The language of a model category

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Outline

Main results

- Given a Quillen model category we build a notion of language from it.
- There is a syntactic and a categorical approach.

The formulas are “invariant”:

Theorem (First invariance)

Homotopic maps satisfy the same of formulas.

Theorem (Second invariance)

Homotopically equivalent objects validate the same formulas.

The language itself is invariant:

Theorem (Third invariance)

Two Quillen model categories that are Quillen equivalent have equivalent languages.

This is inspired from Makkai’s FOLDS, but the connection is subtle.

Motivating result

Theorem (Blanc-Freyd)

An elementary property on categories is invariant under equivalence of categories if and only if it is a diagrammatic property.

Take the following sentence of categories in context x, y, w objects and $f \in \hom(x, y), k, l \in \hom(y, w)$:

$$Eq_{k,l}(f) := kf = lf, \forall z \in \mathrm{Ob}, \forall g \in \hom(z, y), kg = lg$$

$$\exists h \in \hom(z, x), fh = g, \forall h \in \hom(z, x), fh' = g, h = h'$$

These are the kind of sentences for which the theorem applies.

Key point: Such sentences do not include the equality between objects.

Outline

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Generalized algebraic theories

- They were introduce by Cartmell to encode dependency.
- We will use an infinitary version.
- Roughly, a generalized algebraic theory T consists of:
  - A collection S of sorts.
  - A collection O of operations.

The important point, we get **contexts**:

$$x_1 : \Delta_1, x_2 : \Delta_2(x_1), \cdots, x_\alpha : \Delta_\alpha(x_1, x_2, \cdots).$$

Example

The theory of categories:

1. Type of objects: $\vdash \mathrm{Ob}\,\mathrm{Type}$.
2. Type of morphisms: $x : \mathrm{Ob},\ y : \mathrm{Ob} \vdash \mathrm{Hom}(x, y)\,\mathrm{Type}$.
3. Plus composition and axioms.

Contextual categories

These were also introduced by Cartmell. A category is called **contextual** if:

- It is equipped with a class of maps, called **display**.
- Display maps pullback stable, and such pullbacks are strict (substitution).

Additionally:

- Limits of transfinite chains of display maps exists.

Given an generalized algebraic theory T, one can associate a contextual category $\mathbb{C}_T$:

- Objects are the contexts.
- Display maps are of the form

$$\begin{array}{c} \Gamma.x : \Delta \\ \big\downarrow\!\!\!\downarrow \\ \Gamma \end{array}$$

Clans

We extend Joyal's definition. A category is a **clan** if:

- It is equipped with a class of maps, called **fibrations**.
- Every object is fibrant.
- Fibrations are pullback stable.

Additionally:

- Limits of transfinite chains of fibrations exists.

$\mathcal{C}$ is a **coclan** if $\mathcal{C}^{\mathrm{op}}$ is a clan.

The contextual category $\mathbb{C}_T$ has a clan structure where fibrations are the display maps.

Models of a clan are functors $\mathcal{C} \to \mathbf{Set}$ which preserve the structure of $\mathcal{C}$.

Via the Yoneda embedding, for each $A \in \mathcal{C}$ we get a model $\text{よ}_A : \mathcal{C} \to \mathbf{Set}$.

Model categories

A cateogry $\mathcal{C}$ is said to be a **Quillen model category** if is complete and complete with three classes of maps; **cofibrations** $\textsc{Cof}(\mathcal{C})$, **fibrations** $\textsc{Fib}(\mathcal{C})$ and **weak equivalences** $\mathcal{W}$, such that:

1. $\mathcal{W}$ contains all isomorphisms, it is closed under compositions. If f, g are maps such that any two of f, g, gf is in $\mathcal{W}$, so the third.
2. $(\textsc{Cof}(\mathcal{C}) \cap \mathcal{W}, \textsc{Fib}(\mathcal{C}))$ and $(\textsc{Cof}(\mathcal{C}), \textsc{Fib}(\mathcal{C}) \cap \mathcal{W})$ are weak factorization systems *i.e.*

   2.1 Every map $f : a \to b$ factors as

   $$a \xrightarrow{\in\mathcal{L}} \bullet \xrightarrow{\in\mathcal{R}} b$$

   2.2 Lifting problem solutions:

   $$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ {\scriptstyle \mathcal{L}\ni}\big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow{\scriptstyle \in\mathcal{R}} \\ \bullet & \longrightarrow & \bullet \end{array}$$

Outline

The language: syntactic definition

Given a generalized algebraic theory T. The set $\mathcal{L}^T(\Gamma)$ of T-formulas in context Γ is defined inductively:

1. For each context Γ, the true formula $\top$ and false formula $\bot$ are in $\mathcal{L}^T(\Gamma)$.
2. If $\Phi \in \mathcal{L}^T(\Gamma)$ then $\neg\Phi \in \mathcal{L}^T(\Gamma)$.
3. The conjunction and disjunction

$$\bigvee_{i \in I} \Phi_i \qquad \bigwedge_{i \in I} \Phi_i$$

   are in $\mathcal{L}^T(\Gamma)$, where each $\Phi_i \in \mathcal{L}^T(\Gamma)$

4. For any context extension $\Gamma' := \Gamma, x_1 : \Delta_1, \cdots, x_\lambda : \Delta_\lambda$ and any formula $\Phi \in \mathcal{L}^T(\Gamma')$ we have formulas

$$\exists x_1, \cdots, x_\lambda \Phi \qquad \forall x_1, \cdots, x_\lambda \Phi$$

   in $\mathcal{L}^T(\Gamma)$.

The language: categorical approach

Given $\mathcal{C}$ a clan, a **boolean algebra over** $\mathcal{C}$ is a functor

$$\mathcal{B} : \mathcal{C}^{op} \to \textbf{comBOOL}$$

such that:

1. For each fibration $\pi : Z \twoheadrightarrow X$ in $\mathcal{C}$, $\pi^* : \mathcal{B}(X) \to \mathcal{B}(Z)$ has a left adjoint $\exists_\pi : \mathcal{B}(Z) \to \mathcal{B}(X)$.
2. The Beck-Chevalley condition holds:

$$\begin{array}{ccc} Z' & \xrightarrow{f'} & Z \\ {\scriptstyle \pi'} \downarrow & \lrcorner & \downarrow {\scriptstyle \pi} \\ X' & \xrightarrow[f]{} & X \end{array}$$

   we have $f^* \exists_\pi = \exists_{\pi'} f'^*$.

Examples

These examples will be relevant later on:

- The contravariant power-set functor $\mathcal{P} : \mathbf{Set}^{op} \to \mathbf{Bool}$ is a boolean algebra over **Set**.

- Given $F : \mathcal{C} \to \mathcal{D}$ a morphism of clans, if $\mathcal{B}$ is a boolean algebra over $\mathcal{D}$, then $F^*\mathcal{B}$ defined by $F^*\mathcal{B}(\Gamma) = \mathcal{B}(F(\Gamma))$ is a boolean algebra over $\mathcal{C}$.

- Given a model M of a clan $\mathcal{C}$, i.e. $M : \mathcal{C} \to \mathbf{Set}$, one has a boolean algebra $\mathcal{P}(M)$ over $\mathcal{C}$:

$$\begin{array}{rccc} \mathcal{P}(M) : & \mathcal{C}^{\mathrm{op}} & \to & \mathbf{Set} \\ & \Gamma & \mapsto & \mathcal{P}(M(\Gamma)) \end{array}$$

Language of a clan

The **language** of a clan $\mathcal{C}$ is the initial boolean algebra over $\mathcal{C}$, $\mathbb{L}^{\mathcal{C}}$, it always exists but it is large.

For a generalized algebraic theory $\mathbb{L}^{\mathbb{C}_T}$ can be obtained as a quotient from $\mathcal{L}^T$.

By initiality, there exists a unique morphism of boolean algebras over $\mathcal{C}$:

$$|-|_M : \mathbb{L}^{\mathcal{C}} \to \mathcal{P}(M).$$

This morphism associates to each formula ϕ in context Γ, a subset $|\phi|_M \subseteq M(\Gamma)$.

An interpretation $x \in M(\Gamma)$ is said to **satisfy** ϕ if $x \in |\phi|_M$. We write

$$M \vdash \phi(x).$$

Outline

The language of a model category

Given $\mathcal{M}$ a Quillen model category, the category $\mathcal{M}^{\mathrm{Cof}}$ of cofibrant objects with cofibration between them forms a coclan.

The language of $\mathcal{M}$ is the language of the clan $(\mathcal{M}^{\mathrm{Cof}})^{\mathrm{op}}$. We denote it by $\mathbb{L}^{\mathcal{M}}$.

If $X \in \mathcal{M}$ then we can define the following model of $(\mathcal{M}^{\mathrm{Cof}})^{\mathrm{op}}$

$$よ_X : \begin{array}{ccc} (\mathcal{M}^{\mathrm{Cof}})^{\mathrm{op}} & \to & \mathbf{Set} \\ \Gamma & \mapsto & \mathrm{Hom}(\Gamma, X) \end{array}$$

For $\Gamma \in \mathcal{M}$ a cofibrant, and $X \in \mathcal{M}$ any object, $v : \Gamma \to X$ and $\phi \in \mathbb{L}^{\mathcal{M}}(\Gamma)$ we write

$$X \vdash \phi(v)$$

to mean

$$よ_X \vdash \phi(v)$$

Outline

Invariance theorems

Theorem (B-Henry)

Let $\mathcal{M}$ be a Quillen model category, $\Gamma \in \mathcal{M}$ a cofibrant object, X, Y two fibrant objects and $\phi \in \mathbb{L}^{\mathcal{M}}(\Gamma)$ then:

- **First invariance theorem:** *Let $v_1, v_2 : \Gamma \to X$ be two homotopically equivalent maps with X fibrant. Then*

$$X \vdash \phi(v_1) \quad \Leftrightarrow \quad X \vdash \phi(v_2)$$

- **Second invariance theorem:** *Let $f : X \to Y$ be a weak equivalence between two fibrant objects and $v : \Gamma \to X$ any map then*

$$X \vdash \phi(v) \quad \Leftrightarrow \quad Y \vdash \phi(fv)$$

For the third invariance theorem we need to introduce a "semantical" equivalence relation on formulas.

Equivalence relation on formulas

Let Γ be a cofibrant object of $\mathcal{M}$.

- Two formulas $\phi, \psi \in \mathbb{L}^{\mathcal{M}}(\Gamma)$ are said to be **semantically equivalent** if for all fibrant objects $X \in \mathcal{M}$ we have $|\phi|_X = |\psi|_X$.
  This defines a relation equivalence $\phi \approx \psi$ on $\mathbb{L}^{\mathcal{M}}(\Gamma)$.
- We get boolean algebra over the homotopy category

$$h\mathbb{L}^{\mathcal{M}} : Ho(\mathcal{M}) \to \mathbf{comBOOL}$$

- For a functor $F : \mathcal{M} \to \mathcal{N}$ between Quillen model categories. There is a map of boolean algebras

$$h\mathbb{L}F_\Gamma : h\mathbb{L}^{\mathcal{M}}(\Gamma) \to h\mathbb{L}^{\mathcal{N}}(F\Gamma)$$

Third invariance

Theorem (B.-Henry)

Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ a Quillen equivalence. Then for any cofibrant object $A \in \mathcal{M}$. The induced map $h\mathbb{L}F_A : h\mathbb{L}^{\mathcal{M}}(A) \to h\mathbb{L}^{\mathcal{N}}(FA)$ is an isomorphism.

Proof.

1. The result holds for functors $F : \mathcal{M} \to \mathcal{N}$ which lift cofibrations up to isomorphism: for any A cofibrant object of $\mathcal{M}$, Y cofibrant in $\mathcal{N}$ and $f : FA \hookrightarrow Y$ cofibration in $\mathcal{N}$, there is a cofibration $g : A \hookrightarrow X$ in $\mathcal{M}$ projects onto f.
2. Any left Quillen equivalence admits a Brown factorization, therefore we get a span $\mathcal{M} \leftarrow \mathcal{P} \to \mathcal{N}$. Where each leg satisfies the property from step 1.

$\square$

Note: For the second step we need *weak model categories* (Henry).

Outline

Examples

- The language associated to the canonical model structure on **Cat** coincides with the one defined by Freyd and Blanc. In particular, it allows to talk about equality between morphisms but not between objects.
- This language extends straighforwardly to bicategories or 2-categories. We are allowed to speak about equality between 2-cells, but not between 1- or 0-cells.
- The language associated to the projective model structure on chain complexes allows to talk about chains with a specified boundary. There is no equality between chains x, x', we only say $\partial y = x - x'$.
- In the language of the Joyal or the Kan-Quillen model on simplicial sets we can talk about simplicies which satisfy some boundary condition. The point that differenciate them is the fibrant objects in each model structure. Equality is similar to 3.

Thank you!

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