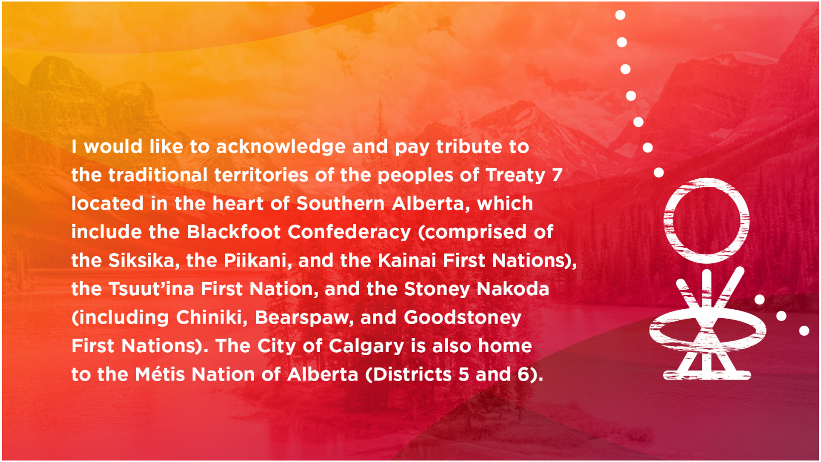
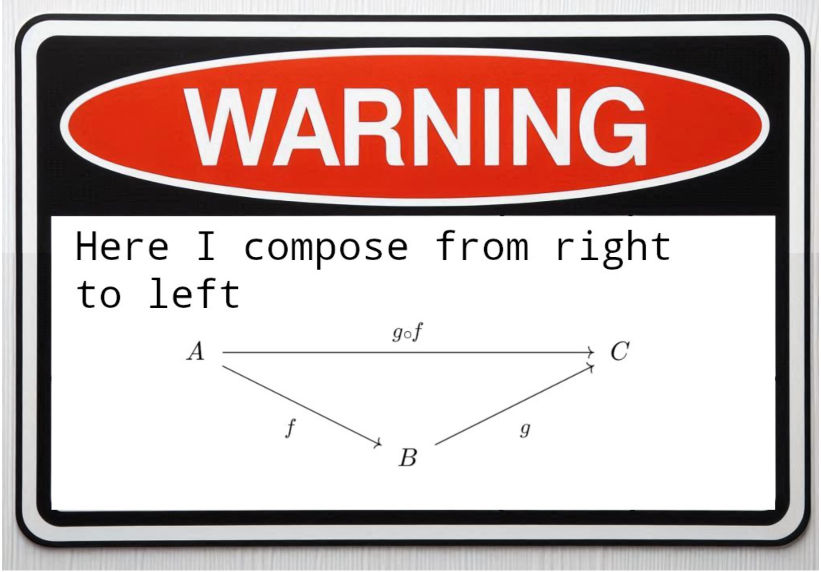
Differential bundles in tangent (infinity) categories

Florian Schwarz, work with Kristine Bauer, idea by Michael Ching

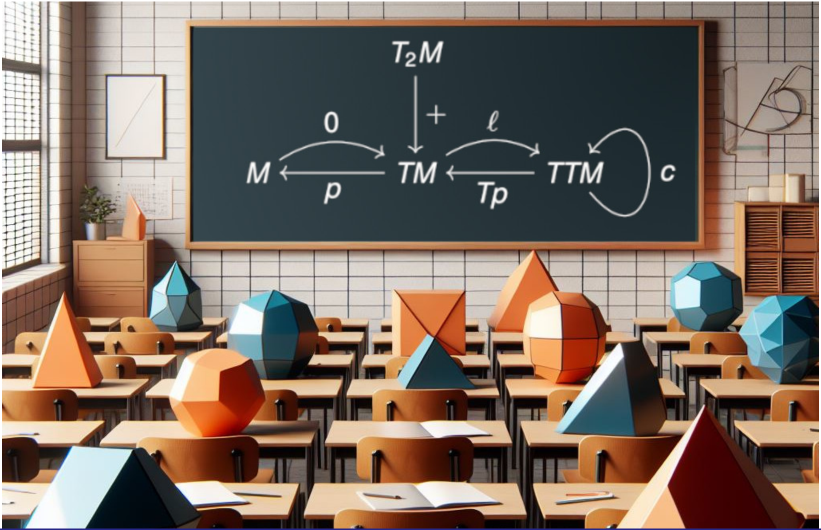
FMCS 2024


I would like to acknowledge and pay tribute to the traditional territories of the peoples of Treaty 7 located in the heart of Southern Alberta, which include the Blackfoot Confederacy (comprised of the Siksika, the Piikani, and the Kainai First Nations), the Tsuut'ina First Nation, and the Stoney Nakoda (including Chiniki, Bearspaw, and Goodstoney First Nations). The City of Calgary is also home to the Métis Nation of Alberta (Districts 5 and 6).


Tangent categories
Differential bundles
Classification of Differential bundles
Outlook towards infinity

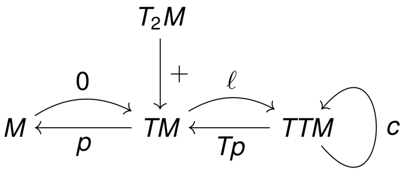

WARNING
Here I compose from right to left
$g \circ f$
A
C
f
g
B

Tangent categories



A **tangent category** is a category $\mathbb{X}$ with:

- A tangent functor $T : \mathbb{X} \to \mathbb{X}$
- A projection natural transformation $p : T \to 1$ with pullback powers $T_n M$ preserved by T.
- A addition natural transformation $+ : T_2 \to T$
- A zero natural transformation $0 : 1 \to T$
- A vertical lift natural transformation $\ell : T \to T^2$
- A canonical flip natural transformation $c : T^2 \to T^2$



and some conditions

Example: $\mathbb{N}^\bullet$

Let $\mathbb{N}^\bullet$ be the category with

- The free f.g. $\mathbb{N}$-modules $\mathbb{N}^k$, $k \in \mathbb{N}$ as objects
- $\mathbb{N}$-linear maps as morphisms

There is a tangent structure on $\mathbb{N}^\bullet$ given by

$$T_2(\mathbb{N}^k) = \mathbb{N}^k \times \mathbb{N}^{2k} \xrightarrow{+_{\mathbb{N}^k} = 1 \times \mathrm{add}} T(\mathbb{N}^k)$$

$$\mathbb{N}^k \underset{p_{\mathbb{N}^k} = \pi_0}{\overset{0_{\mathbb{N}^k} = 1 \times 0}{\rightleftarrows}} T(\mathbb{N}^k) = \mathbb{N}^k \times \mathbb{N}^k \underset{T(p_{\mathbb{N}^k}) = \langle \pi_0, \pi_2 \rangle}{\overset{\ell_{\mathbb{N}^k} = \langle \pi_0, 0, 0, \pi_1 \rangle}{\rightleftarrows}} T^2(\mathbb{N}^k) = \mathbb{N}^k \times \mathbb{N}^k \times \mathbb{N}^k \times \mathbb{N}^k$$

$$c_{\mathbb{N}^k} = \langle \pi_0, \pi_2, \pi_1, \pi_3 \rangle : T^2(\mathbb{N}^k) \to T^2(\mathbb{N}^k)$$

Tangent functors

A **lax tangent functor** between tangent categories $(\mathbb{X}, T_{\mathbb{X}})$ and $(\mathbb{Y}, T_{\mathbb{Y}})$ is a pair (F, α) of a functor $F : \mathbb{X} \to \mathbb{Y}$ and a natural transformation $\alpha : F \circ T_{\mathbb{X}} \to T_{\mathbb{Y}} \circ F$ compatible with the tangent structure, e.g.

$$\begin{array}{ccc} F\circ T_{\mathbb{X}} & \xrightarrow{\alpha} & T_{\mathbb{Y}}\circ F \\ & \searrow^{F(p_{\mathbb{X}})} & \downarrow{\scriptstyle p_{\mathbb{Y}}} \\ & & F \end{array} \qquad \begin{array}{ccc} F\circ (T_{\mathbb{X}})_2 & \xrightarrow{\alpha_2} & (T_{\mathbb{Y}})_2\circ F \\ {\scriptstyle F(+_{\mathbb{X}})}\downarrow & & \downarrow{\scriptstyle +_{\mathbb{Y}}} \\ F\circ T_{\mathbb{X}} & \xrightarrow[\alpha]{} & T_{\mathbb{Y}}\circ F \end{array} \qquad \cdots$$

It is called a **strong tangent functor** if α is an isomorphisms and F preserves the pullbacks that are part of the definition of a tangent structure.

Tangent natural transformations

Let $(\mathbb{X}, T_{\mathbb{X}})$ and $(\mathbb{Y}, T_{\mathbb{Y}})$ be tangent categories and $(F, \alpha), (F', \alpha') : (\mathbb{X}, T_{\mathbb{X}}) \to (\mathbb{Y}, T_{\mathbb{Y}})$ be lax tangent functors. A **tangent natural transformation** $(F, \alpha) \Rightarrow (F', \alpha')$ is a natural transformation

$$\varphi : F \Rightarrow F'.$$

It is called **linear** if

$$\begin{array}{ccc} F \circ T_{\mathbb{X}} & \xrightarrow{\varphi_{T_{\mathbb{Y}}}} & F' \circ T_{\mathbb{X}} \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ T_{\mathbb{Y}} \circ F & \xrightarrow[T_{\mathbb{Y}}(\varphi)]{} & T_{\mathbb{Y}} \circ F' \end{array}$$

commutes.

Biategories of tangent Categories

We can now look at

- lax tangent functors with tangent natural transformations,
- lax tangent functors with linear tangent natural transformations,
- strong tangent functors with tangent natural transformations, or
- strong tangent functors with linear tangent natural transformations.

This gives us *four variants of bicategories* of tangent categories, e.g. $\mathrm{TANG}_{\mathrm{lax}}$, the bicategory with

- Tangent categories as objects
- Lax tangent functors as 1-morphisms
- Tangent natural transformations as 2-morphisms.

Given two tangent categories $(\mathbb{X}, T_{\mathbb{X}}), (\mathbb{Y}, T_{\mathbb{Y}})$ there are 1-categories of lax tangent functors $(\mathbb{X}, T_{\mathbb{X}}) \to (\mathbb{Y}, T_{\mathbb{Y}})$

$$\mathrm{LaxFun}(\mathbb{X}, \mathbb{Y}) \qquad \mathrm{LaxFun}_{\mathrm{lin}}(\mathbb{X}, \mathbb{Y})$$

Differential bundles

Differential bundles

In a tangent category one can define a **differential bundle** E over M:

$$\begin{array}{ccccc} & & E_2 & & \\ & & \downarrow{\scriptstyle \sigma} & & \\ M & \underset{q}{\overset{\zeta}{\leftrightarrows}} & E & \xrightarrow{\lambda} & T(E) \end{array}$$

such that finite pullback powers of q exist and are preserved by T^n,

$$\begin{array}{ccc} E_2 & \xrightarrow{\langle \pi_0 0, \pi_1 \lambda \rangle T(\sigma)} & TE \\ {\scriptstyle \pi_0 q}\downarrow & & \downarrow{\scriptstyle T(q)} \\ M & \xrightarrow{0} & TM \end{array}$$

is a pullback and some additional conditions hold.

First examples

The tangent bundle $TM \xrightarrow{p} M$ is a differential bundle.

Ben MacAdam showed that differential bundles in SmMan are exactly vector bundles. For example $\mathbb{R} \times \mathbb{R}$ over $\mathbb{R}$:

$$E_2 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \xrightarrow{\sigma = 1_{\mathbb{R}} \times \mathrm{add}} E = \mathbb{R} \times \mathbb{R}$$

$$M = \mathbb{R} \underset{q = \pi_0}{\overset{\zeta = 1_{\mathbb{R}} \times 0}{\rightleftarrows}} E = \mathbb{R} \times \mathbb{R} \xrightarrow[\lambda = 1_{\mathbb{R}} \times 0 \times 0 \times 1_{\mathbb{R}}]{} T(E) = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

Example: $\mathbb{N}^1 \to \mathbb{N}^0$

In the tangent category $\mathbb{N}^\bullet$, the following is a differential bundle:

$$
\begin{array}{ccccc}
 & & E_2 = \mathbb{N}^2 & & \\
 & & \big\downarrow \sigma = \mathrm{add} & & \\
M = \mathbb{N}^0 & \underset{q = !}{\overset{\zeta = 0}{\rightleftarrows}} & E = \mathbb{N}^1 & \xrightarrow[\lambda = \langle 0,1 \rangle]{} & T(E) = \mathbb{N}^2
\end{array}
$$

A differential bundle over the terminal object is called a **differential object**.

Proposition (Differential bundles as differential objects)

A differential bundle is a differential object in the slice category.

Morphisms of differential bundles

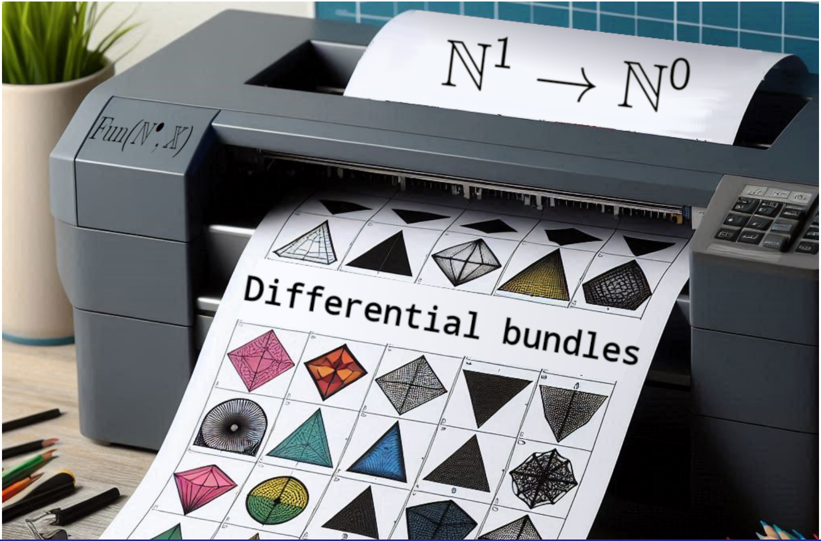
A **morphism of differential bundles** from $(E, M, q, \sigma, \zeta, \lambda)$ to $(E', M', q', \sigma', \zeta', \lambda')$ in $\mathbb{X}$ is a pair of morphisms $f : E \to E', g : M \to M'$ such that

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ M & \xrightarrow[g]{} & M' \end{array}$$

commutes. It is called additive if the first two diagrams commute and linear if the third diagram commutes.

$$\begin{array}{ccc} E_2 & \xrightarrow{f_2} & E'_2 \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma'} \\ E & \xrightarrow[f]{} & E' \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{g} & M' \\ {\scriptstyle \zeta}\downarrow & & \downarrow{\scriptstyle \zeta'} \\ E & \xrightarrow[f]{} & E' \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \lambda'} \\ T(E) & \xrightarrow[T(f)]{} & T(E') \end{array}$$

Classification of differential bundles



Differential functors

A lax tangent functor (F, α) between tangent categories $(\mathbb{X}, T_{\mathbb{X}})$ and $(\mathbb{Y}, T_{\mathbb{Y}})$ is called **lax differential** if it

- $T^n \circ F$ preserves pullbacks over the terminal object $\forall n \in \mathbb{N}$, and
- $\alpha : F \circ T_{\mathbb{X}} \Rightarrow T_{\mathbb{Y}} \circ F$ is Cartesian, i.e. its naturality squares are pullbacks.

It is called **strong differential** if it also preserves the terminal object.

Proposition (preservation)

Strong differential functors send differential objects to differential objects. Lax differential functors send differential objects to differential bundles.

Classification

Proposition (Induced functor)

For any differential bundle $E \xrightarrow{q} M$ in any tangent category $\mathbb{X}$ there is a lax differential functor from $\mathbb{N}^\bullet$.

$$F_E : \mathbb{N}^\bullet \to \mathbb{X}$$

$$\mathbb{N}^0 \mapsto M \qquad \mathbb{N}^1 \mapsto E \qquad \mathbb{N}^k \mapsto E_k$$

$$\Big(n \cdot (-) : \mathbb{N} \to \mathbb{N}\Big) \mapsto \Big(\sigma_n \circ \Delta_n : E \to E\Big)$$

The natural transformation $\alpha_E : F_E \circ T_{\mathbb{N}^\bullet} \Rightarrow T_{\mathbb{X}} \circ F_E$ is

$$(\alpha_E)_{\mathbb{N}^1} := T(\sigma) \circ \langle 0 \circ \pi_0, \lambda \circ \pi_1 \rangle : E_2 \to T(E), \quad (\alpha_E)_{\mathbb{N}^k} := ((\alpha_E)_{\mathbb{N}^1})_k.$$

Equivalence of Categories

Theorem

There is an equivalence of categories

$$\mathrm{DiffFun}_{\mathrm{lin}}(\mathbb{N}^\bullet, \mathbb{X}) \simeq \mathrm{DiffBun}_{\mathrm{lin}}(\mathbb{X})$$

$$\mathrm{DiffFun}(\mathbb{N}^\bullet, \mathbb{X}) \simeq \mathrm{DiffBun}_{\mathrm{add}}(\mathbb{X})$$

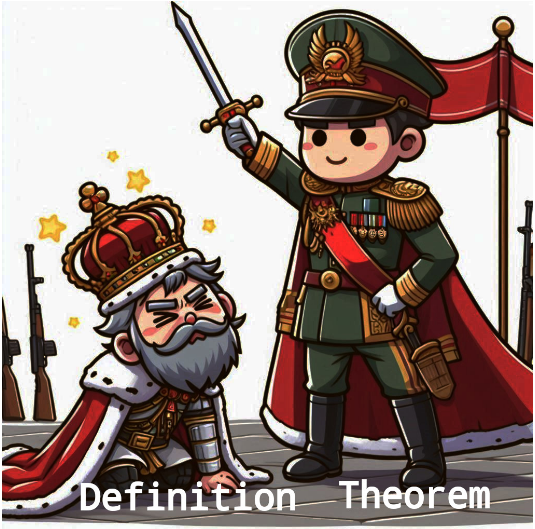
where $\mathrm{DiffFun}$ *and* $\mathrm{DiffFun}_{\mathrm{lin}}$ *are the full subcategory of differential functors in* LaxFun *and* $\mathrm{LaxFun}_{\mathrm{lin}}$.

An additive morphism $f : E \to E', g : M \to M'$ between differential bundles $E \xrightarrow{q} M, E' \xrightarrow{q'} M'$ induces

$$\varphi : F_E \Rightarrow F_{E'} \qquad \varphi_{\mathbb{N}^k} : F(\mathbb{N}^k) = E_k \mapsto E'_k$$

additivity $\leftrightarrow$ naturality (ζ and σ are images of maps in $\mathbb{N}^\bullet$)
linearity $\leftrightarrow$ compatibility with α_E and α'_E

Outlook towards infinity



What about $\mathbb{N}^\bullet$

$\mathbb{N}^\bullet$ encodes commutative monoids

$$\mathrm{Mon}(\mathbb{X}) \simeq \mathrm{Fun}_\times(\mathbb{N}^\bullet, \mathbb{X})$$

James Cranch showed that the Duskin-Nerve of the bicategory $\mathrm{Span}(\mathrm{FinSet})$ encodes homotopy commutative monoids.

$$E_\infty := N(\mathrm{Span}(\mathrm{FinSet}))$$

So E_∞ is the ∞-version of $\mathbb{N}^\bullet$.

In tangent ∞-categories

Make this equivalence a definition

Definition

The category of **differential bundles** in a tangent infinity Category $\mathbb{X}$ is the category of lax tangent functors $E_\infty \to \mathbb{X}$ preserving pullbacks over the terminal object with cartesian natural transformations.

What properties do such differential bundles have?
Is the tangent bundle still a differential bundle? *Yes* (Michael Ching)
Are the differential objects in the slice category still the differential bundles?

Even in the 1-category case
Can one understand connections through this perspective?
How does this relate to Michael Ching's $E \cong M \times_{TM} TE \times_E M$?

References

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