Double Category Sites for Grothendieck Topoi

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Background: Grothendieck Topoi and Etendues

Definition: • A Grothendieck topos $\mathcal{T}$ is a category that admits a geometric embedding into a presheaf category

$$\begin{tikzcd} \mathcal{T} \arrow[rr, hook, shift right, "\text{full and faithful}"'] \arrow[rr, phantom, "\perp"] & & \mathrm{PSh}(\mathcal{C}) \arrow[ll, shift right, "\text{left ex.}"'] \end{tikzcd}$$

• A spatial étendue is a Grothendieck topos $\mathcal{E}$ that has an object E such that:

1. $E \longrightarrow 1$ is epic
2. the slice topos $\mathcal{E}/E$ is localic/spatial.

Sites for Etendues

- A Grothendieck topos is an étendue iff it can be described as sheaves on a left-cancellative site. [Kock-Moerdijk, Rosenthal]

- Definition An ordered groupoid is an internal groupoid $\mathcal{G}$ in the category of posets whose source $s : \mathcal{G}_1 \to \mathcal{G}_0$ is a discrete fibration.

[DeWolf-P]: ordered groupoids as double categories:

order in $\mathcal{G}_1$

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[d, "\wedge"'] \arrow[dr, phantom, "\wedge"] & B \arrow[d, "\wedge"] \\ A' \arrow[r, "\cong"] & B' \end{tikzcd} \qquad + \qquad \forall \quad \begin{tikzcd} A \arrow[r, "f|_A"] \arrow[d] \arrow[dr, phantom, "\exists!\, \wedge"] & B \arrow[d] \\ A' \arrow[r, "\cong", "f"'] & B' \end{tikzcd}$$

objects in $\mathcal{G}_0$

order in $\mathcal{G}_0$

objects in $\mathcal{G}_1$

From Left-Cancellative Categories to Ordered Groupoids

$\mathcal{C}$ left-cancellative $\longmapsto$ $E(\mathcal{C})$ ordered groupoid:

<u>obj</u>: subobjects in $\mathcal{C}$: $[m]$, where

$m : A' \rightarrowtail A$ in $\mathcal{C}$

<u>vert. arrow</u>: $[m] \longrightarrow [n]$ iff

$$\exists\ \begin{tikzcd} A'' \arrow[rr, tail, "k"] \arrow[dr, tail, "m"'] & \arrow[d, phantom, "="] & A' \arrow[dl, tail, "n"] \\ & A & \end{tikzcd} \text{ in } \mathcal{C}.$$

<u>hor. arrows</u>: $[m] \xrightarrow{[m', f, n']} [n]$

given by
$$\begin{tikzcd} \cdot \arrow[r, phantom, "\cong"] \arrow[dr, tail, "m"'] & \cdot \arrow[d, tail, "m'"] \arrow[r, "f", "\cong"'] & \cdot \arrow[r, phantom, "\cong"] \arrow[d, tail, "n'"'] & \cdot \arrow[dl, tail, "n"] \\ & \cdot & \cdot & \end{tikzcd} \text{ in } \mathcal{C}$$

<u>double cells</u>:

$$\begin{tikzcd} {[m]} \arrow[r, "{[m, f, n]}"] \arrow[d] \arrow[dr, phantom, "\wedge"] & {[n]} \arrow[d] \\ {[m']} \arrow[r, "{[m', f', n']}"'] & {[n']} \end{tikzcd} \quad \text{given by} \quad \begin{tikzcd} & \cdot \arrow[dl, tail, "m"'] \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "="] & \cdot \arrow[d] \arrow[dr, tail, "n"] & \\ \cdot & \cdot \arrow[l, tail, "m'"] \arrow[r, "f'"'] & \cdot \arrow[r, tail, "n'"'] & \cdot \end{tikzcd}$$

From Ordered Groupoids to Left-Cancellative Categories

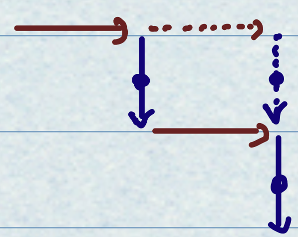
$C(\mathcal{G})$ - category of "corners" in $\mathcal{G}$: $\longleftarrow\!\!\shortmid$ $\mathcal{G}$ ordered groupoid

<u>objects</u>: those of $\mathcal{G}$

<u>arrows</u>: formal composites

$\xrightarrow{\cong} \; \bullet\!\!\!\longrightarrow$

Composition:



<u>Thm</u>. •
$$\begin{tikzcd} \underline{\mathrm{lcCat}} \arrow[r, shift left=2, "E"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{oGpd}} \arrow[l, shift left=2, "C"] \end{tikzcd}$$
is a 2-adjunction

•
$$\begin{tikzcd} \underline{\mathrm{lcCat}} \arrow[r, shift left=2, "E"] \arrow[r, phantom, "\perp"] & (\underline{\mathrm{oGpd}})_{max} \arrow[l, shift left=2, "C"] \end{tikzcd}$$
is a bi-equivalence

Sites for Etendues

$\underline{\text{lc Cat}} \simeq \underline{\text{oGpd}}_{max}$

Grothendieck top. $\xrightarrow{E}$ Ehresmann top. on the
$\xleftarrow[C]{}$ vertical arrows

Sheaves : $\mathcal{C}^{op} \longrightarrow \underline{Set}$
satisfying the amalgamation condition

sheaves : $\mathcal{G}^{op.op} \longrightarrow \underline{Set}$
satisfying the <u>vertical</u> amalgamation condition

$$\begin{tikzcd}
\underline{\text{lc - Sites}} \arrow[rr, phantom, "\simeq"] \arrow[dr, "Sh"'] & \arrow[d, phantom, "\cong"] & \underline{\text{E - Sites}}_{max} \arrow[dl, "Sh."] \\
& \underline{\text{Etendues}} &
\end{tikzcd}$$

Remarks

- An Ehresmann topology on an ordered groupoid gives rise to a Grothendiech topology on its category of vertical arrows.

- To define the 2-equivalence between categories of sites we need to translate the notions of covering flatness and covering preservation to double functors between E-sites.

- We can also translate the notions of the Comparison lemma, but we won't obtain topoi as a category of fractions, because there are not enough left-cancellative sites to satisfy the Ore condition.

Goal for Today

- Introduce a notion of generalized Ehresmann sites such that:
  - every Grothendieck topos is represented by a generalized Ehresmann site
  - Grothendieck topoi can be obtained as a bicategory of fractions w.r.t. the comparison Lemma maps

* if I have time I will tell you what additional sites we obtain for étendues.

The Idea

$$\begin{tikzcd}
\text{lc site} \arrow[r, shift left=2, "\text{subobjects, isos, monos}"] \arrow[r, phantom, "\perp"] & \text{E-site} \arrow[l, shift left=2, "\text{corners}"]
\end{tikzcd}$$

Corners are known to give a factorization system

$$\begin{tikzcd}
\text{Gr. sites with factorization system} \arrow[r, shift left=2, "\text{subobjects, left class, right class}"] \arrow[r, phantom, "\perp"] & \text{generalized E-sites} \arrow[l, shift left=2, "\text{corners}"]
\end{tikzcd}$$

<u>Note:</u> although this looks a lot like the adjunctions studied by M. Štěpán, this is not the same; in particular, in our case corners provide a right adjoint, for him a left adjoint.

Grothendieck Topoi as Sites

Let $\mathcal{G}$ be a Grothendieck topos.

- $(\mathcal{G}, J_{can})$ is a Grothendieck site with J_{can} the canonical topology: covering families = jointly-epic families
- Since $\mathcal{G}$ is regular, (epi, mono) is a stable orth. fact. system.
- J_{can} is fully determined by:

covers:

(m) families of monics that are covering (jointly epic) $\twoheadrightarrow \rightarrowtail$

(c) single covering arrows (epi's) $\twoheadrightarrow \rightarrowtail$

(s) epi's are stable under pbs along epi's and monics $\twoheadrightarrow \rightarrowtail$

→ left cancellative sites (all arrows are monic)

→ atomic sites (sites where every single arrow covers, one needs $\begin{tikzcd} \bullet \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\exists"] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$)

Covering-Mono Grothendieck Sites

<u>Definition</u> A covering-mono site is a Grothendieck site $(\mathcal{C}, J_{\mathcal{C}})$ with an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ on the category $\mathcal{C}$ such that:

- $\mathcal{E} \subseteq \{\text{arrows that are single-arrow } J_{\mathcal{C}}\text{-coverings}\}$ $\quad \twoheadrightarrow$
- $\mathcal{M} \subseteq \{\text{monomorphisms}\}$ $\quad \rightarrowtail$
- $\mathcal{E}$ is left-quadrable: $\forall$ $\begin{tikzcd} \bullet \arrow[r, dashed, two heads] \arrow[d, dashed, two heads] \arrow[dr, phantom, "\exists"] & \cdot \arrow[d, two heads] \\ \cdot \arrow[r, two heads] & \cdot \end{tikzcd}$ $\quad \forall$ $\begin{tikzcd} \cdot \arrow[r, dashed, two heads] \arrow[d, dashed, tail] \arrow[dr, phantom, "\exists"] & \cdot \arrow[d, tail] \\ \cdot \arrow[r, two heads] & \cdot \end{tikzcd}$

<u>Examples:</u>

(1) $(\mathcal{C}, J_{\mathcal{C}})$ left-cancellative site + $(\mathrm{Iso}, \mathrm{Mor})$-factorization

(2) $(\mathcal{C}, J_{\mathcal{C}})$ atomic site + $(\mathrm{Mor}, \mathrm{Iso})$-factorization

(3) $(\mathcal{G}, J_{can})$ Grothendieck topos + $(\mathrm{Epi}, \mathrm{Mono})$-factorization

(4) X_{et}, the small étale site over a scheme X + (surjective, open immersion) factorization

Grothendieck Topoi and CM Sites

<u>Proposition</u> For every Grothendieck topos $\mathcal{G}$, there exists a CM site $(\mathcal{A}, J_{\mathcal{A}})$ such that $\mathcal{G} \simeq Sh(\mathcal{A}, J_{\mathcal{A}})$.

<u>Sketch of the Proof:</u> Take $(\mathcal{C}, J_{\mathcal{C}})$ with $\mathcal{G} \simeq Sh(\mathcal{C}, J_{\mathcal{C}})$.

Consider the functor

$$(\mathcal{C}, J_{\mathcal{C}}) \xrightarrow{\#\not{\mathcal{L}}} \mathcal{G}$$

- Let $(\mathcal{A}', J_{can})$ be the closure of $Im(\#\not{\mathcal{L}})$ under finite limits in $\mathcal{G}$.
- Add the subobjects in the epi-mono factorizations of arrows in $\mathcal{A}'$ to obtain $\mathcal{A}$, with the canonical topology from $\mathcal{G}$.

Morphisms Between CM-Sites

Note: morphisms between sites correspond to the algebraic part of the geometric morphisms — they are left adjoint and left exact.

left adjoint $\Rightarrow$ preserves epis

left exact $\Rightarrow$ preserves monics

Definition A CM morphism of CM sites is a functor that is covering preserving, covering flat and sends the $\begin{cases} \text{left} \\ \text{right} \end{cases}$ class to the $\begin{cases} \text{left} \\ \text{right} \end{cases}$ class.

Notation: CMSite := 2-category of CM sites with CM morphisms and natural transformations.

There are enough CM sites

Definition A covering-flat, covering-preserving morphism of sites is

$$f : (\mathcal{C}, J_{\mathcal{C}}) \longrightarrow (\mathcal{D}, J_{\mathcal{D}})$$

LC (Lemme de Comparaison) if it is covering-reflecting and satisfies:

(G) for all B in $\mathcal{D}$ there is a $J_{\mathcal{D}}$-cover

$$\left(f(A_i) \longrightarrow B\right)_{i \in I}$$

locally surjective

(F) for all $c : f(A) \longrightarrow f(A')$ there is a $J_{\mathcal{C}}$-cover

$$\left(A_i \xrightarrow{a_i} A\right)_{i \in I} \quad \text{s.t.} \quad c \circ f(a_i) \in \mathrm{Im}(f)$$

locally full

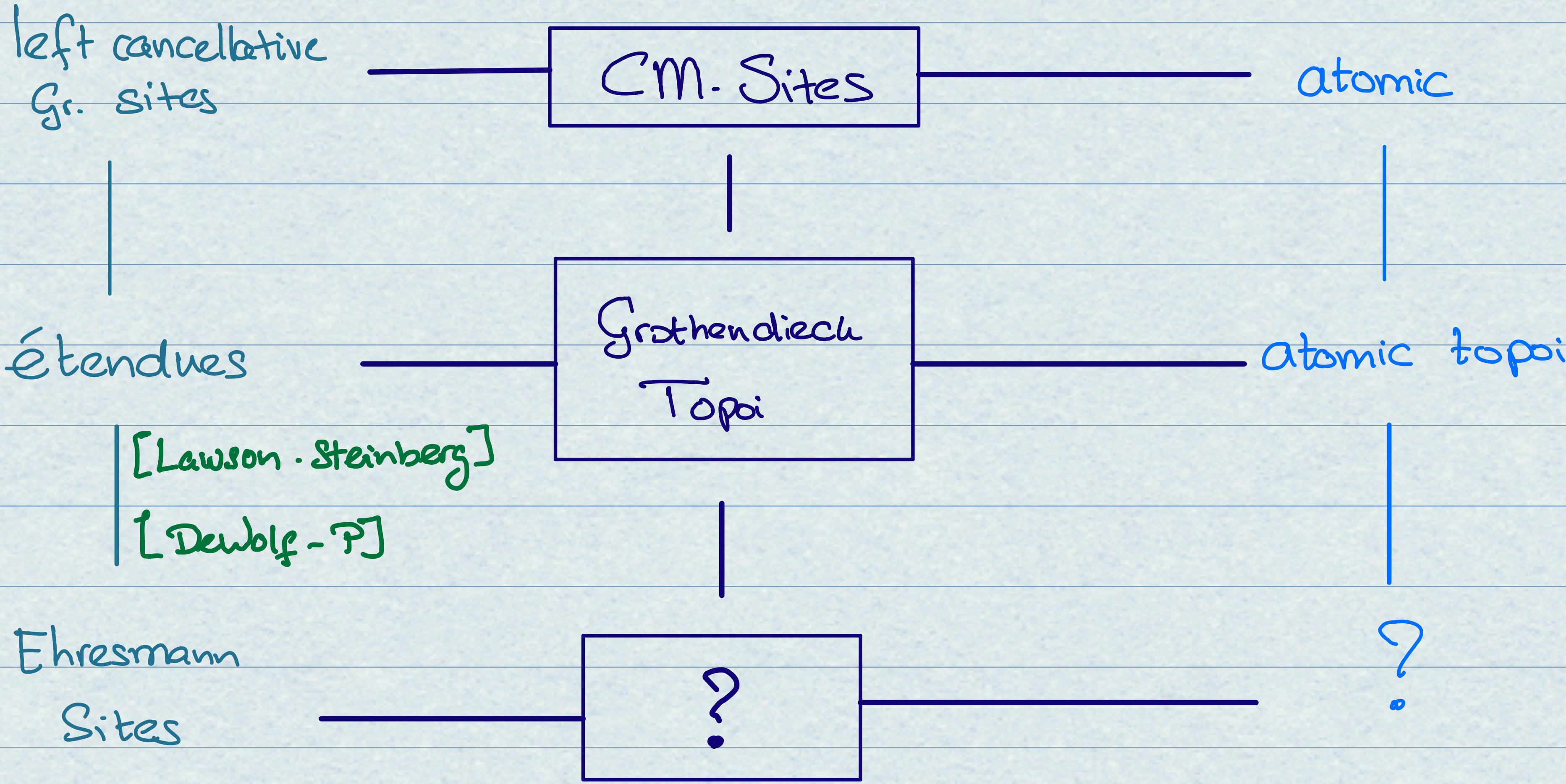
(FF) for all $c, c' : A \rightrightarrows A'$ s.t. $f(c) = f(c')$ there is a $J_{\mathcal{C}}$-cover

$$\left(A_i \xrightarrow{a_i} A\right)_{i \in I} \quad \text{s.t.} \quad c \circ a_i = c' \circ a_i$$

locally faithful

Theorem: The pseudofunctor $\mathrm{Sh} : \mathrm{CM\text{-}Site} \longrightarrow \mathrm{GrTop}^{\mathrm{coop}}$ sends LC CM morphisms to equivalences and induces: $\mathrm{CMSite}[LC_{cm}^{-1}] \underset{bi}{\simeq} \mathrm{GrTop}^{\mathrm{coop}}$.

Sites for Grothendieck Topoi



Generalized Ehresmann Sites

- Replace ordered groupoids by ordered categories :

<u>Definition</u>: An ordered <u>category</u> is an internal category $\mathcal{P}$ in the category of posets whose source $s : \mathcal{P}_1 \longrightarrow \mathcal{P}_0$ is a discrete fibration.

objects in $\mathcal{P}_0$

order in $\mathcal{P}_1$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\wedge\!\|"'] \arrow[dr, phantom, "\wedge\!\|"] & B \arrow[d, "\wedge\!\|"] \\ A' \arrow[r] & B' \end{tikzcd}$$

order in $\mathcal{P}_0$

objects in $\mathcal{P}_1$

$$+ \quad \forall \quad \begin{tikzcd} A \arrow[r, "f|_A"] \arrow[d] \arrow[dr, phantom, "\exists!\ \wedge\!\|"] & B \arrow[d] \\ A' \arrow[r, "\cong", "f"'] & B' \end{tikzcd}$$

<u>Definition</u>: An ordered category is left quadrable if

$$\forall \quad \begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\exists\ \wedge\!\|"] & B \arrow[d] \\ A' \arrow[r] & B' \end{tikzcd} \quad \text{and} \quad \forall \quad \begin{tikzcd} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\exists\ ="] & Y \arrow[d] \\ X \arrow[r] & A \end{tikzcd}$$

Generalized Ehresmann Sites

l. guadr.

<u>Definition</u> A generalized Ehresmann site is an ordered category $\mathbb{P}$ with an Ehresmann topology, a collection of vertical sieves

$(\mathcal{T}(A),\ A \in Obj(\mathbb{P}_0))$:

(ET.1) the trivial sieve $(\downarrow A) \in \mathcal{T}(A)$

(ET.2) if $\mathcal{B} \in \mathcal{T}(B)$ and $A \xrightarrow{f} B' \overset{\bullet}{\longrightarrow} B$

then
$$f^*\mathcal{B} = \left(\begin{tikzcd} A' \arrow[r] \arrow[d, "\bullet"] & B'' \arrow[d, "\bullet"] \\ A \arrow[r, "f"] & B' \arrow[d, "\bullet"] \\ & B \end{tikzcd} \in \mathcal{B} \right)$$
is in $\mathcal{T}(A)$.

in $f^*\mathcal{B}$

(ET.3) for $\mathcal{A} \in \mathcal{T}(A)$ and $\mathcal{D}$ any vertical sieve on A : if for all $C \xrightarrow{f} A' \overset{\bullet}{\longrightarrow} A$ with $A' \overset{\bullet}{\longrightarrow} A$ in $\mathcal{A}$, $f^*\mathcal{D} \in \mathcal{T}(C)$ then $\mathcal{D} \in \mathcal{T}(A)$.

<u>Remark</u>: $\mathcal{T}$ restricts to a Grothendieck topology on the cat^y of vertical arrows.

Sheaves on generalized Ehresmann sites

To define the amalgamation conditions for sheaves $F : \mathbb{P}^{op,op} \longrightarrow \mathbb{Q}\underline{Set}$ we need the following notion wrt horizontal arrows in $\mathbb{P}$:

<u>Definition:</u> For an ordered cat^y $\mathbb{P}$, double functor $F : \mathbb{P}^{op,op} \longrightarrow \mathbb{Q}\underline{Set}$, and horizontal arrow $f : B \longrightarrow A$ in $\mathbb{P}$:

An element $x \in F(B)$ is <u>self-compatible</u> for F and f if

for all
$$\begin{tikzcd} C \arrow[r, "g"] & B' \arrow[d, "\bullet"] \\ & B \end{tikzcd} \quad \& \quad \begin{tikzcd} C \arrow[r, "h"] & B'' \arrow[d, "\bullet"] \\ & B \end{tikzcd}$$
with $f|_{B'} \circ g = f|_{B''} \circ h$ ...

$$\begin{tikzcd} C \arrow[r, "g"] & B' \arrow[r, "f|_{B'}"] \arrow[d, "\bullet"] & A' \arrow[d, "\bullet"] \\ & B \arrow[r, "f"'] & A \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "h"] & B'' \arrow[r, "f|_{B''}"] \arrow[d, "\bullet"] & A' \arrow[d, "\bullet"] \\ & B \arrow[r, "f"'] & A \end{tikzcd}$$

we have:

$$F(g) \circ F(B' \overset{\bullet}{\longrightarrow} B)(x) = F(h) \circ F(B'' \overset{\bullet}{\longrightarrow} B)(x)$$

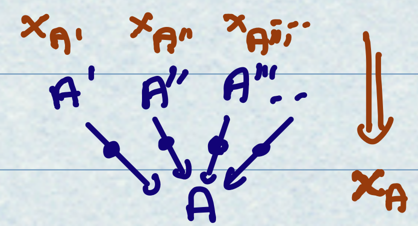
Sheaves on generalized Ehresmann sites

<u>Definition:</u> A sheaf on a gen. E-site $(\mathcal{G}, \mathcal{J})$ is a functor $F : \mathcal{G}^{op,op} \longrightarrow \mathbb{Q}\,\underline{Set}$ satisfying the following conditions:

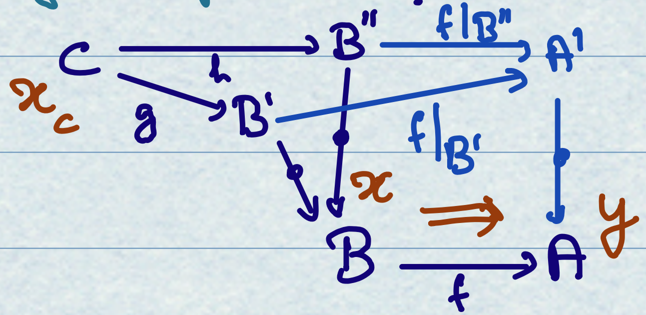
① A is a sheaf on the vertical category with the induced Gr. topology:

for each $S \in \mathcal{J}(A)$ and comp. family $(x_{A'})_{(A' \to A) \in S} \in F(A')$,

$\exists !\ x \in F(A)$ s.t. $F(A' \to A)(x) = x_{A'}$ for all $A' \to A \in S$.



② for each horizontal arrow $f : B \longrightarrow A$ and $x \in F(B)$ which is self-compatible for F and f, $\exists !\ y \in F(A)$ s.t. $F(f)(y) = x$



A morphism of sheaves is a horizontal (equiv. vertical) transformation of double functors.

From ~~Left-Cancellative Categories~~ CM-Sites to Ordered ~~Groupoids~~ generalized E-Sites

$(\mathcal{C}, J_{\mathcal{C}}, \mathcal{E}, \mathcal{M})$ a CM-site $\longmapsto$ $E(\mathcal{C})$ ordered ~~groupoid~~ category:

obj: subobjects in $\mathcal{M}$: $[m]$, where

$m : A' \rightarrowtail A$ in $\mathcal{M}$

vert. arrow: $[m] \to [n]$ iff

$$\begin{tikzcd} \exists\ A'' \arrow[rr, tail, "k"] \arrow[dr, tail, "m"'] & \arrow[d, phantom, "="] & A' \arrow[dl, tail, "n"] \\ & A & \end{tikzcd} \quad \text{in } \mathcal{M}$$

hor. arrows: $[m] \xrightarrow{[m', f, n']} [n]$

given by

$$\begin{tikzcd} \cdot \arrow[dr, tail, "m"'] & \cong & \cdot \arrow[dl, tail, "m'"] \arrow[r, "f"] & \cdot \arrow[dr, tail, "n'"'] & \cong & \cdot \arrow[dl, tail, "n"] \\ & \cdot & & & \cdot & \end{tikzcd} \quad \text{in } \mathcal{E}$$

double cells:

$$\begin{tikzcd} {[m]} \arrow[r, "{[m,f,n]}"] \arrow[d] \arrow[dr, phantom, "\wedge"] & {[n]} \arrow[d] \\ {[m']} \arrow[r, "{[m',f',n']}"'] & {[n']} \end{tikzcd}$$

given by

$$\begin{tikzcd} & \cdot \arrow[dl, tail, "m"'] \arrow[r, "f"] \arrow[dd, tail] \arrow[ddr, phantom, "="] & \cdot \arrow[dr, tail, "n"] \arrow[dd, tail] & \\ \cdot & & & \cdot \\ & \cdot \arrow[ul, tail, "m'"] \arrow[r, "f'"'] & \cdot \arrow[ur, tail, "n'"'] & \end{tikzcd}$$

f in $\mathcal{E}$, f' in $\mathcal{E}$; m, m' in $\mathcal{M}$; n, n' in $\mathcal{M}$

Ehresmann topology on $E(\mathcal{C})$:

$\left([m_i] \to [m]\right)_{i \in I}$ s.t.

$$\begin{tikzcd} \cdot \arrow[rr, tail, "s_i"] \arrow[dr, tail, "m_i"'] & & \cdot \arrow[dl, tail, "m"] \\ & \cdot & \end{tikzcd} \qquad (s_i)_{i \in I}$$

is a $J_{\mathcal{C}}$-cover

From ~~Ordered Groupoids~~ gen. E. sites to ~~Left-Cancellative Categories~~ CM-sites

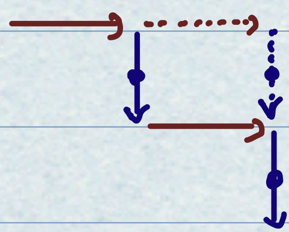
$C(\mathcal{P})$ - category of "corners" in $\mathcal{P}$: $\longleftarrow\!\!\shortmid$ $(\mathcal{P}, \mathcal{T})$ generalized E-site.

<u>objects</u>: those of $\mathcal{P}$

<u>arrows</u>: formal composites

$$\longrightarrow \;\; \bullet\!\!\longrightarrow$$

Composition:



<u>topology</u>: $(B_i \xrightarrow{f_i} A_i' \bullet\!\!\rightarrow A)_{i\in I}$ covers iff $(A_i' \bullet\!\!\rightarrow A)_{i\in I}$ covers in $\mathcal{T}$

<u>factorization system</u>:

- $\mathcal{E} = \{B \longrightarrow A = A\}$
- $\mathcal{M} = \{B \xrightarrow{\cong} A' \bullet\!\!\rightarrow A\}$

left quadrability from that of $\mathcal{P}$.

<u>Note</u>: We could also have defined a <u>strict</u> factorization system:
$\mathcal{E} = \{B \longrightarrow A = A\}$
$\mathcal{M} = \{B \xrightarrow{id} B \bullet\!\!\rightarrow A\}$

Sites for Grothendieck Topoi

left cancellative Gr. sites	C.M. Sites	atomic
étendues	Grothendieck Topoi	atomic topoi
Ehresmann Sites	generalized Ehresmann sites	horizontal generalized E.-Sites:

[Lawson. Steinberg]

[DeWolf - P]

left-quadrable ordered cat^s with an Ehresmann topology

The Correspondence – What we want ...

__Thm.__ •

$$\begin{tikzcd} \underline{\text{CM-Sites}} \arrow[rr, shift left, "E"] \arrow[dr, "Sh"'] & & \underline{\text{Gen.E.Sites}} \arrow[ll, shift left, "C"] \arrow[dl, "Sh"] \arrow[ll, phantom, "\perp"] \\ & \underline{\text{G.Topoi}} & \end{tikzcd}$$

is a 2-adjunction

•

$$\begin{tikzcd} \underline{\text{CM-Sites}} \arrow[rr, shift left, "E"] \arrow[dr, "Sh"'] & & \underline{\text{Gen.E.Sites}}_{\text{max}} \arrow[ll, shift left, "C"] \arrow[dl, "Sh"] \arrow[ll, phantom, "\perp"] \\ & \underline{\text{G.Topoi}} & \end{tikzcd}$$

is a bi-equivalence

__arrows:__

factorization-system preserving

Covering preserving, __Covering flat__ $\rightsquigarrow$ translate ...

The Correspondences we have ...

- $$\text{2-adj.}\qquad \text{Left-cancellative categories} \;\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}\; \text{Ordered groupoids}$$

- $$\text{Left-cancellative categories} \;\simeq\; (\text{Ordered Groupoids})_{max}$$
  bieq.

- $$\text{2-adj.}\qquad \underset{\text{with fact. syst. pres. functors}}{\text{CM-Cats}} \;\underset{\underset{\mathcal{G}}{\longleftarrow}}{\overset{\overset{E}{\longrightarrow}}{\perp}}\; \underset{\text{with all double functors}}{\text{Left Quadr. Ordered Categories}}$$

  $$\text{CM-cats} \;\simeq\; (\text{Left Quadr. Ord. Cats})_{max}$$
  bieq.

Morphisms between underlying (double) categories

$$\begin{tikzcd} \text{Left-cancellative categories} \arrow[r, shift left, "\text{2-adj.}"] \arrow[r, phantom, "\perp"] & \text{Ordered groupoids} \arrow[l, shift left] \end{tikzcd}$$

- Left-cancellative categories $\underset{\text{bieg.}}{\simeq}$ $(\text{Ordered Groupoids})_{max}$

$$\begin{tikzcd} \text{CM-Cats with fact. syst. pres. functors} \arrow[r, shift left, "E"] \arrow[r, phantom, "\perp"] & \text{Left Quadr. Ordered Categories with all double functors} \arrow[l, shift left, "\mathcal{G}"] \end{tikzcd}$$

2-adj.

CM-cats $\underset{\text{bieg.}}{\simeq}$ $(\text{Left Quadr. Ord. Cats})_{max}$

so "just" translate:

covering preserving, covering flat $\Longrightarrow$ covering pres., covering flat

Related Results on Fact^n Systems and Double Categories

- All double categories we consider are domain discrete as in "Factorization Systems and Double Categories" by M. Štěpán, who gives an equivalence

$$\underline{\text{Cats}} \text{ with SFS } \underset{\text{corners}}{\overset{\mathbb{D}}{\rightleftarrows}} \text{ Dom.Discr.Dbl.Cats}$$

- Our functor $\mathcal{C}$ is calculated by taking corners $\longrightarrow \downarrow$

  $\mathbb{D}$ is not $\mathbb{E}$ in general

- Hence, we obtain categories with a strict factorization system inside an OFS: SFS $\subseteq$ OFS.

- $\mathbb{D}$: $\underline{\text{Cats with SFS}} \longrightarrow \underline{\text{Dom.Discr.Dbl.Cats}}$

  keeps obj.^{s} the same, sends $\mathcal{L}$ to hor. arrows, $\mathcal{R}$ to vertical

Covering-Flat Morphisms

<u>Definition:</u> For a Grothendieck site $\mathfrak{a}$, a morphism $F : \mathfrak{b} \longrightarrow \mathfrak{a}$ is covering flat, if for any finite diagram

$$D : \mathcal{I} \longrightarrow \mathfrak{b}$$

and any cone T over $F \circ D$ in $\mathfrak{a}$, with vertex U, the sieve

$$\{ h : V \longrightarrow U \mid T \cdot h \text{ factors through the } F\text{-image of a cone over } D \}$$

is a covering sieve over U in $\mathfrak{a}$.

$$\begin{tikzcd}
 & W_k \arrow[dl, "\omega_i"'] \arrow[d, "\omega_j"] \arrow[dr, "\omega_k"] & \\
D_i \arrow[r, "D\alpha"'] & D_j & D_k
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & V \arrow[dll, dotted] \arrow[d, "h"] & \\
FW_k \arrow[d, "FW_i"'] \arrow[drr] & & U \arrow[dll, "\tau_i"'] \arrow[d, "\tau_j"] \arrow[dr, "\tau_k"] & \\
FD_i \arrow[rr, "FD\alpha"'] & & FD_j & FD_k
\end{tikzcd}$$

T

$T \cdot h$

Two Questions

- What are cones for the double category sites?
    - hv-cones
- What indexing (double) categories do we need?

hv-cones

- for an ordered category $\mathcal{G}$, and $D : \mathbb{I} \longrightarrow \mathcal{G}$, an hv-cone T over D with vertex U consists of:

$$\begin{tikzcd} U \arrow[r, "\tau_i"] & T_i \arrow[d, "\bullet" marking] \\ & D_i \end{tikzcd} \quad \text{for } i \in \mathrm{Obj}(\mathbb{I})$$

s.t. for $i \xrightarrow{\alpha} j$ in $\mathbb{I}$, and for $\begin{tikzcd} i' \arrow[d, "\bullet" marking] \\ i \end{tikzcd}$ in $\mathbb{I}$,

$$\begin{tikzcd} U \arrow[r, "\tau_i"'] \arrow[rr, bend left, "\tau_j"] & T_i \arrow[r, "D_\alpha | T_i"'] \arrow[d, "\bullet" marking] & T_j \arrow[d, "\bullet" marking] \\ & D_i \arrow[r, "D_\alpha"'] & D_j \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\tau_i' = \tau_i"'] & T_{i'} = T_i \arrow[d, "\bullet" marking] \\ & D_{i'} \arrow[d, "\bullet" marking] \\ & D_i \end{tikzcd}$$

Covering-Flat Morphisms

Definition: For a gener. E-site $\mathcal{P}$, a morphism $F: \mathcal{P}' \longrightarrow \mathcal{P}$ is covering flat, if for any $\boxed{\text{suitable finite diagram}}$

$$D : \mathbb{I} \longrightarrow \mathcal{P}'$$

and any hv-cone T over $F \circ D$ in $\mathcal{P}$, with vertex U, there is a family

$$\begin{tikzcd} \bar{V}_k \arrow[r, "h_k"'] & V_k \arrow[d, "\bullet"] \\ & U \end{tikzcd}$$

with $\{V_k \xrightarrow{\bullet} U\}_{k \in K}$ covering, s.t.

$T|_{V_k} \cdot h_k$ factors through the F-image of an hv-cone over D in $\mathcal{P}'$.

The Indexing Diagrams for the Cones

For left-cancellative sites

let $\mathcal{A}$, $\mathcal{B}$ left cancellative sites

$$\begin{tikzcd} \mathcal{I} \arrow[r, "J_D"] \arrow[d, "\forall D"'] & \tilde{\mathcal{I}} \arrow[dl, "\exists\, \tilde{D}"] \\ \mathcal{B} \arrow[r, "F"'] & \mathcal{A} \end{tikzcd}$$

- J_D is bijective on objects, full on arrows
- $\tilde{D}$ is fully faithful
- by choosing a representative for each subobject in $\tilde{\mathcal{I}}$ we can give it a strict fact^n system.

$\Longrightarrow$ $\tilde{\mathcal{I}}$ is left cancellative

$\Longrightarrow$ cones over D (resp. FD) are cones over $\tilde{D}$ (resp. $F\tilde{D}$)

$\therefore$ We may assume for left-cancellative sites that the diagrams in the definition of "covering-flat arrow" are left cancellative, and have a strict factorization system.

Indexing Categories for Diagrams in CM-sites

<u>Proposition:</u> For $\mathfrak{A}$ a category with an OFS $(\mathcal{E}, \mathcal{M})$, $\mathcal{M} \subseteq$ monic arrows, and $D : \mathcal{I} \longrightarrow \mathfrak{A}$ a finite diagram, there is a factorization

$$\begin{tikzcd} \mathcal{I} \arrow[rr, "J"] \arrow[dr, "D"'] & & \tilde{\mathcal{I}} \arrow[dl, "\tilde{D}"] \\ & \mathfrak{A} & \end{tikzcd}$$

with strict factorization $(\mathcal{L}, \mathcal{R})$ on $\tilde{\mathcal{I}}$, s.t.

- J is injective on objects
- $\tilde{D}(\mathcal{L}) \subseteq \mathcal{E}$, $\tilde{D}(\mathcal{R}) \subseteq \mathcal{M}$
- Cones over D extend uniquely to cones over $\tilde{D}$
- $\tilde{\mathcal{I}}$ is finite

if $\mathfrak{A}$ has an SFS $\subseteq$ OFS, we may take $\tilde{D}$ SFS-preserving.

Construction of $\tilde{\mathcal{I}}$:

objects: $\quad Obj(\mathcal{I}) \cup \{ \hat{f} \mid f \in Arr(\mathcal{I}) \}$

arrows: $\quad$ generated by:

* $A \xrightarrow{l_f} \hat{f} \xrightarrow{r_f} B \quad$ for $\quad A \xrightarrow{f} B$ in $\mathcal{I}$
* $\hat{f} \xrightarrow{\lambda_{g,f}} \widehat{gf} \xrightarrow{\rho_{g,f}} \hat{g} \quad$ for $\quad A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{I}$

Subject to:
- $r_{I_A} \circ l_{I_A} = I_A$, the empty string at A
- $\lambda_{I_B,f} = I_{\hat{f}} = \rho_{f,I_A} \quad$ for $f: A \to B$ in $\mathcal{I}$.

$$\begin{tikzcd}
 & & \widehat{gf} \arrow[dr, "\rho_{g,f}"] \arrow[drr, bend left, "r_{gf}"] & & \\
A \arrow[r, "l_f"'] \arrow[urr, bend left, "l_{gf}"] & \hat{f} \arrow[r, "r_f"'] \arrow[ur, "\lambda_{g,f}"] & B \arrow[r, "l_g"'] & \hat{g} \arrow[r, "r_g"'] & C
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \widehat{hgf} \arrow[dr, "\rho_{hg,f}"] \arrow[drr, "\rho_{h,gf}"] & & \\
\hat{f} \arrow[r, "\lambda_{g,f}"'] \arrow[urr, "\lambda_{hg,f}"] & \widehat{gf} \arrow[r, "\rho_{g,f}"'] \arrow[ur, "\lambda_{h,gf}"] & \hat{g} \arrow[r, "\lambda_{h,g}"'] & \widehat{hg} \arrow[r, "\rho_{h,g}"'] & \hat{h}
\end{tikzcd}$$

$\mathcal{L} = \{ l_f \} \cup \{ \lambda_{g,f} \} \cup \{ id \} \qquad \mathcal{R} = \{ r_f \} \cup \{ \rho_{g,f} \} \cup \{ id \}$

Indexing Double Categories

<u>Claim:</u> for double categorical covering-flatness between (generalized) Ehresmann sites, it is sufficient to use diagrams indexed by finite domain-discrete double categories.

$$\boxed{F \text{ covering flat} \Rightarrow \mathcal{G}(F) \text{ covering flat}}$$

For: $\tilde{\mathcal{I}} \xrightarrow{\tilde{D}} \big(\mathcal{G}(\mathcal{P}, \mathcal{T}_p), (\mathcal{H}, \mathcal{V}) \subseteq \mathcal{OFS}\big) \xrightarrow{\mathcal{G}(F)} \big(\mathcal{G}(\mathcal{P}', \mathcal{T}_{p'}), \ldots\big)$

$\Rightarrow \quad \mathcal{S}(\tilde{\mathcal{I}}) = \mathbb{I} \xrightarrow[\text{dbl functor}]{\delta} (\mathcal{P}, \mathcal{T}_p) \xrightarrow{F} (\mathcal{P}', \mathcal{T}_{p'})$

- F covering flat: each hv-cone $\overline{T}$ with vertex U over $F \circ \delta$ has an "hv-covering" $V_k \xrightarrow{\xi_k} U_k$ s.t. each hv-cone $T|_{U_k} \circ \xi_k$
$$\begin{array}{c} U_k \\ \downarrow \\ U \end{array}$$
factors through the F-image of an hv-cone over δ.
- these hv-cones over $F \circ \delta$ and δ (respectively) correspond precisely to cones over $\mathcal{G}(F) \circ \tilde{D}$ and $\tilde{D}$ (respectively).

$\therefore$ F covering-flat $\Rightarrow$ $\mathcal{G}(F)$ covering flat when F is cov. flat for diagrams indexed by finite dom. discr. dbl. cats

$$\boxed{\varphi \text{ covering flat} \Longrightarrow E(\varphi) \text{ covering flat}}$$

for $\mathbb{I} \xrightarrow[\delta]{} E(\mathfrak{a}, \mathcal{J}_{\mathfrak{a}}, (\mathcal{E}_{\mathfrak{a}}, \mathcal{M}_{\mathfrak{a}})) \xrightarrow[E(\varphi)]{} E(\mathfrak{b}, \mathcal{J}_{\mathfrak{b}}, (\mathcal{E}_{\mathfrak{b}}, \mathcal{M}_{\mathfrak{b}}))$ $(*)$

we get:

$$\mathrm{Corners}(\mathbb{I}) \xrightarrow[D]{} \mathcal{G}E(\mathfrak{a}, \mathcal{J}_{\mathfrak{a}}, (\mathcal{E}_{\mathfrak{a}}, \mathcal{M}_{\mathfrak{a}})) \xrightarrow[\mathcal{G}E(\varphi)]{} \mathcal{G}E(\mathfrak{b}, \mathcal{J}_{\mathfrak{b}}, \cdots)$$

objects:

$[X' \xrightarrow{x} X]$

for x in $\mathfrak{a}$

$\Rightarrow$ choose representatives

$$\Rightarrow \mathrm{Corners}(\mathbb{I}) =: \mathcal{I} \xrightarrow[D']{} (\mathfrak{a}, \mathcal{J}_{\mathfrak{a}}, (\mathcal{E}_{\mathfrak{a}}, \mathcal{M}_{\mathfrak{a}})) \xrightarrow[\varphi]{} (\mathfrak{b}, \mathcal{J}_{\mathfrak{b}}, (\mathcal{E}_{\mathfrak{b}}, \mathcal{M}_{\mathfrak{b}}))$$

and cones here correspond to hv-cones in $(*)$

$\therefore$ Suitable finite $\mathbb{I}$:= finite domain discrete double cat's $\mathbb{I}$

Covering Flat for CM-sites and Generalized Ehresmann Sites

- For CM-sites we can define covering-flatness of arrows in term of cones over diagrams with a strict factorization system.

- For generalized Ehresmann sites we can define covering-flatness of arrows in terms of hv-cones over diagrams with a finite double category $\mathbb{I}$ for which the source is a discrete fibration.

2-Adjunctions and Bi-equivalences

$$\underline{\text{lc Cats}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underline{\text{oGpds}}$$

$$\underline{\text{lc Cats}} \simeq (\underline{\text{oGpds}})_{max}$$

each component of the vertical category has a maximal object

$$\underline{\text{lc Sites}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underline{\text{E-Sites}}$$

$$\underline{\text{lc Sites}} \simeq (\underline{\text{E-Sites}})_{max}$$

morphisms: cov. pres. cov. flat pres. left class pres. right class

$$\underline{\text{CM sites}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underline{\text{gen E. Sites}}$$

$$\underline{\text{CM sites}} \simeq (\underline{\text{gen E. Sites}})_{max}$$

morphisms cov. preserving cov. flat in terms of fin. dbl. cats with src. a discr. fibration.

We can now translate the comparison lemma conditions

Comparison Lemma Maps

For Grothendieck sites	For Ehresmann Sites	For Generalized Ehresmann Sites
$F:(\mathcal{A},J_{\mathcal{A}})\longrightarrow(\mathcal{B},J_{\mathcal{B}})$	$\varphi:(\mathbb{E},\mathcal{T}_{\mathbb{E}})\longrightarrow(\mathbb{F},\mathcal{T}_{\mathbb{F}})$	$\varphi:(\mathbb{C},\mathcal{T}_{\mathbb{C}})\longrightarrow(\mathbb{D},\mathcal{T}_{\mathbb{D}})$

locally surjective on objects:

for each B in $\mathcal{B}$ there is a cover

$$\{FA_i \xrightarrow{f_i} B;\ i\in I\} \text{ in } J_{\mathcal{B}}$$

for each F in $\mathbb{F}$ there is a family

$$\{\varphi E_i \xrightarrow{h_i} F_i \xrightarrow{\bullet} F\}$$

such that $\{F_i \xrightarrow{\bullet} F\}\in\mathcal{T}_{\mathbb{F}}$

$$\begin{tikzcd}
\varphi E_{i'} \arrow[r, "h_{i'}"] & F_{i'} \arrow[dd, "\bullet" description] \\
\varphi E_i \arrow[r, "h_i"'] & F_i \arrow[u, dotted, no head] \arrow[d, "\bullet" description] \\
 & F
\end{tikzcd}$$

for each D in $\mathbb{D}$ there is a family

$$\{\varphi C_i \xrightarrow{h_i} D_i \xrightarrow{\bullet} D\}$$

such that $\{D_i \xrightarrow{\bullet} D\}\in\mathcal{T}_{\mathbb{D}}$

$$\begin{tikzcd}
\varphi C_{i'} \arrow[r, "h_{i'}"] & D_{i'} \arrow[dd, "\bullet" description] \\
\varphi C_i \arrow[r, "h_i"'] & D_i \arrow[u, dotted, no head] \arrow[d, "\bullet" description] \\
 & D
\end{tikzcd}$$

Comparison Lemma Maps

For Grothendieck sites

$F: (\mathfrak{A}, J_{\mathfrak{A}}) \longrightarrow (\mathfrak{B}, J_{\mathfrak{B}})$

for each pair of arrows $A \overset{f}{\underset{g}{\rightrightarrows}} A'$ in $\mathfrak{A}$

such that $F(f) = F(g)$

there is a cover

$\{A_i \xrightarrow{h_i} A;\ i \in I\}$ in $J_{\mathfrak{A}}$

such that $f h_i = g h_i$.

For Ehresmann Sites

$\varphi: (\mathbb{E}, \mathcal{T}_{\mathbb{E}}) \longrightarrow (\mathbb{F}, \mathcal{T}_{\mathbb{F}})$

locally faithful:

for each pair of horizontal arrows $E \xrightarrow{f} C_f$, $E \xrightarrow{g} C_g$

with $C_f \longrightarrow C \longleftarrow C_g$

such that $\varphi(f) = \varphi(g)$

there is a cover

$\{E_i \longrightarrow E\}$ in $\mathcal{T}_{\mathbb{E}}$

such that $f|_{E_i} = g|_{E_i}$.

$$
\begin{tikzcd}
E_i \arrow[rr] \arrow[d] \arrow[drr, phantom, "f|_{E_i} = g|_{E_i}"] & & D \arrow[d] \arrow[dd, bend right] \\
E \arrow[rr, "g"] \arrow[drr, "f"'] & & C_g \\
& & C_f
\end{tikzcd}
$$

For Generalized Ehresmann Sites

$\varphi: (\mathbb{C}, \mathcal{T}_{\mathbb{C}}) \longrightarrow (\mathbb{D}, \mathcal{T}_{\mathbb{D}})$

for each pair of horizontal arrows $B \xrightarrow[f]{} C_f$, $B \xrightarrow[g]{} C_g$

with $C_f \longrightarrow C \longleftarrow C_g$

such that $\varphi(f) = \varphi(g)$

there is a cover

$\{B_i \longrightarrow B\}$ in $\mathcal{T}_{\mathbb{C}}$

with horizontal arrows

$A_i \xrightarrow[h_i]{} B_i$ such that

$f|_{B_i} \circ h_i = g|_{B_i} \circ h_i$

$$
\begin{tikzcd}
A_i \arrow[r, "h_i"] & B_i \arrow[rr, shift left, "f|_{B_i}"] \arrow[rr, shift right, "g|_{B_i}"'] \arrow[d] & & D \arrow[d] \arrow[dd, bend right] \\
& E \arrow[rr, "g"] \arrow[drr, "f"'] & & C_g \\
& & & C_f
\end{tikzcd}
$$

Comparison Lemma Maps

For Grothendieck sites

$F: (\mathcal{A}, J_{\mathcal{A}}) \longrightarrow (\mathcal{B}, J_{\mathcal{B}})$

for every arrow

$F(A) \xrightarrow{h} F(A')$ in $\mathcal{B}$

there is a cover

$\{A_i \xrightarrow{a_i} A\}$ in $J_{\mathcal{A}}$

with arrows

$A_i \xrightarrow{f_i} A'$

such that

$h\, F(a_i) = F(f_i)$

$$\begin{tikzcd}
F(A_i) \arrow[r, "F(a_i)"] \arrow[rr, bend right, "Ff_i"'] & FA \arrow[r, "h"] & FA'
\end{tikzcd}$$

For Ehresmann Sites

$\varphi: (\mathbb{E}, T_{\mathbb{E}}) \longrightarrow (\mathbb{F}, T_{\mathbb{F}})$

locally full

for every diagram

$\varphi(E) \xrightarrow{h} F \overset{\bullet}{\longrightarrow} \varphi(D)$

in $\mathbb{F}$, there is a cover

$\{E_i \overset{\bullet}{\longrightarrow} E\}$ in $T_{\mathbb{E}}$

and a family

$\{E_i \xrightarrow{e_i} D_i\}$

such that

$$\begin{tikzcd}
\varphi E_i \arrow[r, "\varphi e_i"] \arrow[d, "\bullet"'] \arrow[dr, phantom, "\wedge"] & \varphi D_i \arrow[d, "\bullet"] \\
\varphi E \arrow[r, "h"'] & F \arrow[d, "\bullet"] \\
 & \varphi(D)
\end{tikzcd}$$

i.e. $h|_{\varphi E_i} = \varphi e_i$

For Generalized Ehresmann Sites

$\varphi: (\mathbb{C}, T_{\mathbb{C}}) \longrightarrow (\mathbb{D}, T_{\mathbb{D}})$

for every diagram

$\varphi C \xrightarrow{h} D \overset{\bullet}{\longrightarrow} \varphi E$ in $\mathbb{D}$

there is a cover

$\{C_i \overset{\bullet}{\longrightarrow} C\}$ in $T_{\mathbb{C}}$

and a family

$\{C_i' \xrightarrow{c_i} C_i\}$

with a family

$\{C_i' \xrightarrow{k_i} E_i\}$

such that

$$\begin{tikzcd}
\varphi C_i' \arrow[r, "\varphi c_i"'] \arrow[rr, bend left, "\varphi k_i"] & \varphi C_i \arrow[r, "h|_{\varphi C_i}"'] \arrow[d, "\bullet"] & \varphi E_i \arrow[d, "\bullet"] \\
 & \varphi C \arrow[r, "h"'] & D \arrow[d, "\bullet"] \\
 & & \varphi E
\end{tikzcd}$$

i.e. $h|_{\varphi C_i} \circ \varphi c_i = \varphi k_i$

Comparison Lemma Maps

For Grothendieck sites

$F : (\mathcal{A}, J_{\mathcal{A}}) \longrightarrow (\mathcal{B}, J_{\mathcal{B}})$

for each cover

$\{ \xi_i : B_i \longrightarrow FA \}$ in $J_{\mathcal{B}}$

the set of arrows

$f : A' \longrightarrow A$

s.t.

$$\exists \quad \begin{tikzcd} FA' \arrow[r, "g"] \arrow[dr, "Ff"'] & B_i \arrow[d, "\xi_i"] \\ & FA \end{tikzcd}$$

covers A

For Ehresmann Sites

$\varphi : (\mathbb{E}, T_{\mathbb{E}}) \longrightarrow (\mathbb{F}, T_{\mathbb{F}})$

Co-continuous

for each cover

$\{ F_i \xrightarrow{\bullet} \varphi E \}$ in $T_{\mathbb{F}}$

the set of arrows

$E' \xrightarrow{\bullet} E$

s.t.

$\varphi E' \xrightarrow{\bullet} F_i$

for some i,

covers E in $T_{\mathbb{E}}$

For Generalized Ehresmann Sites

$\varphi : (\mathbb{C}, T_{\mathbb{C}}) \longrightarrow (\mathbb{D}, T_{\mathbb{D}})$

as for Ehresmann sites

$+$

for each

$F \xrightarrow[h]{} \varphi E$

the set of arrows

$E' \xrightarrow{\bullet} E$

s.t.

$\exists \quad E'' \xrightarrow{k} E'$

and

$$\begin{tikzcd} \varphi E'' \arrow[r] \arrow[rr, bend left, "\varphi k"] & F' \arrow[r, "h|_{F'}"'] \arrow[d, "\bullet"] & \varphi E' \arrow[d, "\bullet"] \\ & F \arrow[r, "h"'] & \varphi E \end{tikzcd}$$

covers E.

So we have

①
$$\begin{tikzcd} \text{CM-Sites}[LC_S^{-1}] \arrow[rr, "\cong"] \arrow[dr, "\sim"'] & & \text{Gen. E.-Sites}[LC_E^{-1}] \arrow[dl, "\sim"] \\ & \text{Gr.Topoi} & \end{tikzcd}$$

and an observation:

② The factorization system $(\mathcal{E}, \mathcal{M})$ on the topos $\mathcal{G}$ itself gives us $\mathcal{G}$ as a wreath product of $\mathcal{E}$ and $\mathcal{M}$.

Etendues and Fractions

$$\mathrm{Gr.Site}[LC^{-1}] \overset{[RG]}{\simeq} \mathrm{Gr\,Topoi}^{coop} \qquad \mathrm{CM\,Site}[LC_{cm}^{-1}] \overset{[D.P,RG]}{\simeq} \mathrm{Gr\,Topoi}^{coop}$$

$$\mathrm{lc\,Sites} \xrightarrow{Sh} \mathrm{Etendues}^{coop}$$

↑ not enough for $LC = LC_{cm}$ to admit a calculus of fractions

Inspired by [Kock-Moerdijk]:

lc Sites		Ehresmann Sites
∩l		∩l
CM Sites where every morphism is locally monic	Etendues	Gen. Ehresmann Sites with locally monic hor. arrows
∩l	LC_{cm} admits a calculus of fractions	∩l
Torsion free generated CM sites		Torsion free generated gen. Ehresmann sites

Work in Progress

- Is the wreath product a topos theoretic wreath product?
- Consider further examples of CM sites (for manifolds, topological spaces, restriction categories, persistence diagrams).
- Consider larger (double) categories of sheaves with values in

  $\mathbb{S}\mathrm{pan}(\underline{\mathrm{Set}})$ or $\mathbb{R}\mathrm{el}(\underline{\mathrm{Set}})$

  $\longrightarrow$ double topoi
- Connections with groupoid representations for étendues/topoi.

Thank You!