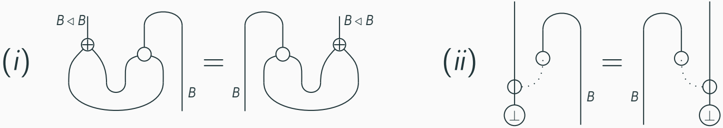
- The functor $M \mapsto y^M$ is strong monoidal, hence preserves comonoids. Every set has a unique comonoid structure given by the diagonal map.

Linear comonoids

In an LDC, a **linear comonoid**[8] $A \underset{\circ}{\dashv\!\!\mid} B$, contains a:

- a $\otimes$-comonoid $(A, \curlywedge : A \longrightarrow A \otimes A, \; \downarrow : A \longrightarrow \top)$

- cyclic duals, $A \dashv\!\!\mid B$ and $B \dashv\!\!\mid A$



We apply the same idea of left and right linear monoids to linear comonoids, and get similar examples in Poly.

[8] Priyaa Varshinee Srinivasan. (PhD Thesis) Dagger linear logic for categorical quantum mechanics (2021)

Left and right linear comonoids

In an LDC, a left linear comonoid, $A \underset{\otimes\ \triangleleft}{\dashv\!\mid} B$, contains a:

- a $\otimes$-comonoid on the left dual A, and

- a dual, $A \dashv\!\mid B$.

The $\otimes$-comonoid on A induces a $\triangleleft$-monoid on B.

In right linear comonoids, right duals carry the $\otimes$-comonoid structure.

Examples:

In Poly, for any monoid $(M, *, u)$ in Set, $My \underset{\triangleleft\ \otimes}{\dashv\!\mid} y^M$ is a right linear comonoid.

In Poly, for any set A, $Ay \underset{\otimes\ \triangleleft}{\dashv\!\mid} y^A$ is a left linear comonoid.

Part V: Linear bialgebras

Priyaa Varshinee Srinivasan.
Dagger linear logic for categorical quantum mechanics (Thesis 2021)

In a symmetric monoidal category $(\mathbb{X}, \otimes, I)$, a **bialgebra** consists of a

- a monoid $(A,$  $,$  $)$

- a comonoid $(A,$  $,$  $)$

satisfying the following rules:

Linear bialgebra in symmetric LDC

In a symmetric LDC, a **linear bialgebra**[9] consists of:

- a linear monoid, $A \overset{\circ}{\longrightarrow\!\!\shortmid\!\shortmid} B$, and

- a linear comonoid, $A \underset{\bullet}{\longrightarrow\!\!\shortmid\!\shortmid} B$,

such that:

- (A, ) is a $\otimes$-bialgebra, and

- (B, ) is a $\triangleleft$-bialgebra.

We can get linear bialgebras in half symmetric case too!

We apply the same left-right trick again! You get the idea!

[9] Priyaa Varshinee Srinivasan. (PhD Thesis) Dagger linear logic for categorical quantum mechanics (2021)

Left and right linear bialgebras in $\otimes$-symmetric LDC

In a $\otimes$-symmetric LDC, a left linear bialgebra consists of:

- a left linear monoid, $A \overset{\otimes\,\triangleleft}{\longrightarrow\!\!\!\!\shortmid\!\shortmid} B$, and
- a left linear comonoid, $A \underset{\otimes\,\triangleleft}{\longrightarrow\!\!\!\!\shortmid\!\shortmid} B$ (denoted below by $\bullet$),

such that (A, ) is a $\otimes$-bialgebra.

In a $\otimes$-symmetric LDC, a right linear bialgebra consists of:

- a right linear monoid, $A \overset{\triangleleft\,\otimes}{\longrightarrow\!\!\!\!\shortmid\!\shortmid} B$, and
- a right linear comonoid, $A \underset{\triangleleft\,\otimes}{\longrightarrow\!\!\!\!\shortmid\!\shortmid} B$ (denoted below by $\bullet$),

such that (B, ) is a $\otimes$-bialgebra.

Left and right linear bialgebras in Poly

Lemma: If $(M, *, e)$ is a monoid in Set then the dual $(\mathrm{coev}, \mathrm{ev}) : My \dashv\!\!\mid y^M$ in Poly carries all four structures:

left linear monoid	left linear comonoid
$My \overset{\otimes\ \triangleleft}{\dashv\!\!\mid} y^M$	$My \underset{\otimes\ \triangleleft}{\dashv\!\!\mid} y^M$
right linear monoid	right linear comonoid
$My \overset{\triangleleft\ \otimes}{\dashv\!\!\mid} y^M$	$My \underset{\triangleleft\ \otimes}{\dashv\!\!\mid} y^M$

Moreover, we get a left linear bialgebra (1st row) and a right linear bialgebra (2nd row).

Corollary: In Poly, we have that $My \dashv\!\!\mid y^M$ is both a left and right linear bialgebra if and only if M is a monoid in Set.

Summary

- All normal duoidal categories are isomix LDCs.

- (Poly, $\otimes$, $\triangleleft$, y) is an $\otimes$-symmetric isomix LDC.

- In Poly, for any set A, Ay is left dual to y^A.

- In Poly, the only objects with duals are linear polynomials and represenatables.

- Poly has left- and right- linear- monoids, comonoids, and bialgebras.

A bugging question:

What is the connection between the cores and the biclosure property? Is there one?