Internalization of the Symplectic Form

Sam Winnick

University of Waterloo
Supervisors: Jon Yard; Michele Mosca

July 17, 2024

Overview

1. The Stone-von Neumann Theorem
2. Variants of the Heisenberg group
3. Symplectic forms, the symplectic group, and the projective Clifford group
4. Formulating the symplectic form category-theoretically
5. Some interactions between: sufficiency of (enriched) global elements, associativity, linear distributivity, nuclearity, etc.

The Stone-von Neumann Theorem

- The starting point for quantum mechanics is to consider the *canonical commutation relations*. These are encoded by the *Heisenberg group* G, of which there are multiple versions.
- The quotient $G/\mathrm{Z}(G)$, called the *phase space*, is abelian and has a structure called the *symplectic form*, which (if done right) remembers the structure of G, that is, the CCRs.
- The Stone-von Neumann-Mackey Theorem states that the Heisenberg group has a unique center-fixing unitary irrep [1, 2]. This ends up being a crucial tool in the theory. The representation space is the *Hilbert space*.
- Goal is to define a *kind* of category C such that one may reason about objects equipped with a symplectic form, and hopefully get the Stone-von Neumann-Mackey Theorem.

Symplectic forms concretely

A **symplectic form** on a free finitely generated R-module V is an alternating nondegenerate bilinear form $\omega : V \otimes V \to R$.

- **Nondegenerate** means $\lceil \omega \rceil : V \to V^*$ is an isomorphism.
- **Alternating** means $\omega(v, v) = 0$ for all $v \in V$.
- **Skew-symmetric** or **anti-symmetric** means $\omega(u, v) + \omega(v, u) = 0$ for all $u, v \in V$.
- Every alternating form is skew-symmetric, but the converse holds iff R has odd characteristic, complicating the even case.
- **Symplectic group**: $\mathrm{Sp}(V, \omega) = \{\phi \in \mathrm{GL}(V) \,|\, \phi^* \omega = \omega\}$.

Given $Q \subseteq V$, define $Q^\omega = \{v \in V \,|\, \omega(q, v) = 0 \,\forall\, q \in Q\}$. A subspace Q is **isotropic** if $Q \subseteq Q^\omega$, **coisotropic** if $Q^\omega \subseteq Q$, and **Lagrangian** if it is both, or equivalently, if it is maximally isotropic.

Same idea for LCA,[1] but $\omega : V \otimes V \to \mathrm{U}(1)$. In each version, $\mathbb{R}^{2n}$ gets a canonical symplectic form, but in LCA, $\mathbb{Z}^{2n}$ is Lagrangian.

---

[1] LCA = the category of locally compact (Hausdorff) abelian groups.

Examples of Heisenberg groups

Each **Heisenberg group** listed here is a central extension:

$$1 \longrightarrow \Lambda \longrightarrow G \longrightarrow V \longrightarrow 1$$

where the **phase space** $V \in \mathrm{ob}(C)$ is an abelian group/module/ vector space with symplectic form, but G is nonabelian.

$\Lambda = \mathrm{Z}(G)$	G	V	C
$\mathrm{U}(1)$	$\langle e^{i\theta}, X, Z\rangle$ [2]	$(\mathbb{Z}/d)^{2n}$	LCA
$\mathrm{U}(1)$	$\mathrm{Heis}(\mathbb{R}^{2n})$	$\mathbb{R}^{2n}$	LCA
comm. ring R	$\left\{\begin{pmatrix} 1 & q & \lambda \\ & 1 & p \\ & & 1 \end{pmatrix}\right\}$	R^{2n}	$\textsc{fgf}$-R-$\textsc{Mod}$
$\langle e^{2\pi i/d}\rangle \cong \mathbb{Z}/d$ (for d even or odd)	$\langle X, Z\rangle$ iso. to first one	$(\mathbb{Z}/d)^{2n}$	LCA or $\textsc{fgf}$-$\mathbb{Z}_d$-$\textsc{Mod}$
$\langle e^{\pi i/d}\rangle \cong \mathbb{Z}/2d$ (for d even only)	$\langle X, Y, Z\rangle$	$(\mathbb{Z}/d)^{2n}$	LCA or $\textsc{fgf}$-$\mathbb{Z}_d$-$\textsc{Mod}$...

[2] X, Y, Z, and I are the *qudit Pauli matrices*. Noncentral generators are n-fold Kronecker products of these.

The projective Clifford group

The n qu$d$it Pauli matrices are given on the *standard basis* of symbols $|r\rangle$ (for $r \in \mathbb{Z}/d$) for $\mathbb{C}[\mathbb{Z}/d] \cong \mathbb{C}^d$ by:

$$X\,|r\rangle = |r+1\rangle\,, \quad Z\,|r\rangle = \zeta^r\,|r\rangle\,, \quad Y = \sigma XZ, \quad \begin{matrix}\zeta = \text{a cplx } d\text{th root of } 1\\ \sigma = \text{a square root of } \zeta\end{matrix}$$

The groups $\mathrm{P}_{d,n} = \langle X, Z\rangle$ and $\mathrm{P}'_{d,n} = \langle X, Y, Z\rangle$ have orders d^{2n+1} resp. $d'd^{2n}$, where $d' = \left\{\begin{smallmatrix} 2d & : d \text{ even} \\ d & : d \text{ odd}\end{smallmatrix}\right\}$ is the order of σ.

By the S-vN-M Theorem, (which applies to the version with U(1) center,) the center-fixing automorphisms of $\mathrm{P}'_{d,n}$ precisely comprise the **projective Clifford group**:

$$\mathrm{PCl}'_{d,n} = \{[U] \in \mathrm{PU}(d^n) \mid UgU^\dagger \in \mathrm{P}'_{d,n} \, \forall\, g \in \mathrm{P}'_{d,n}\}$$

For the multiqubit case ($d = 2$), the "small version" $\mathrm{PCl}_{d,n}$ doesn't contain the (projectivized) S gate $\left(\begin{smallmatrix}1 & \\ & i\end{smallmatrix}\right)$. Elements of the "large version" $\mathrm{PCL}'_{d,n}$ are in bijection with pairs $(\mu, \psi) \in V^* \times \mathrm{Sp}(V)$.[3]

[3]Nontrivial. See [3, 4] or upcoming math paper with Jennifer Paykin.

Comparison with existing "categories + quantum" ideas

- **Dagger categories**. Recall that FDHILB is an example of a dagger category. The Hilbert space for a system of n qu$d$its is $\mathcal{H} = (\mathbb{C}^d)^{\otimes n} \cong \mathbb{C}^{d^n}$. This is the relevant object in FDHILB. In contrast, the phase space $V = (\mathbb{Z}/d)^{2n}$ lies in LCA or $C = \text{FGF-}\mathbb{Z}_d\text{-MOD}$. The Stone-von Neumann Theorem says there exists a unique projective unitary irrep of V:

$$\rho : V \to \mathsf{PU}(\mathcal{H})$$

  This may be viewed as a functor $\tilde{C} \to D$ where $\tilde{C} = \mathbf{B}V$ and $D =$ Hilbert spaces with projective unitaries (only); $\bullet \mapsto \mathcal{H}$.
- **Weinstein category**. This idea pertains to a symplectic manifold. In contrast, what we're looking at could be seen as looking at a single "tangent space", but the objects don't have to be smooth. Although viewing ρ (above) as a functor suggests suggests a multi-object version (oidification), this direction is beyond my current scope due to complications in *connecting* the tangent spaces (objects).

What properties and structure are needed on C?

To answer this, let's look at how certain aspects of the theory of finite dimensional symplectic vector spaces internalize.

Define nondegenreate bilinear form $\omega : V \otimes V \to \perp$ in any **LDC** $(C, \otimes, \parr)$. Now if $\mathrm{Sub}_C(V^{\otimes 2})$ has **colimits**,[4] define $I^2 V \rightarrowtail V^{\otimes 2}$:

$$(I^2 V \rightarrowtail V^{\otimes 2}) \cong \coprod_{x : \top \to V} x^{\otimes 2} \quad \text{in } \mathrm{Sub}_C(V^{\otimes 2}) = \mathrm{monos}(C) \downarrow V^{\otimes 2}$$

The motivation is that in $\mathbb{R}$-Vect, $(\bigwedge^2 V)^* = \{\text{alt forms}\}$, where:

$$I^2 V = \mathrm{span}\{x^{\otimes 2} \mid x \in V\} \qquad \text{and} \qquad \bigwedge^2 V \cong V^{\otimes 2} / I^2 V$$

So if C is **preabelian**, we may construct ω from a global element $\tilde{\omega} : \top \to (\bigwedge^2 V)^*$. Finally, the theory of f.d. symplectic vector spaces requires direct sums (**biproducts** $\oplus$). Since $\otimes$ is closed, $\oplus$ distributes over $\otimes$ (in the traditional sense). Anything else? (Yes.)

[4] I am brushing size issues under the rug here.

Enrichment and generator

- Sometimes need enrichment and/or sufficiency of global elts.
- Let C and K be categories and let $\top \in \mathrm{ob}(C)$.
- Suppose $\mathrm{el} : C \to K$ is full, and we have faithful functors $\mathrm{El} : C \to \mathrm{SET}$ and $U : K \to \mathrm{SET}$ where this commutes:

$$\begin{tikzcd} C \arrow[d, "\mathrm{el}"'] \arrow[dr, "\mathrm{El}"] & \\ K \arrow[r, "U"'] & \mathrm{SET} \end{tikzcd}$$

  Then el is (full and) faithful.
- **Def.** If C is K-enriched, let's say C is **determined by its global elements** if $\mathrm{el} = \hom_C(\top, -) : C \to K$ is ff.
- Concretely, the main task is to find suitable K; prove el is full.
- Abstractly, use the fact that ff functors reflect structure.
- Relevant examples: Each of FGF-R-MOD and LCA is enriched over itself, and el is chosen to be the *internal* hom.

Isotropy – example of internalization

Idea: before we had $Q^\omega = \{v \in V \mid q^\flat(v) = 0 \,\forall\, q \in Q\}$, where $q^\flat := \omega(q, -) : V \to \mathbb{R}$. Things work out nicely if we define:

$$\begin{tikzcd} U \arrow[r, "\lceil q^\flat \rceil"] \arrow[d, "q"'] & V^* \\ Q \arrow[r, "x"'] & V \arrow[u, "\lceil \omega \rceil"'] \end{tikzcd}$$

$\lceil - \rceil$ denotes "left currying"

$\lfloor - \rfloor$ denotes "right currying"

Writing $\cup$ and $\cap$ for the coproduct and product in $\mathrm{SUB}_C(V)$, let:

$$Q^\omega_U \overset{\mathrm{def}}{=\!=\!=} \bigcap_{q \in_U Q} \ker \lfloor q^\flat \rfloor \xrightarrow{\pi_q} \ker \lfloor q^\flat \rfloor \xrightarrow{k_q} V \xrightarrow{\lfloor q^\flat \rfloor} {}^*U$$

This lets us define $Q^\omega_{\mathrm{gen}} = \bigcup_{U \subseteq Q} Q^\omega_U$, and then $(-)^\omega_\top$ and $(-)^\omega_{\mathrm{gen}}$.

Proposition 1. If monos(C) is determined by its global elements, and $(-)^\omega$ is a contravariant endofunctor on $\mathrm{SUB}_C(V)$ with double negation translation (such as $(-)^\omega_\top$), then maximally isotropic subobjects are coisotropic.

Lax functors respecting “magmal” products $\otimes$ and $\boxtimes$

- Some properties (de Morgan duality, etc.) of $(-)^\omega$ and as well as negations can be understood in the following framework.
- Let C and D be categories, and consider an adjunction:

$$\begin{tikzcd} D \arrow[rr, bend right, "R"'] \arrow[rr, phantom, "\perp"] & & C \arrow[ll, bend right, "L"'] \end{tikzcd}$$

- Further let $\otimes : C \times C \to C$ and $\boxtimes : D \times D \to D$. There could be associators or not; units and unitors or not.
- In any case, L is oplax iff R is lax:

$$L(Y \otimes X) \to LX \boxtimes LY \qquad RX \otimes RY \to R(Y \boxtimes X)$$

- The same holds in $\textsc{OplaxMonCat}$ in place of $\textsc{Cat}$, and likewise for the other variants, by “doctrinal adjunction” [5] or seen directly. In this setting, ≥ 3 of 4 directions hold.

Negations

- Let C be a K-enriched category, and consider:

$$\otimes : C \times C \to C \qquad \top \in \mathrm{ob}(C) \qquad \text{with unitors}$$

We neither impose an associator nor a braiding.

- Further assume C is biclosed (in the K-enriched sense):

$$\hom_C(X \otimes Y, Z) \cong \hom_C(X, Y \multimap Z) \in \mathrm{ob}(K) \quad \text{left currying}$$
$$\hom_C(Y \otimes X, Z) \cong \hom_C(X, Z \multimapinv Y) \in \mathrm{ob}(K) \quad \text{right currying}$$

- Fix some $\perp \in \mathrm{ob}(C)$, and form **negations**:[5]

$$(-)^{\neg} = - \multimap \perp : C \to C^{\mathrm{op}} \qquad \text{left negation}$$
$$^{\neg}(-) = \perp \multimapinv - : C^{\mathrm{op}} \to C \qquad \text{right negation}$$

Then we get the **negation adjunction** $(-)^{\neg} \dashv {}^{\neg}(-)$.

[5] Caution: $(-)^{\omega}_{\mathrm{gen}}$ and $(-)^{\omega}_{\top}$ are not negations on $C \downarrow V$. We are now thinking about the original category C (or a slight generalization).

Reflexivity viewed as an adjoint equivalence

- $(C, \otimes)$ is **left reflexive** if the counit of negation is an iso $X \xrightarrow{\sim} \neg(X^\neg)$. Thus, $(C, \otimes)$ is left reflexive iff $(-)^\neg : C \rightarrowtail C^{\mathrm{op}}$ is an embedding; **reflexive** iff $(-)^\neg \dashv \neg(-)$ is an equivalence.
- The counit is double negation translation. Caution: its **opposite** is the unit, whereas its **inverse** (if it exists) is double negation elimination.
- Intermediately, negation may be an **idempotent adjunction** (generalizing the Triple Dual Problem [6]):

$$(-)^\neg \eta_X : X^\neg \xrightarrow{\sim} (\neg(X^\neg))^\neg$$

  Example: C is a model of intuitionistic logic ($\otimes = \wedge$, $\boxtimes = \vee$).
- Negation becomes an equivalence $C \simeq C^{\mathrm{co\ op}}$ of monoidal categories with strong monoidal functors in the presence of strong de Morgan duality...

Nuclearity and the de Morgan laws (without distribution)

- Given C, $\otimes$, $\top$, $\bot$ as before, give C^{op} a "unital magmal structure" $\boxtimes : C^{\mathrm{op}} \times C^{\mathrm{op}} \to C^{\mathrm{op}}$ with unit $\bot$.
  We do not require $\otimes$ and $\boxtimes$ to distribute or linearly distribute.
- Call $(C, \otimes, \boxtimes)$ **left nuclear** if it is equipped[6] with a natural isomorphism of the following form (resp. *right* version):

$$Y^{\neg} \boxtimes X^{\neg} \cong X \multimap^{\mathrm{op}} Y^{\neg} \qquad \text{left nuclearizer}$$

$$^{\neg}Y \multimapinv X \cong {}^{\neg}X \boxtimes^{\mathrm{op}} {}^{\neg}Y \qquad \text{right nuclearizer}$$

- There is also a **strong** version: $Y \boxtimes X^{\neg} \cong X \multimap^{\mathrm{op}} Y$. (left)
- $(C, \otimes, \boxtimes)$ has **left (strong) de Morgan law** if $(-)^{\neg}$ is strong, *i.e.* we keep track of a natural isomorphism:

$$(X \otimes Y)^{\neg} \cong Y^{\neg} \boxtimes X^{\neg}$$

The right strong de Morgan duality uses right negations.

[6] Canonical nuclearizers should be used for $*$-autonomous categories [7, 8].

Associativity

- Let $(C, \otimes, \top)$ be a biclosed unital magmal category.
- **Prop 2.** If any (hence all) of the following natural isomorphisms exist, they canonically determine each other:

$$\begin{aligned}
\text{associate}_{XYZ} &: (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \\
\text{internal left curry}_{XYZ} &: (X \otimes Y) \multimap Z \xrightarrow{\sim} X \multimap (Y \multimap Z) \\
\text{internal right curry}_{XYZ} &: Z \multimapinv (Y \otimes X) \xrightarrow{\sim} (Z \multimapinv Y) \multimapinv X \\
\text{lollipop}_{XYZ} &: (X \multimap Z) \multimapinv^{\mathrm{op}} Y \xrightarrow{\sim} X \multimap (Z \multimapinv^{\mathrm{op}} Y)
\end{aligned}$$

- **Prop 3.** Substituting $\perp$ for Z, one may show that if C has an associative structure (above), then a left nuclear structure equivalently defines a de Morgan duality structure.
- **Prop 4.** If C has an associative structure, then a natural isomorphism $Y \cong {}^{\neg}(Y^{\neg})$ equivalently defines a natural isomorphism $X \multimap Y \cong X^{\neg} \multimapinv Y^{\neg}$.
- **Prop 5.** Associativity also enables internal composition, (which appears in the inverse formula for the nuclearizer [8]).

Nuclearity, reflexivity, and the de Morgan laws

- **Prop 6.** If $(C, \otimes)$ is left reflexive, then $\otimes$ has a left bd de Morgan dual $\boxtimes$, which is unique up to iso. Moreover, $\otimes$ is the right bd de Morgan dual of $\boxtimes$.

- **Prop 7.** If C is determined by its global elements as well as by its global co-elements, then $\otimes$ is associative. If it is also left reflexive, then it has a left nuclear structure.

- **Def.** An **LDC** consists of linearly distributing structures $\otimes$ and $⅋$ on a category C, with units $\top$ and $\bot$, respectively. A **$*$-autonomous category** is an LDC where $\otimes$ is biclosed, $⅋$ is bicoclosed, and where each object X has left and right linear adjoints $X^* \dashv X \dashv {}^*X$ in the linear delooping bicat. $\mathbf{B}C$ [9].

- **Thm.** In an LDC, left nuclearity implies left reflexivity [**?**]...

- **Q1:** Is a reflexive biclosed bicoclosed (non-assoc.) LDC always (non-assoc.) $*$-autonomous? **Q2:** Always associative?

References I

David Mumford.
Tata Lectures on Theta III.
Birkhäuser Boston, Boston, January 2006.

Keeley Hoek.
The Stone–von Neumann–Mackey theorem.

D. M. Appleby.
SIC-POVMs and the Extended Clifford Group.
Journal of Mathematical Physics, 46(5):052107, May 2005.

Marcus Appleby, Steven Flammia, Gary McConnell, and Jon Yard.
SICs and Algebraic Number Theory.
Foundations of Physics, 47(8):1042–1059, August 2017.

G. M. Kelly.
Doctrinal adjunction.
In Gregory M. Kelly, editor, Category Seminar, pages 257–280, Berlin, Heidelberg, 1974. Springer.

References II

G. M. Kelly and S. Maclane.
Coherence in closed categories.
Journal of Pure and Applied Algebra, 1(1):97–140, January 1971.

J R B Cockett.
PROOF THEORY FOR FULL INTUITIONISTIC LINEAR LOGIC, BILINEAR LOGIC, AND MIX CATEGORIES.
Theory and Applications of Categories, 3(5).

D.A. Higgs and K.A. Rowe.
Nuclearity in the category of complete semilattices.
Journal of Pure and Applied Algebra, 57(1):67–78, February 1989.

J. R. B. Cockett, J. Koslowski, and R. A. G. Seely.
Introduction to linear bicategories.
Mathematical Structures in Computer Science, 10(2):165–203, April 2000.

R. F. Blute, J. R. B. Cockett, R. A. G. Seely, and T. H. Trimble.
Natural deduction and coherence for weakly distributive categories.
Journal of Pure and Applied Algebra, 113(3):229–296, December 1996.