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Products, Joins, Meets, and Ranges in Restriction Categories

by

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Abstract

Restriction categories provide a convenient abstract formulation of partial functions. However, restriction categories can have a variety of structures such as finite partial products (cartesianess), joins, meets, and ranges which are of interest in computability theory, semigroup theory, topology, and algebraic geometry. This thesis studies these structures.

For finite partial products (cartesianess), a construction to add finite partial products to an arbitrary restriction category freely is provided.

For joins, we introduce the notion of join restriction categories, describe a construction for the join completion of a restriction category, and show the completeness of join restriction categories in partial map categories using $\mathcal{M}$-adhesive categories and $\mathcal{M}$-gaps. As the join completion for inverse semigroups is well-known in semigroup theory, we show the relationships between the join completion for restriction categories and the join completion for inverse semigroups by providing adjunctions among restriction categories, join restriction categories, inverse categories, and join inverse categories.

For meets, we introduce the notion of meet restriction categories, show the completeness of meet restriction categories in partial map categories whose $\mathcal{M}$-maps include the regular monics, and provide a meet completion for restriction categories and discuss its connections with the meet completion for inverse semigroups.

Finally, for ranges, Schein's representation theorem for a certain class of semigroups (called type 3 function systems) is generalized to range categories and when a partial map category satisfies Schein's condition ([**RR**.**6**]) that guarantees each map is an epimorphism onto its range is studied.

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Dedication

To my wife Julie and my daughter Jenny

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Chapter 1

Introduction

1.1 Background

In 1987, Di Paolo and Heller [17] introduced *dominical categories* as an abstract setting in which to study recursion theory. They used zero morphisms and near products to abstract the notion of *partiality* of partial functions and showed that the basic results of recursion theory could be obtained from these simple assumptions and the presence of a Turing object.

In 1988, Robinson and Rosolini [37] noticed that the zero structure was not necessary for obtaining a notion of partiality and introduced the notion of P*-categories* (categories with a near product structure) as the basis for a more general theory of partiality. All P-categories are essentially the same as Cockett's *copy categories* [5].

In 2002, Cockett and Lack [14] introduced *restriction categories* as an even more general and more convenient framework for working with abstract categories of partial maps. In a restriction category, the notion of partiality is captured abstractly by a single combinator $\overline{(\ )}$ and four restriction axioms ([**R.1**], [**R.2**], [**R.3**], and [**R.4**] in Section 1.6.1 below). As claimed in [14], "the intuition for the combinator $\overline{f}$ is provided by thinking of the maps as programs: the restriction combinator modifies a program so that, rather than returning its output, it returns its input unchanged when it terminates." Dominical categories, P-categories, and copy categories are all restriction categories.

In 2006, Blute, Cockett, and Seely [1] introduced the notion of a differential category to provide a basic axiomatization for differential operators in monoidal categories. In 2009, they [2] introduced the notion of a cartesian differential category to directly axiomatize differentiable maps and thus to move the emphasis from the linear notion to the cartesian and classical notion. In [8], Cockett, Cruttwell, and Gallagher introduced differential restriction categories that combined restriction categories and cartesian differential categories to axiomatize categories like the smooth maps defined on open subsets of $\mathcal{R}^n$ in a way that is completely algebraic.

In [35], Moggi studied formal systems for reasoning about partial functions in the setting of first order logic and the lambda calculus with the particular emphasis on the partial lambda calculus. In 2005, Cockett and Hofstra [11] studied the theory of partial combinatory algebras, models of the partial lambda calculus in restriction categories, and proved the Scott-Koymans Theorem [27] linking reflexive objects to lambda algebras.

In [12], Cockett and Hofstra developed a convenient setting for the categorical study of abstract notions of computability. The key concept is *Turing categories*: *cartesian restriction categories* with a universal object, called a *Turing object.* They illustrated how a Turing category is a meeting point for various other areas in logic and computation and gave a detailed exposition of the connection between Turing categories and *partial combinatory algebras* (PCA). Furthermore, Cockett and Hofstra [13] investigated the notion of *simulations* between restriction functors over a fixed base restriction category and showed that the category of Turing categories over a fixed base and simulations between them is 2-equivalent to the category of relative PCAs in the base. A recursion category [6] is a Turing category which has both *joins* and *meets.*

In [10], Cockett, Guo, and Hofstra introduced range categories to begin a systematic study of partial map categories in which both the domain and the range of each map are axiomatized. A range category is a restriction category in which, in addition, there is a range combinator which satisfies four axioms.

Restriction categories not only provide a convenient setting for abstract computability but also have applications in other mathematical areas, such as semigroup theory, topology, algebraic geometry. See, for example, [34], [1, 2], [8], [31, 32], and [24, 43]. Structures in restriction categories, such as partial (restriction) products, joins, meets, and ranges frequently occur in these areas: this thesis is a study of these structures.

1.2 Objectives

In [16], Cockett and Lack observed that cartesian objects in the 2-category **rCat** of restriction categories, restriction functors, and restriction natural transformations are not the right notion of partial (restriction) products for restriction categories. Instead, the cartesian objects in the 2-category **rCatl** of restriction categories, restriction functors, and *lax* restriction natural transformations give the appropriate notion. Cockett-Hofstra's Turing categories are, for example, based on cartesian restriction categories (restriction categories that have finite partial products). Thus partial products are an important structural feature of restriction categories. In this thesis, we start by giving a free construction for adding partial products to an arbitrary restriction category.

Restriction categories are poset enriched: the natural partial order enrichment is given by $f \leq g \Leftrightarrow f = g\overline{f}$ (see Lemma 1.6.3 below). With the enrichment in posets one may wonder about the least upper bound (join) and the greatest lower bound (meet) for each pair of objects. A join restriction category is essentially a restriction category which is enriched in sup-lattices rather than a partial order. We describe the free join completion for restriction categories. The join completion of inverse semigroups is well known: as each inverse semigroup can be viewed as a restriction category with one object, we compare the two constructions by providing adjunctions between restriction categories, join restriction categories, inverse categories, and join inverse categories.

Since the partial map category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ of an $\mathcal{M}$-category is a restriction category (Proposition 1.6.17 below), it is natural to ask when a partial map category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$

is a join restriction category. We answer this question completely using $\mathcal{M}$-adhesive categories and $\mathcal{M}$-gaps. It is somewhat surprising that the notion of joins in partial map categories is related to adhesivity: Lack and Sobocińsk [28, 29] introduced adhesivity in order to provide a general setting in which double-pushout (DPO) rewriting could be performed. $\mathcal{M}$-adhesivity is, however, weaker than the adhesivity of Lack and Sobocińsk.

After discussing joins in restriction categories, we turn to meets and define meet restriction categories. We provide a completeness theorem for meet restriction categories in partial map categories and a construction of the free meet completion of a restriction category.

In [10], we examined categories of partial maps in which not only is the domain of the partial map abstractly defined but also the image of the partial map. This occurs frequently in practice: for example, in partial recursive functions, enumerable sets can be described not only as the domains of partial recursive functions but also as their images. We call restriction categories in which images are defined *range categories* and they require, in addition to the restriction combinator, another combinator called the *range* combinator which satisfies just four axioms: [**RR.1**], [**RR.2**], [**RR.3**], and [**RR.4**] (see Chapter 5 below). Range categories with split restrictions are essentially partial map categories of a category with a system of monics which are the $\mathcal{M}$-maps of an $\mathcal{M}$-stable factorization system (Theorem 4.5 [10]). In [43] (see also [24]), Schein embedded a certain class of semigroups (which he called type 3 function systems) faithfully in the partial function category. Every type 3 function system is an example of a range category with only one object. In Chapter 5 we generalize Schein's representation theorem to range categories.

1.3 Contributions

The main contributions of this thesis include:

1. The partial product completion for restriction categories so that a partial product can be added to a given restriction category freely.
2. The join completion for inverse categories using compatibility is described and its relationship to the join completion for restriction categories is provided.
3. The completeness of join restriction categories in partial map categories is proved using $\mathcal{M}$-adhesive categories and $\mathcal{M}$-gaps.
4. The completeness of meet restriction categories in partial map categories using equalizers and the meet completion for restriction categories using parallel map pairs.
5. The generalization of Schein's representation theorem to range categories and reasons for the condition [**RR.6**] to be required.

1.4 Outline

Before actually presenting the results of this thesis, we first review in Chapter 1 some categorical notions (Section 1.5) and restriction category basics (Section 1.6) that are used in the thesis.

In Chapter 2, we first observe some basic properties of cartesian objects in restriction categories (Section 2.1). Then we construct a free cartesian restriction category over a given restriction category (Section 2.2).

We start Chapter 3 by introducing the notions of $\smile$-compatibility and join restriction categories and showing the basic relation to partial map categories by describing the very elegant construction of the free join restriction category of a restriction category, called the join completion for restriction categories (Section 3.1). The join completion for inverse semigroups is well known, where $\smile$-compatibility (that is $\sim$-compatibility in semigroup theory) is used. We show relationships between the join completion for restriction categories and the join completion for inverse semigroups by providing adjunctions among restriction categories, join restriction categories, inverse categories, and join inverse categories.

Next we discuss, starting with Van Kampen squares, some general properties of Van Kampen colimits. This facilitates the definition of an $\mathcal{M}$-adhesive category and $\mathcal{M}$-gaps and allows us to establish the completeness theorem for join restriction categories (Section 3.2).

Then, for the completeness of join restriction categories, we show the main result of this chapter that states $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a join restriction category if and only if $\mathbf{C}$ is an $\mathcal{M}$-adhesive category and $\mathcal{M}_{\text{gap}} \subseteq \mathcal{M}$ (Subsection 3.3.1).

In any $\mathcal{M}$-adhesive category the $\mathcal{M}$-gaps themselves form a stable system of monics which, furthermore, always contains its gaps. Thus, with respect to these monics the partial map category is a join restriction category. This suggests an alternative technique for forming a join completion of a restriction category: namely, use the gap completion of the standard sheaf embedding of the restriction category. We show that the two constructions coincide, a fact that underlies the construction of schemes in algebraic geometry (Subsection 3.3.2).

We begin Chapter 4 by defining meet restriction categories and showing some basic properties (Section 4.1). Then we characterize when a partial map category is a meet restriction category. The completeness of meet restriction categories in partial map categories using equalizers follows by showing that a category is a meet restriction category if and only if it is a full subcategory of $\mathsf{Par}(\mathbf{X}, \mathcal{M})$ for some $\mathcal{M}$-category $(\mathbf{X}, \mathcal{M})$ in which $\mathbf{X}$ has equalizers and every regular monic of $\mathbf{X}$ is in $\mathcal{M}$ (Section 4.2). We close Chapter

4 by providing a free meet structure over a given restriction category, called a meet completion, and discuss the relationship of this construction to the meet completion already known for inverse semigroups.

In Chapter 5, we first review basic properties of range categories and their completeness in the partial map categories with $\mathcal{M}$-stable factorization systems (Section 5.1). Then we generalize Schein's representation theorem for certain class of semigroups (type 3 function systems) to range categories (Section 5.2). Finally, we study when certain range categories, specially partial map categories, satisfy Schein's condition [**RR.6**] which guarantees each map is an epimorphism onto its range (Section 5.3).

We end this thesis by listing the main results and discussing some possible directions for further work in Chapter 6.

1.5 Category Preliminaries

In this section, we review some basic notions that we shall use from category theory.

1.5.1 Categories and Some Special Maps

A category is a directed graph with composition. More precisely, a *category* $\mathbf{C}$ consists of the following data:

- A class of *objects*, , whose members are called $\mathbf{C}$-*objects*;
- For each pair (A, B) of $\mathbf{C}$-objects, a set $\mathrm{map}_{\mathbf{C}}(A, B)$, whose members are called $\mathbf{C}$-*maps from A to B*. We also write $\mathrm{map}_{\mathbf{C}}(A, B)$ by $\mathbf{C}(A, B)$. The class of all $\mathbf{C}$-maps (denoted by) $\mathrm{map}(\mathbf{C})$ is defined to be the union of all the sets $\mathrm{map}_{\mathbf{C}}(A, B)$;
- Two operations assigning to each map $f \in \mathrm{map}(\mathbf{C})$ its *domain* $\mathrm{dom}(f)$ which is an object of $\mathbf{C}$ and its *codomain* $\mathrm{cod}(f)$ also an object of $\mathbf{C}$. We indicate that f has domain A and codomain B by writing $f : A \to B$;

- Maps f and g are *composable* if $\mathrm{cod}(f) = \mathrm{dom}(g)$. There is an operation assigning to each pair of composable maps f and g their *composition* which is a map denoted by gf such that $\mathrm{dom}(gf) = \mathrm{dom}(f)$ and $\mathrm{cod}(gf) = \mathrm{cod}(g)$. There is also an operation assigning to each object $A \in \mathrm{ob}(\mathbf{C})$ an identity map $1_A : A \to A$. These operations are required to satisfy the following axioms:

  [C.1] (identity law) if $f : A \to B$ is a map in $\mathbf{C}$ then $f1_A = f = 1_B f$,

  [C.2] (association law) if $f : A \to B$, $g : B \to C$, and $h : C \to D$ are maps in $\mathbf{C}$ then $(hg)f = h(gf)$.

In a category $\mathbf{C}$, a map $f : A \to B$ is *monic* if $fg_1 = fg_2$ implies $g_1 = g_2$ and a map $f : A \to B$ is a *section* if there is a map g in $\mathbf{C}$ such that $gf = 1_A$. Dually, a map $f : A \to B$ is *epic* if $g_1 f = g_2 f$ implies $g_1 = g_2$ and a map $f : A \to B$ is a *retraction* if there is a map g in $\mathbf{C}$ such that $fg = 1_B$. A map is called an *isomorphism* if it is both a section and a retraction. The collection of special maps, for instance, monics, in $\mathbf{C}$, can be denoted by the following notation:

$$\mathrm{Monics}_{\mathbf{C}} = \{\text{monics in } \mathbf{C}\}.$$

A *subcategory* $\mathbf{C}'$ of a category $\mathbf{C}$ is given by any subcollections of the objects and maps of $\mathbf{C}$ which is a category under the domain, codomain, composition, and identity operations of $\mathbf{C}$.

Given a category $\mathbf{C}$, if we flip the directions of all maps in $\mathbf{C}$ then we obtain its *dual category*, denoted by $\mathbf{C}^{\mathrm{op}}$. Clearly, $(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{C}$.

A category $\mathbf{C}$ is said to be *small* if tts class of objects, $\mathrm{ob}(\mathbf{C})$, is a set.

1.5.2 Functors, Natural Transformations, and Adjunctions

A *functor* F from a category $\mathbf{C}$ to a category $\mathbf{D}$, written as $F : \mathbf{C} \to \mathbf{D}$, is specified by

- an operation taking each object A in $\mathbf{C}$ to an object $F(A)$ in $\mathbf{D}$,
- an operation sending each map $f : A \to B$ in $\mathbf{C}$ to a map $F(f) : F(A) \to F(B)$ in $\mathbf{D}$,

such that

$$F(1_A) = 1_{F(A)} \text{ and } F(gf) = F(g)F(f)$$

for any maps $f : A \to B$ and $g : B \to C$ in $\mathbf{C}$. So, functors are *structure preserving* maps between categories.

Let $F, G : \mathbf{C} \to \mathbf{D}$ be two functors. A *natural transformation* α from F to G, written as $\alpha : F \to G$, is specified by an operation which assigns each object C of $\mathbf{C}$ a map $\alpha_C : F(C) \to G(C)$, called a *component* of α, such that for each map $f : A \to B$ in $\mathbf{C}$

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\alpha_B"] & G(B) \end{tikzcd}$$

commutes in $\mathbf{D}$, which means that $G(f)\alpha_A = \alpha_B F(f)$. Natural transformations are maps between functors. A natural transformation α is called *a natural isomorphism*, denoted by $\alpha : F \cong G$, if each component α_C is an isomorphism.

An *equivalence* between categories $\mathbf{C}$ and $\mathbf{D}$ is defined to be a pair of functors $S : \mathbf{C} \to \mathbf{D}$ and $T : \mathbf{D} \to \mathbf{C}$ together with natural isomorphisms $1_{\mathbf{C}} \cong TS$ and $1_{\mathbf{D}} \cong ST$. Two categories $\mathbf{C}$, $\mathbf{D}$ are *equivalent*, written $\mathbf{C} \approx \mathbf{D}$, if there is an equivalence between them.

An *adjunction* from $\mathbf{C}$ to $\mathbf{D}$ is a triple $\langle F, G, \varphi \rangle : \mathbf{C} \to \mathbf{D}$, where F and G are functors:

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "F"] & \mathbf{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

and φ is a function which assigns to each pair of objects $C \in \mathbf{C}, D \in \mathbf{D}$ a bijection of sets

$$\varphi = \varphi_{C,D} : \mathbf{D}(F(C), D) \cong \mathbf{C}(C, G(D))$$

which is natural in C and D:

$$\frac{F(C) \to D}{\overline{C \to G(D)}} \ .$$

If $F : \mathbf{C} \to \mathbf{D}$ is a functor and $D \in \mathrm{ob}(\mathbf{D})$, a *universal arrow* from D to F is a pair (C, u) with $C \in \mathrm{ob}(\mathbf{C})$ and $u : D \to F(C)$ being in $\mathrm{map}(\mathbf{D})$ such that for each pair (C', f) with $C' \in \mathrm{ob}(\mathbf{C})$ and $f : D \to F(C') \in \mathrm{map}(\mathbf{D})$ there is a unique $\mathbf{C}$-map $f^* : C \to C'$ such that

$$\begin{tikzcd}
D \arrow[r, "u"] \arrow[dr, "f"'] & F(C) \arrow[d, "F(f^*)"] & C \arrow[d, dotted, "f^*"] \\
& F(C') & C'
\end{tikzcd}$$

commutes. Equivalently, $u : D \to F(D)$ is universal from D to F provided that the pair (C, u) is an initial object (see Subsection 1.5.3 below) in the comma category $(D \downarrow F)$ that has maps $D \to F(C)$ as its objects.

If $G : \mathbf{D} \to \mathbf{C}$ is a functor and $C \in \mathrm{ob}(\mathbf{C})$, dually, a *universal arrow* from G to C is a pair (D, v) with $D \in \mathrm{ob}(\mathbf{D})$ and $v : G(D) \to C \in \mathrm{map}(\mathbf{C})$ such that for each pair (D', f) with $D' \in \mathrm{ob}(\mathbf{D})$ and $f : G(D') \to C \in \mathrm{map}(\mathbf{C})$ there is a unique $\mathbf{D}$-map $f^\sharp : D' \to D$ making

$$\begin{tikzcd}
D' \arrow[d, dotted, "f^\sharp"] & G(D') \arrow[d, "G(f^\sharp)"'] \arrow[dr, "f"] & \\
D & G(D) \arrow[r, "v"] & C
\end{tikzcd}$$

commute.

By [[33], p.83, Theorem 2], each adjunction $\langle F, G, \varphi \rangle : \mathbf{C} \to \mathbf{D}$ is completely determined by one of five conditions. Here we only record some of them, which we shall use in this thesis:

(ii) *The functor* $G : \mathbf{D} \to \mathbf{C}$ *and for each* $C \in \mathrm{ob}(\mathbf{C})$ *a* $F_0(C) \in \mathrm{ob}(\mathbf{C})$ *and a universal arrow* $\eta_C : C \to GF_0(C)$ *from* C *to* G*. Then the functor* F *has object function* F_0 *and is given by sending* $f : C \to C'$ *to* $GF(f)\eta_C = \eta_{C'} f$*.*

(iv) *The functor* $F : \mathbf{C} \to \mathbf{D}$ *and for each* $D \in \mathrm{ob}(\mathbf{D})$ *a* $G_0(D) \in \mathrm{ob}(\mathbf{C})$ *and a universal arrow* $\varepsilon_D : FG_0(D) \to D$ *from* F *to* D*.*

(v) *Functors* F*,* G *and natural transformations* $\eta : 1_{\mathbf{C}} \to GF$ *and* $\varepsilon : FG \to 1_{\mathbf{D}}$ *such that* $G\varepsilon \cdot \eta G = 1_G$ *and* $\varepsilon F \cdot F\eta = 1_F$*.*

Hence we often denote the adjunction $\langle F, G, \varphi \rangle : \mathbf{C} \to \mathbf{D}$ by $(\eta, \varepsilon) : F \dashv G : \mathbf{C} \to \mathbf{D}$ or by $\langle F, G, \eta, \varepsilon \rangle : \mathbf{C} \to \mathbf{D}$. In this case, we say that F is a *left adjoint* to G or G is a *right adjoint* to F and that F has a right adjoint G and G has a left adjoint F. We also say that $F \dashv G$ is an *adjoint pair.*

1.5.3 Limits and Colimits

Limits and colimits are an example of universals. Given a category $\mathbf{C}$, an $\mathbf{I}$-indexed *diagram* in $\mathbf{C}$ is a functor $D : \mathbf{I} \to \mathbf{C}$, where the category $\mathbf{I}$ is thought of as index category. A D-*cone* is a natural transformation $\phi : L \to D$, where $L : \mathbf{I} \to \mathbf{C}$ is a constant functor that sends each $\mathbf{I}$-map $f : I \to J$ to a constant $\mathbf{C}$-map $1_L : L \to L$. Each D-cone can be specified by a $\mathbf{C}$-object L together with a family of $\mathbf{C}$-maps $(\phi_I : L \to D(I))_{I \in \mathrm{ob}(\mathbf{I})}$ such that $D(f)\phi_I = \phi_J$:

$$\begin{tikzcd}
 & L \arrow[dl, "\phi_I"'] \arrow[dr, "\phi_J"] & \\
D(I) \arrow[rr, "D(f)"] & & D(J)
\end{tikzcd}$$

for each $\mathbf{I}$-map $f : I \to J$. A *limit* of the diagram $D : \mathbf{I} \to \mathbf{C}$ is a D-cone (L, ϕ) such that for each other D-cone (J, ψ) there is a unique $\mathbf{C}$-map $u : J \to L$ making the following diagram

$$\begin{tikzcd}
J \arrow[dr, "\psi_I"] \arrow[dd, dotted, "u"'] & \\
& D(I) \\
L \arrow[ur, "\phi_I"'] &
\end{tikzcd}$$

commute for each **I**-object I. If **I** is specified by the following graphs

$$\begin{tikzcd}
 & & & & & \bullet \arrow[d] \\
\bullet, & & \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet, & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

then the limit of $D : \mathbf{I} \to \mathbf{C}$ is called a *terminal object*, an *equalizer*, a *pullback (square)* in $\mathbf{C}$, respectively.

Explicitly, a **C**-object 1 is a *terminal object* provided for each **C**-object X there is a unique **C**-map $!_X : X \to 1$.

A commutative square

$$\begin{tikzcd}
P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

in **C** is called a *pullback (square)* provided given any **C**-maps $w_1 : W \to X$ and $w_2 : W \to Y$ with $fw_1 = gw_2$ there is a unique **C**-map $w : W \to P$ such that

$$p_1 w = w_1 \text{ and } p_2 w = w_2 :$$

$$\begin{tikzcd}
W \arrow[dr, dotted, "w"] \arrow[drr, bend left, "w_2"] \arrow[ddr, bend right, "w_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

For two parallel **C**-maps $f, g : X \to Y$, the *equalizer* of f and g is a **C**-map $e : E \to X$ such that $fe = ge$ and e is unique with this property: if a **C**-map $z : Z \to X$ is such that

$fz = gz$ then there is a unique **C**-map $d : Z \to E$ such that $ed = z$:

$$\begin{tikzcd}
Z \arrow[d, dotted, "d"'] \arrow[dr, "z"] & & \\
E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

If maps between D-cones are defined properly, then limits can be characterized as terminal objects in the category of all D-cones.

The dual notions of cone, limit, terminal object, pullback (square), equalizer are *cocone*, *colimit*, *initial object*, *pushout (square)*, and *coequalizer*, respectively.

An initial object 0 is call *strict* provided any map $X \to 0$ must be an isomorphism.

1.5.4 2-Categories, 2-Functors, and 2-Natural Transformations

A 2-*category* **K** consists of

- A class of objects or 0-cells: $A, B, \cdots$
- A class of maps or 1-cells: $f : A \to B, \cdots$
- A class of 2-cells: $\alpha : f \Rightarrow g, \cdots$
- The objects and maps form a category $\mathbf{K}_0$, called the *underlying category* of **K**.
- For any objects A and B, the maps $f : A \to B$ and the 2-cells between them form a map-category $\mathbf{K}(A, B)$ under *vertical composition*, denoted by $\beta \circ \alpha$. The identity 2-cell on $f : A \to B$ is denoted by 1_f.
- There is an operation of *horizontal composition* of 2-cells:

$$(\beta \star \alpha : uf \to vg) = (\beta : u \to v) \star (\alpha : f \to g)$$

as displayed by

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\beta"] & C & = & A \arrow[r, bend left, "uf"] \arrow[r, bend right, "vg"'] \arrow[r, phantom, "\Downarrow\beta\star\alpha"] & C
\end{tikzcd}$$

Under this operation the 2-cells form a category with identities:

$$\begin{tikzcd} A \arrow[r, bend left, "1_A"] \arrow[r, bend right, "1_A"'] \arrow[r, phantom, "\Downarrow 1_{1_A}"] & A. \end{tikzcd}$$

- In the situation:

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & B \arrow[r, bend left=50, "u"] \arrow[r] \arrow[r, bend right=50, "v"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\gamma"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\delta"] & C, \end{tikzcd}$$

  the *interchange law*

$$(\delta \star \gamma) \circ (\beta \star \alpha) = (\delta \circ \beta) \star (\gamma \circ \alpha)$$

  holds true and for any pair of composable 1-cells f and g,

$$1_g \star 1_f = 1_{gf}.$$

A basic example of a 2-category is **Cat**, whose objects are small categories, 1-cells are functors and 2-cells are natural transformations. Also, for any small category **C**, the slice category **Cat**/**C** is again a 2-category.

A 2-*functor* $F : \mathbf{K} \to \mathbf{L}$ between 2-categories $\mathbf{K}$ and $\mathbf{L}$ is a triple of functions sending objects, 1-cells, and 2-cells of $\mathbf{K}$ to items of the same types in $\mathbf{L}$ preserving domains, codomains, compositions, and identities.

A 2-*natural transformation* $\alpha : F \Rightarrow G$ between 2-functors $F, G : \mathbf{K} \to \mathbf{L}$ assigns to each object A of $\mathbf{K}$ a map $\alpha_A : F(A) \to G(A)$ in $\mathbf{L}$ such that for each map $f : A \to B$ in $\mathbf{K}$,

$$\alpha_B F(f) = G(f)\alpha_A$$

and for each 2-cell $\theta : f \Rightarrow g$ in $\mathbf{K}$,

$$\begin{tikzcd} F(A) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Downarrow F(\theta)"] & F(B) \arrow[r, "\alpha_B"] & G(B) & = & F(A) \arrow[r, "\alpha_A"] & G(A) \arrow[r, bend left, "G(f)"] \arrow[r, bend right, "G(g)"'] \arrow[r, phantom, "\Downarrow G(\theta)"] & G(B). \end{tikzcd}$$

In a 2-category, transposing the classical definitions for adjoints in **Cat** one has the notion of adjoint pairs in a 2-category using 2-cells. Clearly, each 2-functor preserves such adjoint pairs.

Many categorical notions/constructions are defined *up to isomorphism.* A *pseudo-functor* is defined in such a way: if we require that those equalities in the definition of a functor hold only up to isomorphism, then we get a pseudo-functor.

1.5.5 Factorization Systems and Fibrations

In **Set**, each function $f : X \to Y$ can be factored through its image, i.e.,

$$X \overset{f}{\to} Y \qquad = \qquad X \overset{e}{\to} f(X) \overset{m}{\to} Y,$$

where $e : X \to f(X)$ is the codomain restriction of f and $m : f(X) \hookrightarrow Y$ is the inclusion. This says that **Set** admits an $(\mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$-factorization system. We shall use factorization systems in range categories.

A *factorization system* on a category **C** consists of two classes $\mathcal{E}, \mathcal{M}$ of maps in **C** such that

(*i*) *every isomorphism is both in* $\mathcal{E}$ *and in* $\mathcal{M}$*;*

(*ii*) $\mathcal{E}$ *and* $\mathcal{M}$ *are closed under composition;*

(*iii*) *every map* f *of* **C** *factors as* $f = m_f e_f$ *with* $e_f \in \mathcal{E}$ *and* $m_f \in \mathcal{M}$*;*

(*iv*) *for each commutative square where* $e \in \mathcal{E}$ *and* $m \in \mathcal{M}$*, there exists a unique diagonal map making both triangles commutative:*

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d] & \bullet \arrow[d] \arrow[dl, dashed] \\
\bullet \arrow[r, "m"'] & \bullet
\end{tikzcd}$$

Given a category $\mathbf{C}$, if $(\mathcal{E}, \mathcal{M})$ is a factorization system on $\mathbf{C}$, then there is a special codomain functor $\partial : \mathcal{M} \to \mathbf{C}$, called a fibration. Formally, we have:

Definition 1.5.1 *Let* $P : \mathbf{E} \to \mathbf{C}$ *be a functor and* $p : E \to B$ *a map of* $\mathbf{C}$. The fibre *of* P *at* B *is the non-full subcategory* $\mathbf{E}(B)$ *of* $\mathbf{E}$ *whose objects are in* $P^{-1}(B)$ *(i.e., those objects* A *of* $\mathbf{E}$ *with* $P(A) = B$*) and whose maps* $f : A \to A'$ *are* $\mathbf{E}$*-maps such that* $P(f) = 1_B$*. If* $X \in \mathbf{E}(B)$*, then a map* $\vartheta_p X : p^*X \to X$ *of* $\mathbf{C}$ *is* a *cartesian lifting* over p at X *if*

[F.1] $P(\vartheta_p X) = p$*;*

[F.2] *For any map* $v : Y \to X$ *of* $\mathbf{E}$ *and any map* $h : P(Y) \to E$ *in* $\mathbf{C}$ *satisfying* $ph = P(v)$*, there is a unique* $w : Y \to p^*X$ *in* $\mathbf{E}$ *such that*

$$\vartheta_p X \cdot w = v \;\; \textit{and} \;\; P(w) = h.$$

$$\begin{tikzcd}
 & Y \arrow[dl, dashed, "w"'] \arrow[dr, "v"] & & \\
p^*X \arrow[rr, "\vartheta_p X"] & & X & \text{in } \mathbf{E} \\
 & {} \arrow[d, "P"] & & \\
 & {} & & \\
 & P(Y) \arrow[dl, "h"'] \arrow[dr, "P(v)"] & & \\
E \arrow[rr, "p"] & & B & \text{in } \mathbf{C}
\end{tikzcd}$$

A functor $P : \mathbf{E} \to \mathbf{C}$ *is called* a fibration *if for any map* $p : E \to B$ *in* $\mathbf{C}$ *and every object* X *in* $\mathbf{E}(B)$ *there is a cartesian lifting* $(p^*X, \ \vartheta_p X)$ *over* p *at* X*. A functor* $P : \mathbf{E} \to \mathbf{C}$ *is called* an opfibration *if* P^{op} *is a fibration. A functor* P *is a* bifibration *if both* P *and* P^{op} *are fibrations.*

1.6 Restriction Category Basics

In this section, we review the fundamentals of Cockett-Lack's restriction theory.

1.6.1 Definitions and Basic Properties of Cockett-Lack's Restrictions

First, we recall the definition of Cockett-Lack's restriction.

A *restriction structure* on a category $\mathbf{C}$ is an assignment of a map $\overline{f} : X \to X$ to each map $f : X \to Y$ such that the following four restriction axioms are satisfied:

[**R.1**] $f\overline{f} = f$ for each map f,

[**R.2**] $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ whenever $\mathrm{dom}(f) = \mathrm{dom}(g)$,

[**R.3**] $\overline{g\overline{f}} = \overline{g}\,\overline{f}$ whenever $\mathrm{dom}(f) = \mathrm{dom}(g)$,

[**R.4**] $\overline{g}f = f\overline{gf}$ whenever $\mathrm{cod}(f) = \mathrm{dom}(g)$,

A category with a restriction structure is called a *restriction category.* A category $\mathbf{X}$ is called a *co-restriction category* if $\mathbf{X}^{\mathrm{op}}$ is a restriction category.

Now, we record some basic properties of restriction categories in Lemmas 1.6.1, 1.6.2, and 1.6.4, which are Lemmas 2.1, 2.2, and 2.3 in [14], respectively.

Lemma 1.6.1 *In a restriction category,*

(*i*) $\overline{f}$ *is an idempotent for each map* f*;*

(*ii*) $\overline{f}\,\overline{gf} = \overline{gf}$ *if* $\mathrm{codom}(f) = \mathrm{dom}(g)$*;*

(*iii*) $\overline{\overline{g}f} = \overline{gf}$ *if* $\mathrm{codom}(f) = \mathrm{dom}(g)$*;*

(*iv*) $\overline{\overline{f}} = \overline{f}$ *for each map* f*;*

(*v*) $\overline{\overline{g}\,\overline{f}} = \overline{g}\,\overline{f}$ *if* $\mathrm{dom}(f) = \mathrm{dom}(g)$*;*

(*vi*) *if* $f : A \to B$ *is monic then* $\overline{f} = 1_A$*;*

(*vii*) $f\overline{g} = f$ *implies* $\overline{f} = \overline{f}\overline{g}$*.*

In a given restriction category, a map f such that $f = \overline{f}$ is called a *restriction idempotent.* Restriction idempotents are precisely the maps of the form $\overline{f}$ by Lemma 1.6.1 (*iv*). A map $f : A \to B$ is called *total* if $\overline{f} = 1_A$.

Lemma 1.6.2 *In a restriction category,*

(*i*) *if* f *is monic, then* f *is total;*

(*ii*) *if* f *and* g *are total and* $\mathrm{codom}(f) = \mathrm{dom}(g)$ *then so is* gf*;*

(*iii*) *if* gf *is total then so is* f*;*

(*iv*) *the total maps form a subcategory.*

The subcategory of total maps of a restriction category $\mathbf{C}$ *is denoted by* $\mathsf{Total}(\mathbf{C})$*.*

A restriction category is a partial order enriched category as shown in the following Lemma.

Lemma 1.6.3 *Let* $\mathbf{C}$ *be a restriction category. For any* $A, B \in \mathrm{ob}(\mathbf{C})$*,*

(*i*) $\mathrm{map}_{\mathbf{C}}(A, B)$ *is a poset with the order given by*

$$f \leq g \Leftrightarrow f = g\overline{f};$$

(*ii*) $f \leq g$ *in* $\mathrm{map}_{\mathbf{C}}(A, B)$ *implies* $\overline{f} \leq \overline{g}$ *in* $\mathrm{map}_{\mathbf{C}}(A, A)$*.*

PROOF: Lemma 3.2 [9]. □

A restriction idempotent $\overline{f}$ is called *split* if $\overline{f} = mr$ for some maps m and r with $rm = 1$. In such a case, m and r are called *the monic part* and *the epic part* of the split restriction idempotent $\overline{f}$, respectively. The monic part of a split restriction idempotent is also called a *restriction monic*. Note that if $\overline{f}$ splits by m and r then $\overline{f} = \overline{r}$ as $\overline{f} = \overline{mr} = \overline{\overline{m}r} = \overline{r}$ since m is monic. A restriction structure on a category is said to be *split* if all of its restriction idempotents split. Split idempotents are determined completely by their monic parts or epic parts as shown by the following lemma:

Lemma 1.6.4 *In any restriction category:*

- *(i) if $rm = 1$ and $sm = 1$ with $mr = \overline{r}$ and $ms = \overline{s}$ then $r = s$;*
- *(ii) if $rm = 1$ and $rn = 1$ with $mr = \overline{r}$ and $nr = \overline{r}$ then $m = n$.*

In a given restriction category, a map f is a *restricted isomorphism* (or *partial isomorphism*) if there is a map g such that $gf = \overline{f}$ and $fg = \overline{g}$. A restriction category is called an *inverse category* provided each map is a restricted isomorphism. As shown in Theorem 2.20 of [14], for a given category $\mathbf{X}$, $\mathbf{X}$ is an inverse category if and only if each $\mathbf{X}$-map f has a unique map g, denoted by $f^{(-1)}$, such that $fgf = f$ and $gfg = g$ in $\mathbf{X}$. The opposite category of an inverse category is also an inverse category.

Lemma 1.6.5 *If $\mathbf{I}$ is an inverse category, then so is $\mathbf{I}^{\mathrm{op}}$.*

PROOF: Each $\mathbf{I}^{\mathrm{op}}$-map $f : X \to Y$ is an $\mathbf{I}$-map $f : Y \to X$ so that there is an $\mathbf{I}$-map $f^{(-1)}$ such that

$$ff^{(-1)}f = f \text{ and } f^{(-1)}ff^{(-1)} = f^{(-1)},$$

that is

$$f \cdot^{\mathrm{op}} f^{(-1)} \cdot^{\mathrm{op}} f = f \text{ and } f^{(-1)} \cdot^{\mathrm{op}} f \cdot^{\mathrm{op}} f^{(-1)} = f^{(-1)}.$$

Hence $\mathbf{I}^{\mathrm{op}}$ is an inverse category. □

1.6.2 Examples of Restriction Categories

Some examples of restriction categories are as follows:

Example 1.6.6 1. $\mathbf{Par}(\mathbf{Set}, \mathrm{Monics}_{\mathbf{Set}})$ is a restriction category if for any partial map $f : X \to Y$, one defines the partial map $\overline{f} : X \to X$ given by

$$\overline{f}(x) = \begin{cases} x & \text{whenever} \quad f(x) \text{ is defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

to be the restriction of f.

2. Every category is a restriction category with the restriction given by $\overline{f} = 1_X$ for any map $f : X \to Y$. The restriction is called *the trivial restriction structure.* So a restriction structure is not a property of a category but an extra structure.

3. If C is an object of a given restriction category $\mathbf{C}$, then the slice category $\mathbf{C}/C$ with objects all pairs (f, X), where $f : X \to C$ is a map of $\mathbf{C}$, and with maps $h : (f, X) \to (g, Y)$ those maps $h : X \to Y$ of $\mathbf{C}$ for which $gh = f$, is also a restriction category with the same restriction as $\mathbf{C}$.

4. Every inverse category is a restriction and co-restriction category with $\overline{f} = f^{(-1)}f$ and $\overline{f}^{\mathrm{op}} = ff^{(-1)}$. Every inverse semigroup with an identity can be regarded as the one object restriction and co-restriction category with $\overline{x} = x^{(-1)}x$ and $\overline{x}^{\mathrm{op}} = xx^{(-1)}$ (See Proposition 1.6.8 below).

We recall some definitions and properties of *inverse semigroups.* A *semigroup* $(S, \cdot)$ is a nonempty set S with an associative binary operation $\cdot$. An *identity* 1 is an element $1 \in S$ such that $1 \cdot s = s \cdot 1 = s$ for all $s \in S$. Let S be a semigroup. An element $a \in S$ is called *regular* if there is $x \in S$, called *a regular-inverse of* a, such that $axa = a$. A semigroup S is called *regular* if all of its elements are regular. An *inverse* of an element a is $x \in S$ such that $axa = a$ and $xax = x$. A regular semigroup can be characterized by the inverse defined above: a semigroup S is regular if and only if each $a \in S$ has at least one inverse x.

An *inverse semigroup* is a semigroup in which each element a has a unique inverse, denoted by $a^{(-1)}$. Let x, y, z be elements of an inverse semigroup S. Then one has the following equalities:

$$\begin{aligned} x(yz) &= (xy)z, \\ (x^{(-1)})^{(-1)} &= x, \\ (xy)^{(-1)} &= y^{(-1)}x^{(-1)}, \\ xx^{(-1)}yy^{(-1)} &= yy^{(-1)}xx^{(-1)}. \end{aligned}$$

For example, any group is an inverse semigroup. But the inverse semigroup $\{0, a, b, 1\}$, defined by

$$ab = ba = 0, a^2 = a, \text{ and } b^2 = b,$$

is not a group.

The following can be used to test when a semigroup is an inverse semigroup.

Proposition 1.6.7 *A semigroup is an inverse semigroup if and only if it is regular and any two idempotents commute.*

Proof: See [36], p.78. □

Clearly, a one-object inverse category is precisely an inverse semigroup with identity. Each inverse category can be viewed as a restriction and co-restriction category that as shown in the following proposition.

Proposition 1.6.8 *Every inverse category is a restriction and co-restriction category with the restriction and the co-restriction category given by* $\overline{f} = f^{(-1)}f$ *and* $\overline{f}^{\text{op}} = ff^{(-1)}$.

Proof: By the definition of inverse categories, each inverse category is a restriction category with $\overline{f} = f^{(-1)}f$. It suffices to check the four co-restriction axioms.

[R.1]$^{\text{op}}$ $\overline{f}^{\text{op}}f = ff^{(-1)}f = f$.

[R.2]$^{\mathrm{op}}$ $\overline{f}^{\mathrm{op}}\overline{g}^{\mathrm{op}} = ff^{(-1)}gg^{(-1)} = gg^{(-1)}ff^{(-1)} = \overline{g}^{\mathrm{op}}\overline{f}^{\mathrm{op}}$.

[R.3]$^{\mathrm{op}}$

$$\begin{aligned} \overline{\overline{g}^{\mathrm{op}}f}^{\mathrm{op}} &= (gg^{(-1)}f)(gg^{(-1)}f)^{(-1)} \\ &= (gg^{(-1)}f)(f^{(-1)}gg^{(-1)}) \\ &= gg^{(-1)}ff^{(-1)} \\ &= \overline{g}^{\mathrm{op}}\overline{f}^{\mathrm{op}}. \end{aligned}$$

[R.4]$^{\mathrm{op}}$

$$\begin{aligned} \overline{gf}^{\mathrm{op}}g &= (gf)(gf)^{(-1)}g \\ &= gff^{(-1)}g^{(-1)}g \\ &= gg^{(-1)}gff^{(-1)} \\ &= g\overline{f}^{\mathrm{op}}. \end{aligned}$$

□

1.6.3 Category of Restriction Categories

A functor $F : \mathbf{C} \to \mathbf{D}$ between two restriction categories is said to be a *restriction* functor if $F(\overline{f}) = \overline{F(f)}$ for any map f in $\mathbf{C}$. Restriction categories and restriction functors form a category, denoted by $\mathbf{rCat}_0$. Clearly, there is a forgetful functor $U_r : \mathbf{rCat}_0 \to \mathbf{Cat}_0$ which forgets restriction structures by sending any restriction functor $F : \mathbf{C} \to \mathbf{D}$ to the functor $F : \mathbf{C} \to \mathbf{D}$.

A natural transformation between restriction functors is called a *restriction natural transformation* if all of its components are total. Restriction categories, restriction functors, and restriction transformations form a 2-category, called **rCat**. The category **rCat** has an important full 2-subcategory, comprising those objects with split restriction structures, denoted by $\mathbf{rCat}_s$. The category **rCat** has also an important full sub-2-category **invCat** consisting of the inverse categories. As observed in [14], any functor between inverse categories is a restriction functor, since the structure of a restricted inverse is algebraic. Cockett and Lack proved:

Proposition 1.6.9 ([14], Proposition 2.24) *The 2-category* **invCat** *is a full coreflective sub-2-category of* **rCat** *with the right adjoint* inv *sending each restriction category to the subcategory of restricted isomorphisms.*

Given two restriction functors $F, G : \mathbf{X} \to \mathbf{Y}$, a *lax restriction natural transformation* from F to G consists of a total map $\alpha_X : F(X) \to G(X)$ in $\mathbf{Y}$ for each $X \in \text{map}(\mathbf{X})$ such that for each map $f : X \to Y$ in $\mathbf{X}$

$$\begin{tikzcd}
F(X) \arrow[r, "F(\overline{f})"] \arrow[d, "F(f)"'] & F(X) \arrow[r, "\alpha_X"] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[rr, "\alpha_Y"] & & G(Y)
\end{tikzcd}$$

commutes. The commutativity of the last diagram is equivalent to inequality of

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\leq"] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"] & G(Y)
\end{tikzcd}$$

That is: $\alpha_Y F(f) \leq G(f)\alpha_X$. The 2-category **rCatl** has the same objects and maps as **rCat** but with lax restriction natural transformations as the larger class of 2-cells.

Given a restriction category $\mathbf{X}$, one has a category $\mathsf{Total}(\mathbf{X})$ and conversely each category $\mathbf{X}$ can be viewed as a restriction category with the trivial restriction, denoted by $\mathsf{Triv}(\mathbf{X})$. This leads to a 2-adjunction ([14], Proposition 2.17): $\mathsf{Triv} \dashv \mathsf{Total} : \mathbf{Cat} \to \mathbf{rCat}$.

Let $\mathsf{Total} : \mathbf{rCatl} \to \mathbf{Cat}$ be given by sending each restriction functor $F : \mathbf{X} \to \mathbf{Y}$ to its restriction to $\mathsf{Total}(\mathbf{X})$, $\mathsf{Total}(F) : \mathsf{Total}(\mathbf{X}) \to \mathsf{Total}(\mathbf{Y})$, where $\mathsf{Total}(F)(f) = F(f)$, and sending each lax restriction natural transformation $\alpha : F \to G$ to $\mathsf{Total}(\alpha) : \mathsf{Total}(F) \to \mathsf{Total}(G)$ with $\mathsf{Total}(\alpha)_X = \alpha_X$ for each $X \in \mathrm{ob}(\mathsf{Total}(\mathbf{X}))$. Since $\mathsf{Total}(\mathbf{X})$ has only total maps, the last lax commutative square turns to a usual commutative square. Hence $\mathsf{Total} : \mathbf{rCatl} \to \mathbf{Cat}$ is a 2-functor.

1.6.4 Splitting Restriction Idempotents

Given a restriction category $\mathbf{C}$, $\mathsf{Split}(\mathbf{C})$ is defined as follows:

objects: restriction idempotents of $\mathbf{C}$;

maps: a map f from $(e_1 : A \to A)$ to $(e_2 : B \to B)$ is given by a map $f : A \to B$ in $\mathbf{C}$ such that both triangles in the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "e_1"'] \arrow[rd, "f"] & B \arrow[d, "e_2"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

are commutative;

composition: as in $\mathbf{C}$;

identities: $1_e = e$ for any object e of $\mathsf{Split}(\mathbf{C})$.

If $f : e_1 \to e_2$ and $g : e_2 \to e_3$ are maps in $\mathsf{Split}(\mathbf{C})$, then

$$e_3(gf) = (e_3 g)f = gf \text{ and } (gf)e_1 = g(fe_1) = gf,$$

and so gf is a map from e_1 to e_3 in $\mathsf{Split}(\mathbf{C})$. Hence the composition is well-defined. Obviously, the composition is associative. Since $ee = e = ee$, clearly e is a map from e

to e so that identities are well-defined. For any map $f : e_1 \to e_2$ in $\mathsf{Split}(\mathbf{C})$, since all triangles of the diagram

$$\begin{tikzcd}
A \arrow[r, "e_1"] \arrow[d, "e_1"'] \arrow[dr, "e_1"] & A \arrow[r, "f"] \arrow[d, "e_1"] \arrow[dr, "f"] & B \arrow[r, "e_2"] \arrow[d, "e_2"] \arrow[dr, "e_2"] & B \arrow[d, "e_2"] \\
A \arrow[r, "e_1"'] & A \arrow[r, "f"'] & B \arrow[r, "e_2"'] & B
\end{tikzcd}$$

are commutative,

$$f1_{e_1} = fe_1 = f = e_2f = 1_{e_2}f.$$

Therefore, $\mathsf{Split}(\mathbf{C})$ is indeed a category. Furthermore, $\mathsf{Split}(\mathbf{C})$ is a restriction category when we define its restriction structure by the restriction structure in $\mathbf{C}$. To show this, it suffices to show that $\overline{f} : e_1 \to e_1$ is a map of $\mathsf{Split}(\mathbf{C})$.

Lemma 1.6.10 *If $f : e_1 \to e_2$ is a map of $\mathsf{Split}(\mathbf{C})$, then so is $\overline{f} : e_1 \to e_1$.*

Proof: Since

$$\overline{f}e_1 = \overline{f}\,\overline{e_1} = \overline{f\overline{e_1}} = \overline{fe_1} = \overline{f}$$

and

$$e_1\overline{f} = \overline{f}e_1 = \overline{f},$$

all triangles in

$$\begin{tikzcd}
A \arrow[r, "\overline{f}"] \arrow[d, "e_1"'] \arrow[dr, "\overline{f}"] & A \arrow[d, "e_1"] \\
A \arrow[r, "\overline{f}"'] & A
\end{tikzcd}$$

are commutative. Hence $\overline{f} : e_1 \to e_1$ is a map in $\mathsf{Split}(\mathbf{C})$, as desired. $\square$

Proposition 1.6.11 *If $\mathbf{C}$ is a restriction category, then so is $\mathsf{Split}(\mathbf{C})$ with a split restriction structure given by the restriction in the category $\mathbf{C}$, and there is a restriction preserving inclusion $\mathbf{C} \to \mathsf{Split}(\mathbf{C})$ sending $f : X \to Y$ to $f : 1_X \to 1_Y$.*

Proof: It remains to prove that the restriction structure of $\mathsf{Split}(\mathbf{C})$ is split. For any restriction idempotent $\overline{f}$ given by $\mathbf{C}$-map $f : (e_1 : A \to A) \to (e_2 : B \to B)$, since all triangles in

$$\begin{tikzcd}
A \arrow[r, "\overline{f}"] \arrow[d, "e_1"'] \arrow[dr, "\overline{f}"] & A \arrow[r, "\overline{f}"] \arrow[d, "\overline{f}"] \arrow[dr, "\overline{f}"] & A \arrow[d, "e_1"] \\
A \arrow[r, "\overline{f}"'] & A \arrow[r, "\overline{f}"'] & A
\end{tikzcd}$$

are commutative, $\overline{f} : e_1 \to \overline{f}$ and $\overline{f} : \overline{f} \to e_1$ are maps of $\mathsf{Split}(\mathbf{C})$. Note that

$$\overline{f}\,\overline{f} = \overline{f} = 1_{\overline{f}}$$

is in $\mathsf{Split}(\mathbf{C})$. Hence $\overline{f}$ is a split restriction.

Obviously, $\mathbf{C} \to \mathsf{Split}(\mathbf{C})$ sending $f : X \to Y$ to $f : 1_X \to 1_Y$ is a full and faithful restriction functor so that $\mathsf{Split}(\mathbf{C})$ contains $\mathbf{C}$ as a full sub-restriction category. $\square$

Specially, each inverse category $\mathbf{I}$ is a restriction category so that one can form $\mathsf{Split}(\mathbf{I})$ which is an inverse category too, as proved in the following lemma.

Lemma 1.6.12 *If $\mathbf{I}$ is an inverse category, then so is $\mathsf{Split}(\mathbf{I})$ with a full and faithful functor $\eta_{\mathbf{I}} : \mathbf{I} \to \mathsf{Split}(\mathbf{I})$ given by sending $f : X \to Y$ to $f : 1_X \to 1_Y$.*

Proof: Each $\mathsf{Split}(\mathbf{I})$-map $s : (f^{(-1)}f : X \to X) \to (g^{(-1)}g : Y \to Y)$ is given by an $\mathbf{I}$-map $s : X \to Y$ such that

$$\begin{tikzcd}
X \arrow[r, "s"] \arrow[d, "f^{(-1)}f"'] \arrow[dr, "s"] & Y \arrow[d, "g^{(-1)}g"] \\
X \arrow[r, "s"] & Y
\end{tikzcd}$$

commutes. Then

$$sf^{(-1)}f = s, g^{(-1)}gs = s$$

and so

$$s^{(-1)}sf^{(-1)}fs^{(-1)} = s^{(-1)}ss^{(-1)}, s^{(-1)}g^{(-1)}gss^{(-1)} = s^{(-1)}ss^{(-1)}.$$

That is, $f^{(-1)}fs^{(-1)} = s^{(-1)}$ and $s^{(-1)}g^{(-1)}g = s^{(-1)}$. Hence $s^{(-1)} : (g^{(-1)}g : Y \to Y) \to (f^{(-1)}f : X \to X)$ is a $\mathsf{Split}(\mathbf{I})$-map and therefore $\mathsf{Split}(\mathbf{I})$ is an inverse category too.

Now, it is routine to verify that $\eta_{\mathbf{I}} : \mathbf{I} \to \mathsf{Split}(\mathbf{I})$ is a full and faithful functor between inverse categories. $\square$

Proposition 1.6.13 *There is an adjunction with a full and faithful unit* $\eta_{\mathbf{C}} : \mathbf{C} \to E(\mathsf{Split}(\mathbf{C}))$ *given by sending* $f : X \to Y$ *to* $f : 1_X \to 1_Y$*:*

$$\begin{tikzcd}
\mathbf{rCat}_{s0} \arrow[r, hook, shift right, "E"'] \arrow[r, phantom, "\perp"] & \mathbf{rCat_0} \arrow[l, shift right, "\mathsf{Split}"']
\end{tikzcd}$$

where E *is the inclusion.*

Proof: For each restriction category $\mathbf{C}$ and each split restriction category $\mathbf{D}$, clearly $\eta_{\mathbf{C}}$ is a full and faithful restriction functor and each restriction functor $F : \mathsf{Split}(\mathbf{C}) \to \mathbf{D}$ gives rise to a restriction functor $F\eta_{\mathbf{C}} : \mathbf{C} \to \mathbf{D}$. Conversely, each restriction functor $G : \mathbf{C} \to E(\mathbf{D})$ leads to a restriction functor $\mathsf{Split}(\mathbf{C}) \to \mathbf{D}$ that is given by sending $f : e_1 \to e_2$ to $f : \mathrm{dom}(e_1) \to \mathrm{dom}(e_2)$. Clearly, we have

$$\begin{array}{c} \mathsf{Split}(\mathbf{C}) \to \mathbf{D} \\ \hline\hline \mathbf{C} \to E(\mathbf{D}) \end{array} .$$

$\square$

Clearly, $\mathsf{Split} : \mathbf{rCat} \to \mathbf{rCat}$, given by sending each restriction functor $F : \mathbf{X} \to \mathbf{Y}$ to $\mathsf{Split}(F) : \mathsf{Split}(\mathbf{X}) \to \mathsf{Split}(\mathbf{Y})$ with $\mathsf{Split}(F)(f : e_1 \to e_2) = (F(f) : F(e_1) \to F(e_2))$, and sending each restriction natural transformation $\alpha : F \to G$ to $\mathsf{Split}(\alpha) : \mathsf{Split}(F) \to \mathsf{Split}(G)$, where $\mathsf{Split}(\alpha)_{e_1} = \alpha_{\mathrm{dom}(e_1)}$, is a 2-functor since the naturality of α implies that of $\mathsf{Split}(\alpha)$ clearly.

Similarly, there is a 2-functor $\mathsf{Split} : \mathbf{rCatl} \to \mathbf{rCatl}$, having the same assignments on 0-cells, 1-cells, and 2-cells as $\mathsf{Split} : \mathbf{rCat} \to \mathbf{rCat}$ since the lax naturality of $\mathsf{Split}(\alpha)$ follows from the lax naturality of α immediately.

Remark 1.6.14 Given a category $\mathbf{C}$ and a set E of idempotents in $\mathbf{C}$, one may formally split E to form $\mathsf{Split}_E(\mathbf{C})$. The restriction category $\mathsf{Split}(\mathbf{C})$ defined above is the special case, where $\mathbf{C}$ is a restriction category and $E = \{\text{restriction idempotents in } \mathbf{C}\}$. See [14] or [6] for details.

1.6.5 Partial Map Categories

We first recall the notions of system of monics, $\mathcal{M}$-categories, the category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ given in [14].

System of Monics and $\mathcal{M}$-Categories

In a category, a collection $\mathcal{M}$ of monics that includes all isomorphisms and is closed under composition is called *a system of monics*. A system of monics $\mathcal{M}$ is said to be *stable* if for any $m : C \to B \in \mathcal{M}$ and any $f : A \to B$ the pullback m' of m along f exists and belongs to $\mathcal{M}$.

A stable system of monics $\mathcal{M}$ has the following useful property:

Lemma 1.6.15 (Left-cancellable) *For a stable system of monics $\mathcal{M}$, if $mn \in \mathcal{M}$ and m is a monic, then $n \in \mathcal{M}$.*

Proof: If m is a monic, then

$$\begin{tikzcd}
\bullet \arrow[r, "n"] \arrow[d, "1"'] & \bullet \arrow[d, "m"] \\
\bullet \arrow[r, "mn"] & \bullet
\end{tikzcd}$$

is a pullback diagram. So $n \in \mathcal{M}$. ☐

An $\mathcal{M}$-*category* is a pair $(\mathbf{C}, \mathcal{M})$, where $\mathbf{C}$ is a category and $\mathcal{M}$ is a stable system of monics in $\mathbf{C}$.

Example 1.6.16 Let $\mathbf{Set}_{\text{ffib}}$ be the subcategory of $\mathbf{Set}$ with functions $f : A \to B$ such that $|f^{-1}(b)| < +\infty$ for each $b \in B$ as maps. Consider

$$\mathcal{M} = \{\text{injections } i : A \hookrightarrow B \mid |B \setminus i(A)| < +\infty\}.$$

Then $(\mathbf{Set}_{\text{ffib}}, \mathcal{M})$ is an $\mathcal{M}$-category.

An $\mathcal{M}$-*functor* $F : (\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{N})$ between two $\mathcal{M}$-categories $(\mathbf{C}, \mathcal{M})$ and $(\mathbf{D}, \mathcal{N})$ is defined to be a functor $F : \mathbf{C} \to \mathbf{D}$ such that $F(\mathcal{M}) \subseteq \mathcal{N}$ and F preserves $\mathcal{M}$-pullbacks. A natural transformation $\alpha : F \to G$ between two $\mathcal{M}$-functors $F, G : (\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{N})$ is $\mathcal{M}$-*cartesian* if for each $m : A \to B$ in $\mathcal{M}$ the naturality square

$$\begin{tikzcd}
F(A) \arrow[r, "F(m)"] \arrow[d, "\alpha_A"'] & F(B) \arrow[d, "\alpha_B"] \\
F(A) \arrow[r, "G(m)"] & G(B)
\end{tikzcd}$$

is a pullback diagram in $\mathbf{D}$. $\mathcal{M}$-categories, $\mathcal{M}$-functors, and $\mathcal{M}$-cartesian natural transformations form a 2-category $\mathcal{M}\mathbf{Cat}$. The 2-category $\mathcal{M}\mathbf{Catl}$ has the same 0-cells and 1-cells as $\mathcal{M}\mathbf{Cat}$ but has natural transformations as its 2-cells.

Category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$

Given an $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, one may form the category of partial maps $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ as in [14] with:

objects: $A \in \mathbf{C}$;

maps: a map from A to B is a pair (m, f), where $m : A' \to A$ is in $\mathcal{M}$ and $f : A' \to B$ is a map in $\mathbf{C}$:

$$\begin{tikzcd}
& A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\
A & & B
\end{tikzcd}$$

factored out by the equivalence relation: $(m, f) \approx (m', f')$ whenever there exists an isomorphism α in $\mathbf{C}$ such that $m'\alpha = m$ and $f'\alpha = f$;

identities: $(1_A, 1_A) : A \to A$;

composition: $(m', g)(m, f) = (mm'', gf')$, where f' and m'' are given by the pullback diagram $(*)$:

$$\begin{tikzcd}
& & A'' \arrow[dl, "m''"'] \arrow[dr, "f'"] & & \\
& A' \arrow[dl, "m"'] \arrow[dr, "f"'] & (*) & B' \arrow[dl, "m'"] \arrow[dr, "g"] & \\
A & & B & & C
\end{tikzcd}$$

The original maps in $\mathbf{C}$ can be embedded into $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ by $f \mapsto (1, f)$ and are called *total partial maps.* In [14], Cockett and Lack proved that each partial map category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a restriction category with a split restriction and Par is not only a 2-functor but also a part of 2-equivalence by Proposition 1.6.17 and Theorem 1.6.22 below.

Proposition 1.6.17 ([14], Proposition 3.1) *Let* $(\mathbf{C}, \mathcal{M})$ *be an* $\mathcal{M}$*-category. Then the category* $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ *has a split restriction given by* $\overline{(m, f)} = (m, m)$*. Furthermore, a map is total in* $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ *with respect to this restriction if and only if it is total as a partial map.*

Proposition 1.6.18 ([14], Proposition 3.2) *There is a 2-functor* $\mathsf{Par} : \mathcal{M}\mathbf{Cat} \to \mathbf{rCat}_s$ *taking* $F : (\mathbf{C}, \mathcal{M}) \to (\mathbf{C}', \mathcal{M}')$ *to* $\mathsf{Par}(F) : \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to \mathsf{Par}(\mathbf{C}', \mathcal{M}')$.

To provide the "inverse" of $\mathsf{Par} : \mathcal{M}\mathbf{Cat} \to \mathbf{rCat}_s$, let $\mathbf{D}$ be a restriction category with split restriction. Consider

$$\mathcal{M}_{\mathbf{D}} = \{m : X \to Y \, \text{in} \, \mathsf{Total}(\mathbf{D}) \,|\, \exists\, r : Y \to X \text{ in } \mathbf{D},\ rm = 1_X \text{ and } \overline{r} = mr\}$$

and

$$\mathcal{M}\mathsf{Total}(\mathbf{D}) = (\mathsf{Total}(D), \mathcal{M}_{\mathbf{D}}).$$

Cockett and Lack proved:

Proposition 1.6.19 ([14], Proposition 3.3) *If* $\mathbf{D}$ *is a split restriction category, then* $\mathcal{M}\mathsf{Total}(\mathbf{D})$ *is an* $\mathcal{M}$*-category.*

As $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a restriction category, it is poset-enriched. Now, let us look at the partial order $\leq$ in a partial map category.

Lemma 1.6.20 *In a partial map category* $\mathsf{Par}(\mathbf{C}, \mathcal{M})$*, let* $(m, f), (n, g) : X \to Y$ *be two partial maps and let* (π_m, π_n) *be the pullback of* (m, n)*:*

$$\begin{tikzcd}
\bullet \arrow[r, "\pi_m"] \arrow[d, "\pi_n"'] & \bullet \arrow[d, "n"] \\
\bullet \arrow[r, "m"] & \bullet .
\end{tikzcd}$$

Then the following are equivalent:

(a) $(m, f) \leq (n, g)$;

(b) π_n *is an isomorphism and* $g\pi_m\pi_n^{-1} = f$;

(c) π_n *is an isomorphism and* $(m, f)\overline{(n, g)} = (n, g)\overline{(m, f)}$.

Proof: Note that

$$\begin{aligned}
(m, f) \leq (n, g) \quad &\Leftrightarrow \quad (n, g)\overline{(m, f)} = (m, f) \\
&\Leftrightarrow \quad (n, g)(m, m) = (m, f) \\
&\Leftrightarrow \quad (m\pi_n, g\pi_m) = (m, f) \\
&\Leftrightarrow \quad \exists \text{ isomorphism } \alpha \text{ such that } m\pi_n\alpha = m \text{ and } g\pi_m\alpha = f.
\end{aligned}$$

But $m\pi_n\alpha = m$ gives $\pi_n\alpha = 1$. Hence $\pi_n = \alpha^{-1}$ and therefore $(m, f) \leq (n, g)$ if and only if π_n is an isomorphism and $g\pi_m\pi_n^{-1} = f$. Hence $(a) \Leftrightarrow (b)$. By (a), $(b) \Leftrightarrow (c)$ is obvious. $\square$

A crucial observation is that the pullback of a total map along a restriction monic in a split restriction category can be characterized as follows:

Lemma 1.6.21 *In a split restriction category* $\mathbf{X}$*, a commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "m'"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\
C \arrow[r, "m"] & D
\end{tikzcd}$$

in which m *and* m' *are restriction monics and* f *is total, is a pullback diagram in* $\mathsf{Total}(\mathbf{X})$ *if and only if* $\overline{e_m f} = e_{m'}$*, where* $e_m = \overline{mr}$ *and* $rm = 1$*.*

Proof: "⇐" Suppose that $rm = 1$ and $r'm' = 1$ and suppose that $x : X \to C$ and $y : X \to B$ are total maps such that $mx = fy$. Note that

$$m'r'y = e_{m'}y = \overline{e_m f}y = \overline{mrf}y = y\overline{mrfy} = y\overline{mrmx} = y\overline{mx} = y\overline{fy} = \overline{f}y = y$$

and

$$mf'r'y = fm'r'y = fy = mx.$$

Then there is a unique map $r'y : X \to A$ such that $m'r'y = y$ and $f'r'y = x$:

$$\begin{tikzcd}
X \arrow[drr, bend left, "y"] \arrow[dr, dotted, "r'y"] \arrow[ddr, bend right, "x"'] & & \\
& A \arrow[r, "m'"'] \arrow[d, "f'"] & B \arrow[d, "f"] \\
& C \arrow[r, "m"] & D
\end{tikzcd}$$

and so

$$\begin{tikzcd}
A \arrow[r, "m'"] \arrow[d, "f'"'] & B \arrow[d, "f"] \\
C \arrow[r, "m"] & D
\end{tikzcd}$$

is a pullback diagram in $\mathsf{Total}(\mathbf{X})$.

"⇒" For each restriction monic $m : C \to D$ and each total map $f : B \to D$,

$$\begin{tikzcd}
A \arrow[r, "m'"] \arrow[d, "g"'] & B \arrow[d, "f"] \\
C \arrow[r, "m"] & D
\end{tikzcd}$$

is a pullback diagram, where $r : D \to C$ is a map such that $rm = 1$ and $\overline{mr} = mr$, $\overline{mrf} = m'r'$ with $r'm' = 1$, and $g = rfm'$. Then, by the uniqueness of the pullback of m and f, $\overline{e_m f} = \overline{mrf} = m'r' = e_{m'}$, as desired. $\square$

A partial map category can be characterized as a restriction category with a split restriction by the following theorem proved by Cockett and Lack.

Theorem 1.6.22 ([14], Theorem 3.4) *The 2-functors $\mathcal{M}\mathsf{Total}$ and Par give an equivalence of 2-categories between $\mathbf{rCat}_s$ and $\mathcal{M}\mathbf{Cat}$.*

By Proposition 1.6.13 and Theorem 1.6.22, immediately one has:

Theorem 1.6.23 (Completeness of Restriction Categories [14]) *Each restriction category embeds via a full and faithful restriction preserving functor into a restriction category of the form $\mathsf{Par}(\mathbf{C}, \mathcal{M})$.*

As another corollary of Theorem 1.6.22, we can see when two partial map categories $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ and $\mathsf{Par}(\mathbf{D}, \mathcal{N})$, as restriction categories, are equivalent.

Corollary 1.6.24 *For given $\mathcal{M}$-categories $(\mathbf{C}, \mathcal{M})$ and $(\mathbf{D}, \mathcal{N})$, the following are equivalent:*

- (i) $\mathsf{Par}(\mathbf{C}, \mathcal{M}) \approx \mathsf{Par}(\mathbf{D}, \mathcal{N})$ *in* $\mathbf{rCat}$;
- (ii) $(\mathbf{C}, \mathcal{M}) \approx (\mathbf{D}, \mathcal{N})$ *in* $\mathcal{M}\mathbf{Cat}$;
- (iii) *there are category equivalences $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that $F(\mathcal{M}) \subseteq \mathcal{N}$ and $G(\mathcal{N}) \subseteq \mathcal{M}$.*

PROOF: "$(i) \Leftrightarrow (ii)$:" As Par and $\mathcal{M}\mathsf{Total}$ are part of equivalences between $\mathbf{rCat}_s$ and $\mathcal{M}\mathbf{Cat}$, it is clear.

"*(ii)* $\Rightarrow$ *(iii)*:" Assume that $F : (\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{N})$ and $G : (\mathbf{D}, \mathcal{N}) \to (\mathbf{C}, \mathcal{M})$ are such that $GF \approx 1_{(\mathbf{C}, \mathcal{M})}$ and $FG \approx 1_{(\mathbf{D}, \mathcal{N})}$. Then, obviously, $GF \approx 1_{\mathbf{C}}$ and $FG \approx 1_{\mathbf{D}}$ and $F(\mathcal{M}) \subseteq \mathcal{N}$ and $G(\mathcal{N}) \subseteq \mathcal{M}$ as F and G are also $\mathcal{M}$-functors

"*(iii)* $\Rightarrow$ *(ii)*:" Clearly, F and G give rise to $\mathcal{M}$-functors such that $GF \approx 1_{(\mathbf{C}, \mathcal{M})}$ and $FG \approx 1_{(\mathbf{D}, \mathcal{N})}$ as each equivalence of categories preserves pullbacks. ☐

By the same process used in Proposition 1.6.18 and paying attention to the 2-cells, we have:

Proposition 1.6.25 *There is a 2-functor* $\mathsf{Par} : \mathcal{M}\mathbf{Catl} \to \mathbf{rCatl}_s$ *taking* $F \overset{\alpha}{\to} G : (\mathbf{C}, \mathcal{M}) \to (\mathbf{C}', \mathcal{M}')$ *to* $\mathsf{Par}(F) \overset{\mathsf{Par}(\alpha)}{\to}: \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to \mathsf{Par}(\mathbf{C}', \mathcal{M}')$*, where* $\mathsf{Par}(F)(m, f) = (F(m), F(f))$ *and* $\mathsf{Par}(\alpha)_X = (1_{F(X)}, \alpha_X)$*.*

Proof: Given a 2-cell $F \overset{\alpha}{\to} G : (\mathbf{C}, \mathcal{M}) \to (\mathbf{C}', \mathcal{M}')$, since $F(\mathcal{M}) \subseteq \mathcal{M}'$, $\mathsf{Par}(F)$ is well-defined. Since F preserves pullbacks along $\mathcal{M}$-maps, it is easy to see that $\mathsf{Par}(F) : \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to \mathsf{Par}(\mathbf{C}', \mathcal{M}')$ is a restriction functor.

We must check the lax naturality of $\mathsf{Par}(\alpha)$, that is, for each $\mathsf{Par}(\mathbf{C}, \mathcal{M})$-map $(m, f) : X \to Y$,

$$\begin{tikzcd}
F(X) \arrow[r, "(1_{F(X)}{,}\alpha_X)"] \arrow[d, "(F(m){,}F(f))"'] \arrow[dr, phantom, "\leq"] & G(Y) \arrow[d, "(G(m){,}G(f))"] \\
F(Y) \arrow[r, "(1_{F(Y)}{,}\alpha_Y)"] & G(Y).
\end{tikzcd}$$

But it is clear as the following composites of partial maps are the same:

$$\begin{tikzcd}
 & & F(X') \arrow[dl, "1_{F(X')}"'] \arrow[dr, "F(f)"] & & \\
 & F(X') \arrow[dl, "F(m)"'] \arrow[dr, "F(f)"] & & F(Y) \arrow[dl, "1_{F(Y)}"'] \arrow[dr, "\alpha_Y"] & \\
F(X) & & F(Y) & & G(Y)
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & F(X') \arrow[dl, "1_{F(X')}"'] \arrow[ddrr, "\alpha_{X'}"] & & & \\
 & & F(X') \arrow[dl, "1_{F(X')}"'] \arrow[dr, "F(m)"] & & & & \\
 & F(X') \arrow[dl, "F(m)"'] \arrow[dr, "F(m)"] & & F(X) \arrow[dl, "1_{F(X)}"'] \arrow[dr, "\alpha_X"] & & G(X') \arrow[dl, "G(m)"'] \arrow[dr, "G(f)"] & \\
F(X) & & F(X) & & G(Y) & & G(Y)
\end{tikzcd}$$

as

$$\begin{tikzcd}
F(X') \arrow[rr, "\alpha_{X'}"] \arrow[d, "1_{F(X')}"'] & & G(X') \arrow[d, "G(m)"] \\
F(X') \arrow[r, "F(m)"] & F(X) \arrow[r, "\alpha_X"] & G(X)
\end{tikzcd}$$

is a pullback diagram.

Now, it is straightforward to verify the functoriality conditions for Par to be a 2-functor. ☐

Define $\mathcal{M}\mathsf{Total} : \mathbf{rCatl}_s \to \mathcal{M}\mathbf{Catl}$ by sending each restriction functor $F : \mathbf{X} \to \mathbf{Y}$ between restriction categories with split restrictions to $\mathcal{M}\mathsf{Total}(F) : (\mathsf{Total}(\mathbf{X}), \mathcal{M}_{\mathbf{X}}) \to \mathsf{Total}(\mathbf{Y}, \mathcal{M}_{\mathbf{Y}})$, where $\mathcal{M}\mathsf{Total}(F) = F|_{\mathsf{Total}(\mathbf{X})}$. By Lemma 1.6.21, pullbacks along $\mathcal{M}_{\mathbf{X}}$-maps in $\mathsf{Total}(\mathbf{X})$ is completely determined algebraically, $\mathcal{M}\mathsf{Total}(F)$ preserves such pullbacks. Hence $\mathcal{M}\mathsf{Total}(F)$ is an $\mathcal{M}$-functor.

For each lax restriction natural transformation $F \overset{\alpha}{\to} G : \mathbf{X} \to \mathbf{Y}$, define

$$\mathcal{M}\mathsf{Total}(\alpha)_X = \alpha_X$$

for each $X \in \mathrm{ob}(\mathsf{Total}(\mathbf{X}))$. The naturality of $\mathcal{M}\mathsf{Total}(\alpha)$ is given by the following commutative square

$$\begin{tikzcd}
F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\alpha_Y"] & G(Y)
\end{tikzcd}$$

as $F(\overline{f}) = 1$ in $\mathsf{Total}(\mathbf{X})$. These data form a 2-functor $\mathcal{M}\mathsf{Total} : \mathbf{rCatl}_s \to \mathcal{M}\mathbf{Catl}$.

Using the same proof process as in Theorem 1.6.22, we have:

Theorem 1.6.26 *The* 2*-functors* $\mathcal{M}\mathsf{Total}$ *and* Par *give an equivalence of* 2*-categories between* $\mathbf{rCatl}_s$ *and* $\mathcal{M}\mathbf{Catl}$.

Proof: In order to prove $\mathsf{Par} \circ \mathcal{M}\mathsf{Total} \cong 1_{\mathbf{rCatl}_s}$, for each restriction category with split restriction structure $\mathbf{D}$, we define $\Phi_{\mathbf{D}} : \mathbf{D} \to \mathsf{Par}(\mathsf{Total}(\mathbf{D}), \mathcal{E}_{\mathbf{D}}, \mathcal{M}_{\mathbf{D}})$ by

$$\begin{tikzcd}
A \arrow[d, "f"'] & & & A' \arrow[dl, "m"'] \arrow[drr, "fm"] & & \\
B & \mapsto & A & & & B
\end{tikzcd}$$

where m is determined by the conditions $\overline{f} = mr = \overline{r}$ and $rm = 1$. Since

$$\overline{fm} = \overline{\overline{f}m} = \overline{mrm} = \overline{m} = 1,$$

$\Phi_{\mathbf{D}}$ is well-defined. Clearly $\Phi_{\mathbf{D}}(1_A) = (1_A, 1_A)$ for each object A. For any maps $f : A \to B$ and $g : B \to C$ in $\mathbf{D}$, assume that $\overline{f} = mr = \overline{r}, rm = 1$ and $\overline{g} = ns = \overline{s}, sn = 1$. Write $\overline{sfm} = n's' = \overline{s'}$ with $s'n' = 1$ and let $f' = sfmn'$. Then, by Lemma 1.6.21, (n', f') is a pullback of (n, fm) and so $\Phi_{\mathbf{D}}(g)\Phi_{\mathbf{D}}(f) = (mn', f'gn)$ since $(*)$ is a pullback:

$$\begin{tikzcd}
 & & B'' \arrow[dl, "n'"'] \arrow[dr, "f'"] & & \\
 & A' \arrow[dl, "m"'] \arrow[dr, "fm"'] \arrow[rr, phantom, "(*)"] & & B' \arrow[dl, "n"] \arrow[dr, "gn"] & \\
A & & B & & C.
\end{tikzcd}$$

But

$$mn's'r = m\overline{sfm}r = mr\overline{nsfmr} = \overline{f}\,\overline{gf} = \overline{gf}$$

and $(s'r)(mn') = 1$, so mn' is the monic part of $\overline{gf}$. Notice that $gnf' = gnsfmn' = g\overline{g}fmn' = gfmn'$. Then $\Phi_{\mathbf{D}}(gf) = (mn', gfmn') = \Phi_{\mathbf{D}}(g)\Phi_{\mathbf{D}}(f)$. Hence $\Phi_{\mathbf{D}}$ is a functor. Since $\Phi_{\mathbf{D}}$ is the identity on objects, to prove $\mathsf{Par} \circ \mathcal{M}\mathsf{Total} \cong 1_{\mathbf{rCatl}_s}$, it suffices to show that $\Phi_{\mathbf{D}}$ is full and faithful. If (m, f) is a map in $\mathsf{Par}(\mathsf{Total}(\mathbf{D}), \mathcal{M}_{\mathbf{D}})$, then there exists a unique map r such that $rm = 1$ and $mr = \overline{mr}$ and so $\Phi_{\mathbf{D}}(fr) = (m, frm) = (m, f)$ which means that $\Phi_{\mathbf{D}}$ is full. On the other hand, $\Phi_{\mathbf{D}}(g) = (m, f)$ yields $gm = f$ and $mr = \overline{g}$ so that $fr = gmr = g\overline{g} = g$. Faithfulness of $\Phi_{\mathbf{D}}$ follows, as desired.

For an $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, since the total maps in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ are the same as $\mathbf{C}$ and the monic parts of restriction idempotent in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ are just $\mathcal{M}$, we clearly have an isomorphism $\mathcal{M}\mathsf{Total} \circ \mathsf{Par} \cong 1_{\mathcal{M}\mathbf{Catl}}$. Thus, Total and Par are part of an equivalence of 2-categories between $\mathbf{rCatl}_s$ and $\mathcal{M}\mathbf{Catl}$. ☐

1.6.6 Cockett-Lack's Free Restriction Categories

Cockett and Lack [14] gave a large class of examples of restriction categories by providing a left adjoint to the forgetful functor $U_r : \mathbf{rCat}_0 \to \mathbf{Cat}_0$. They proved:

Proposition 1.6.27 ([14], Subsection 2.2.1) *There is a left adjoint F_r to the forgetful functor $U_r : \mathbf{rCat}_0 \to \mathbf{Cat}_0$.*

Explicitly, Cockett-Lack's free restriction categories over categories are described as follows.

Let $\mathbf{C}$ be a category, $K = \{f_i : X \to Z_i \mid i \in I\}$ a set of maps with domain X, and $g : Y \to X$ a map. Then we write Kg for the set $\{f_i g \mid i \in I\}$, and $\Downarrow(K)$ for the set $\{f : X \to Z \mid uf = f_i$ for some $i \in I$ and some $u : Z \to Z_i\}$. Suppose that K and L are sets of maps with domain X. Clearly, if $K \subseteq L$ then $\Downarrow(K) \subseteq \Downarrow(L)$. One has the following lemma.

Lemma 1.6.28 *For any category, $\Downarrow(\_)$ is a Kuratowski closure operator on the maps with domain X. Namely, if K, K_1, and K_2 are sets of maps with domain X, then*

$$\Downarrow(\emptyset) = \emptyset, \Downarrow(K_1 \cup K_2) = \Downarrow(K_1) \cup \Downarrow(K_2), K \subseteq \Downarrow(K), \Downarrow(\Downarrow(K)) = \Downarrow(K).$$

So, $\Downarrow(\_)$ endows $X/\mathbf{C}$ with a topology: the closed sets of this topology are the sets $\Downarrow(K)$ while the open sets (the complements of the closed sets) are *sieves* that are sets O such that $f \in O$ implies $uf \in O$. A map $f : X \to Y$ of $\mathbf{C}$ induces a map $\Downarrow(f)$ in the reverse direction between these topological spaces $Y/\mathbf{C}$ and $X/\mathbf{C}$:

$$\Downarrow(f) : Y/\mathbf{C} \to X/\mathbf{C}; h \mapsto hf.$$

Moreover, $\Downarrow(f) : Y/\mathbf{C} \to X/\mathbf{C}$ is a continuous map.

Now, we can form $\Downarrow : \mathbf{C}^{\mathrm{op}} \to \mathbf{Top}$ by

$$\begin{tikzcd}
Y \arrow[d, "f"'] & \mapsto & Y/\mathbf{C} \arrow[d, "\Downarrow(f)"] \\
X & \mapsto & X/\mathbf{C}
\end{tikzcd}$$

Now, it is routine to verify that $\Downarrow : \mathbf{C}^{\mathrm{op}} \to \mathbf{Top}$ is a functor. So we have:

Proposition 1.6.29 $\Downarrow : \mathbf{C}^{\mathrm{op}} \to \mathbf{Top}$ *is a functor.*

The Cockett-Lack's free restriction category can be generated by certain free fibrations as in [9]. Explicitly, given a category $\mathbf{C}$, Cockett-Lack's free restriction category $F_r(\mathbf{C})$ has:

- the same objects as $\mathbf{C}$;
- a map from C to D being a pair of $(f, \Downarrow(K))$, where $f : C \to D$ is a map of $\mathbf{C}$ and K is a finite set of maps in $\mathbf{C}$ with domain C such that

$$f \in \Downarrow(K);$$

- the composition given by

$$\begin{aligned}
(g, \Downarrow(L))(f, \Downarrow(K)) &= (gf, \Downarrow(\Downarrow(K) \cup (\Downarrow f)\Downarrow(L))) \\
&= (gf, \Downarrow(K \cup Lf));
\end{aligned}$$

- the identities given by

$$1_C = (1_C, \Downarrow\{1_C\});$$

- $\overline{(f, \Downarrow(K))} = (1, \Downarrow(K))$.

Chapter 2

Cartesian Restriction Categories

Recall that a *cartesian object* in a 2-category with finite products is an object $\mathbf{X}$ such that both the diagonal $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$, sending $f : X \to Y$ to $(f, f) : (X, X) \to (Y, Y)$, and the unique functor $! : \mathbf{X} \to \mathbf{1}$, sending each $f : X \to Y$ to $1_1 : 1 \to 1$, have right adjoints. In Section 4 of [16], Cockett and Lack pointed out that cartesian objects in **rCat** do not give the right notion of products in restriction categories since a cartesian object $\mathbf{X}$ in **rCat** must have a trivial restriction. To get the correct notion, Cockett and Lack studied cartesian objects in **rCatl** which does give the right notion of partial (restriction) products in restriction categories. In this chapter, we shall first investigate cartesian objects in a number of 2-categories and then construct a free structure that can provide finite partial products to restriction categories.

2.1 Cartesian Objects

In this section, we shall study cartesian objects in 2-categories related to restriction categories, such as, **rCat**, **rCatl**, $\mathcal{M}$**Cat**, $\mathcal{M}$**Catl** and when a partial map category is a cartesian object.

2.1.1 Cartesian Objects in **Cat**, **rCat**, and **rCatl**

As is well-known, a cartesian object in **Cat** can be characterized by finite products as explained in the following proposition.

Proposition 2.1.1 (Propositions 3.17 and 3.22 [6]) *Given a category* $\mathbf{X}$*,*

(*i*) $! : \mathbf{X} \to \mathbf{1}$ *has a right adjoint in* **Cat** *if and only if* $\mathbf{X}$ *has a terminal object;*

(ii) $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ *has a right adjoint in* **Cat** *if and only if* $\mathbf{X}$ *has binary products;*

(iii) $\mathbf{X}$ *is a cartesian object in* **Cat** *if and only if* $\mathbf{X}$ *has finite products.*

In [16], Cockett and Lack observed:

Lemma 2.1.2 *Given a restriction category* $\mathbf{X}$*,*

(i) *the unique restriction functor* $! : \mathbf{X} \to \mathbf{1}$ *has a right adjoint in* **rCat** *if and only if* $\mathbf{X}$ *has a terminal object as a category and* $\mathbf{X}$ *has the trivial restriction;*

(ii) *the diagonal* $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ *has a right adjoint in* **rCat** *if and only if* $\mathbf{X}$ *has binary products as a category and* $\mathbf{X}$ *has the trivial restriction;*

(iii) $\mathbf{X}$ *is a cartesian object in* **rCat** *if and only if* $\mathbf{X}$ *is a cartesian object in* **Cat** *and* $\mathbf{X}$ *has the trivial restriction.*

PROOF:

(i) If the unique restriction functor $! : \mathbf{X} \to \mathbf{1}$ has a right adjoint U with the unit $t : 1 \to U!$ in **rCat**, then for each $\mathbf{X}$-object X and any $\mathbf{X}$-map $f : X \to U(1)$ there is a unique $\mathbf{1}$-map $f^* :!(X) \to 1$ such that

$$\begin{tikzcd}
X \arrow[r, "t_X"] \arrow[dr, "f"'] & U(!(X)) \arrow[d, "U(f^*)"] & !(X) \arrow[d, dotted, "\exists! f^*"] \\
& U(1) & 1
\end{tikzcd}$$

commutes and so $\mathbf{X}$ has a terminal object $U(1)$ and for each $\mathbf{X}$-object X the unique map $t_X : X \to U(1)$, as a component of the unit $t : 1 \to U!$, is total. Furthermore, for each $\mathbf{X}$-map $f : X \to Y$, $t_Y f = t_X$:

$$\begin{tikzcd}
X \arrow[r, "t_X"] \arrow[d, "f"'] & U(1) \arrow[d, equal] \\
Y \arrow[r, "t_Y"] & U(1)
\end{tikzcd}$$

and so $\overline{f} = \overline{\overline{t_Y} f} = \overline{t_Y f} = \overline{t_X} = 1_X$. Hence each $\mathbf{X}$-map f must be total.

Conversely, if $\mathbf{X}$ has a terminal object T as a category and has the trivial restriction, then $! : \mathbf{X} \to \mathbf{1}$ has a right adjoint in **Cat** by Proposition 2.1.1(i) and so $\mathsf{Triv}(!) : \mathsf{Triv}(\mathbf{X}) \to \mathsf{Triv}(\mathbf{1})$ has a right adjoint in **rCat** applying 2-functor $\mathsf{Triv} : \mathbf{Cat} \to \mathbf{rCal}$. But $\mathsf{Triv}(!) : \mathsf{Triv}(\mathbf{X}) \to \mathsf{Triv}(\mathbf{1})$ is the same as $! : \mathbf{X} \to \mathbf{1}$ as $\mathbf{X}$ has the trivial restriction. So $! : \mathbf{X} \to \mathbf{1}$ has a right adjoint in **rCat**.

(ii) Suppose now that $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ has a right adjoint U with the unit $\eta : 1 \to U\Delta$ and counit $\varepsilon : \Delta U \to 1$ in **rCat**. Then for any $\mathbf{X}$-object X, any $\mathbf{X} \times \mathbf{X}$-object (Y, Z), and any $\mathbf{X} \times \mathbf{X}$-map $(f, g) : \Delta(X) \to (Y, Z)$, there is a unique $\mathbf{X}$-map $(f, g)^\sharp : X \to U(Y, Z)$ such that

$$\begin{tikzcd}
X \arrow[d, dotted, "(f,g)^\sharp"'] & \Delta(X) \arrow[d, "\Delta((f,g)^\sharp)"'] \arrow[dr, "(f,g)"] & \\
U(Y,Z) & \Delta U(Y,Z) \arrow[r, "\varepsilon_{(Y,Z)}"'] & (Y,Z)
\end{tikzcd}$$

commutes. Hence $\mathbf{X}$ has binary products $U(Y, Z)$ as $Y \times Z$ for all objects Y, Z and the projections, as the components of ε, are total and $\overline{f \times g} = \overline{f} \times \overline{g}$.

For each $\mathbf{X}$-map $f : X \to Y$, since $1_X \times f = \langle \pi_1, f\pi_2 \rangle$, we have the following commutative diagram:

$$\begin{tikzcd}
& X \arrow[dl, "1_X"'] \arrow[d, "\Delta_X=\langle 1_X,1_X\rangle"] \arrow[dr, "1_X"] & \\
X \arrow[d, "1_X"'] & X \times X \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\langle \pi_1,f\pi_2\rangle=1_X\times f"] & X \arrow[d, "f"] \\
X & X \times Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Y
\end{tikzcd}$$

Then

$$\overline{1_X \times f} = \overline{\overline{\pi_1}(1_X \times f)} = \overline{\pi_1(1_X \times f)} = \overline{\pi_1} = 1_{X \times X}$$

and so

$$\overline{f} = \overline{f}\pi_2\Delta_X = \pi_2\overline{f\pi_2}\Delta_X = \pi_2\overline{\pi_2(1_X \times f)}\Delta_X = \pi_2\overline{1_X \times f}\Delta_X = \pi_2\Delta_X = 1_X.$$

Thus, f must be total again.

Conversely, if $\mathbf{X}$ has binary products as a category and has the trivial restriction then $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ has a right adjoint in $\mathbf{Cat}$ by Proposition 2.1.1(*ii*). As we did in (*i*), $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ has also a right adjoint in $\mathbf{rCat}$ by applying $\mathsf{Triv} : \mathbf{Cat} \to \mathbf{rCat}$.

(*iii*) By (*i*) and (*ii*) above and Proposition 2.1.1.

$\square$

By the last proposition, if a restriction category $\mathbf{X}$ is a cartesian object in $\mathbf{rCat}$, then $\mathbf{X}$ must have the trivial restriction. To give a proper notion of partial (restriction) products in a given restriction category, Cockett and Lack [16] studied cartesian objects in the category $\mathbf{rCatl}$ instead of $\mathbf{rCat}$. Recall that 2-category $\mathbf{rCatl}$ has restriction categories as 0-cells, restriction functors as 1-cells, and lax natural transformations between 1-cells as 2-cells.

Cockett and Lack [16] defined a *partial (restriction) terminal object* in a restriction category $\mathbf{X}$ to be an object T for which for each $\mathbf{X}$-object X there is a unique total map $!_X : X \to T$ such that $!_T = 1_T$ and $!_Y f = !_X \overline{f}$ for each $\mathbf{X}$-map $f : X \to Y$:

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, "!_X"] \arrow[dr, phantom, "\leq"'] & \\
Y \arrow[r, "!_Y"'] & T
\end{tikzcd}$$

As well, in [16], a *binary partial (restriction) product* of two objects X, Y in a restriction category $\mathbf{X}$ is a $\mathbf{X}$-object $X \times Y$ equipped with two total $\mathbf{X}$-maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, called *projections*, such that for each pair of $\mathbf{X}$-maps $f : Z \to X$ and $g : Z \to Y$ there is a unique map $\langle f, g\rangle : Z \to X \times Y$ satisfying $\pi_X\langle f, g\rangle \leq f$,

$\pi_Y\langle f,g\rangle \leq g$, and $\overline{\langle f,g\rangle} = \overline{f}\,\overline{g}$:

$$\begin{tikzcd}
 & Z \arrow[ld, "f"'] \arrow[rd, "g"] \arrow[d, dotted, "{\langle f,g\rangle}"] & \\
X \arrow[r, phantom, "\geq", shift left=3] & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] \arrow[r, phantom, "\leq", shift left=3] & Y
\end{tikzcd}$$

Similarly, one can define a partial product of any finite objects $X_1, \cdots, X_n$.

A restriction category is called *cartesian* if it has all binary partial (restriction) products and a partial (restriction) terminal object.

A restriction functor $F : \mathbf{X} \to \mathbf{Y}$ between two cartesian restriction categories is called *cartesian* if it preserves both partial (restriction) terminal objects and binary partial (restriction) products. All cartesian restriction categories and cartesian restriction functors between them form a category, denoted by $\mathbf{crCat}_0$.

For partial terminal objects, we have:

Lemma 2.1.3 *For a restriction category* $\mathbf{X}$*, the following are equivalent:*

(*i*) $\mathbf{X}$ *has a partial terminal object* T*;*

(*ii*) *there is a* $\mathbf{X}$*-object* T *such that for each* $\mathbf{X}$*-object* X*, there is a unique total map* $!_X : X \to T$ *such that for each* $\mathbf{X}$*-map* $f : X \to T$*,*

$$f = !_X\overline{f};$$

(*iii*) *there is a* $\mathbf{X}$*-object* T *such that for each* $\mathbf{X}$*-object* X*, there is a total map* $!_X : X \to T$ *such that for each* $\mathbf{X}$*-map* $f : X \to T$

$$f = !_X\overline{f};$$

(*iv*) *there is a* $\mathbf{X}$*-object* T *such that for each* $\mathbf{X}$*-object* X*,* $\mathrm{map}_{\mathbf{X}}(X,T)$ *has a top element* $!_X$ *that is total;*

(*v*) $! : \mathbf{X} \to \mathbf{1}$ *has a right adjoint* $U : \mathbf{1} \to \mathbf{X}$ *in* $\mathbf{rCatl}$.

Proof: $(i) \Rightarrow (ii)$: Consider the following diagram:

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, "!_X"] & \\
1 \arrow[r, "!_1"'] \arrow[ur, phantom, "\leq"] & 1
\end{tikzcd}$$

Then, by (i),

$$f = 1_1 f = !_1 f = !_X \overline{f}.$$

$(ii) \Rightarrow (i)$: For each **X**-map $f : X \to Y$, consider the **X**-map $!_Y f : X \to Y \to 1$. We have

$$!_Y f = !_X \overline{!_Y f} = !_X \overline{\overline{!_Y} f} = !_X \overline{f}.$$

For the **X**-map $1_1 : 1 \to 1$, by (ii),

$$1_1 = !_1 \overline{1_1} = !_1.$$

Hence, the unique total map $!_X : X \to 1$ satisfies $!_1 = 1_1$ and $!_Y f = !_X \overline{f}$ for each **X**-map $f : X \to Y$:

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, "!_X"] & \\
Y \arrow[r, "!_Y"'] \arrow[ur, phantom, "\leq"] & 1
\end{tikzcd}$$

$(i) \Rightarrow (v)$: Define $U : \mathbf{1} \to \mathbf{X}$ by sending $1_1 : 1 \to 1$ to $1_T : T \to T$.

For each **X**-object X, let $\eta_X = !_X : X \to U!(X)$. Then, for each **X**-map $f : X \to Y$, by (i), we have $!_Y f = !_X \overline{f}$ and so

$$\begin{tikzcd}
X \arrow[r, "!_X"] \arrow[d, "f"'] & T \arrow[d, "1_T"] \\
Y \arrow[r, "!_Y"] & T
\end{tikzcd}$$

commutes in **rCatl**. Hence $\eta : 1 \to U!$ is a lax restriction natural transformation. For each **X**-map $f : X \to U(1)$, by $(i) \Rightarrow (ii)$, we have $f = !_X \overline{f}$, which gives rise to the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & U!X \arrow[d, "U(f^*)"] & !X \arrow[d, dotted, "f^*=1_1"'] \\
& U(1) & 1
\end{tikzcd}$$

in **rCatl**. Hence each $\eta_X : X \to U!(X)$ is a universal arrow from X to U and therefore $! \dashv U$ in **rCatl**.

$(v) \Rightarrow (iii)$: Suppose that $! : \mathbf{X} \to \mathbf{1}$ has a right adjoint $U : \mathbf{1} \to \mathbf{X}$. Let $T = U(1)$. Then there is a total $\mathbf{X}$-map $!_X = \eta_X : X \to U!(X)$ such that for each $\mathbf{X}$-maps $f : X \to U(1)$,

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & U!X \arrow[d, "U(f^*)"] & !X \arrow[d, dotted, "f^*=1_1"'] \\
& U(1) & 1
\end{tikzcd}$$

commutes in **rCatl**. That is, $f = 1_X\overline{f}$. Hence (iii) follows.

$(ii) \Rightarrow (iii)$: Clear.

$(iii) \Rightarrow (ii)$: If $t_X : X \to T$ is a total map such that for each $\mathbf{X}$-map $f : X \to T$, $f = t_X\overline{f}$, then

$$!_X = t_X\overline{!_X} = t_X 1_X = t_X$$

and so the uniqueness of $!_X$ follows.

$(iii) \Leftrightarrow (iv)$: Clear. □

If the diagonal $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ has a right adjoint $U : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ in **rCatl**, then, as Cockett and Lack did in [16], we denote $U(Y, Z)$ by $Y \times Z$ and $U(f, g)$ by $f \times g$. Since $\Delta \dashv U$, for each $\mathbf{X} \times \mathbf{X}$-map $(f, g) : \Delta(X) \to (Y, Z)$ there is a unique $\mathbf{X}$-map $(f, g)^\sharp : X \to Y \times Z$ such that

$$\begin{tikzcd}
X \arrow[d, dotted, "\exists!(f{,}g)^\sharp"'] & \Delta(X) \arrow[d, "\Delta((f{,}g)^\sharp)"'] \arrow[dr, "(f{,}g)"] & \\
Y \times Z & \Delta(Y \times Z) \arrow[r, "(\pi_Y{,}\pi_Z)"'] & (Y, Z)
\end{tikzcd}$$

commutes in **rCatl**. Adjunction $\Delta \dashv U$ has some properties as proved in the following lemma.

Lemma 2.1.4 ([16], Proposition 4.3) *If the diagonal* $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ *has a right adjoint* $U : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ *in* **rCatl***, then*

(i) *for each* $\mathbf{X}$*-object* X*, there is a total* $\mathbf{X}$*-map* $\Delta_X : X \to X \times X$ *such that*

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[d, dotted, "\Delta_X"] \arrow[dr, "1_X"] & \\ X & X \times X \arrow[l, "p_X"] \arrow[r, "q_X"'] & X \end{tikzcd}$$

commutes in $\mathbf{X}$*;*

(ii) $(f \times g)\Delta_X = (f, g)^\sharp$ *if* $\mathrm{dom}(f) = \mathrm{dom}(g) = X$*;*

(iii) $(fh, gh)^\sharp = (f, g)^\sharp h$ *if* $\mathrm{cod}(h) = \mathrm{dom}(f) = \mathrm{dom}(g)$*;*

(iv) $\overline{(f \times g)\Delta_X} = \overline{f}\,\overline{g}$ *if* $\mathrm{dom}(f) = \mathrm{dom}(g) = X$*;*

(v) $\Delta_X : X \to X \times X$ *is natural in* X*.*

PROOF:

(i) Since

$$\Delta X \xrightarrow{\Delta(\eta_X)} \Delta U \Delta X \xrightarrow{\varepsilon_{\Delta X}} \Delta X = \Delta X \xrightarrow{1_{\Delta X}} \Delta X$$

for each $\mathbf{X}$-object X by the adjunction $\Delta \dashv U$, there is a unique total $\mathbf{X}$-map $(1_{\Delta X})^\sharp = \eta_X$ such that

$$\begin{tikzcd} \Delta X \arrow[d, "\Delta(\eta_X)"'] \arrow[dr, "1_{\Delta X}"] & \\ \Delta U \Delta X \arrow[r, "\varepsilon_{\Delta X}"] & \Delta X \end{tikzcd}$$

commutes in **rCatl**. But $\varepsilon_{\Delta X}\Delta((1_{\Delta X})^\sharp) \leq 1_{\Delta X}$ implies $\varepsilon_{\Delta X}\Delta((1_{\Delta X})^\sharp) = 1_{\Delta X}$. It follows that there is a total **X**-map $\Delta_X = \eta_X : X \to X \times X$ such that

$$\begin{tikzcd}
 & X \arrow[dl, "1_X"'] \arrow[dr, "1_X"] \arrow[d, dotted, "\Delta_X"] & \\
X & X \times X \arrow[l, "p_X"] \arrow[r, "q_X"'] & X
\end{tikzcd}$$

commutes, where $(p_X, q_X) = \varepsilon_{\Delta X}$ is total in $\mathbf{X} \times \mathbf{X}$.

(ii) Since $\varepsilon : \Delta U \to 1$ is a lax natural transformation in **rCatl**, for each $\mathbf{X} \times \mathbf{X}$-map $(f, g) : (X, Y) \to (X', Y')$,

$$\begin{tikzcd}
\Delta U(X,Y) \arrow[r, "\Delta U\overline{(f,g)}"] \arrow[d, "\Delta U(f,g)"'] & \Delta U(X,Y) \arrow[r, "\varepsilon_{(X,Y)}"] & (X,Y) \arrow[d, "(f,g)"] \\
\Delta U(X',Y') \arrow[rr, "\varepsilon_{(X',Y')}"] & & (X',Y')
\end{tikzcd}$$

commutes. That is

$$\begin{tikzcd}
X \times Y \arrow[d, "\pi_X"'] & X \times Y \arrow[l, "\overline{f}\times\overline{g}"'] \arrow[r, "\overline{f}\times\overline{g}"] \arrow[dd, "f\times g"] & X \times Y \arrow[d, "\pi_Y"] \\
X \arrow[d, "f"'] & & Y \arrow[d, "g"] \\
X' \times Y' & X' \times Y' \arrow[l, "\pi_{X'}"'] \arrow[r, "\pi_{Y'}"] & Y'
\end{tikzcd}$$

If $\mathrm{dom}(f) = \mathrm{dom}(g) = X$, then

$$\begin{aligned}
\pi_{X'}(f \times g)\Delta_X &= f\pi_X(\overline{f} \times \overline{g})\Delta_X \\
&= f\pi_X(\overline{f \times g})\Delta_X \\
&= f\pi_X\Delta\overline{(f \times g)\Delta_X} \\
&= f\overline{(f \times g)\Delta_X} \\
&= f\overline{\pi_{X'}(f \times g)\Delta_X}
\end{aligned}$$

and so $\pi_{X'}(f \times g)\Delta_X \leq f$. Similarly, $\pi_{Y'}(f \times g)\Delta_X \leq g$. Hence

$$\begin{tikzcd}
\Delta(X) \arrow[d, "\Delta((f\times g)\Delta_X)"'] \arrow[dr, "(f{,}g)"] \arrow[dr, phantom, "\leq"' near end] & \\
\Delta(X' \times Y') \arrow[r, "(\pi_{X'}{,}\pi_{Y'})"'] & (X', Y')
\end{tikzcd}$$

anf therefore $(f \times g)\Delta = (f, g)^\sharp$.

(iii) If $\mathrm{cod}(h) = \mathrm{dom}(f) = \mathrm{dom}(g)$, then we have the commutative diagram

$$\begin{tikzcd}
W \arrow[d, "h"] & \Delta(W) \arrow[d, "\Delta(h)"'] \arrow[ddr, "(fh{,}gh)"] & \\
Z \arrow[d, dotted, "\exists!(f{,}g)|\sharp"'] & \Delta(Z) \arrow[d, "\Delta(f{,}g)^\sharp"'] \arrow[dr, "(f{,}g)"] & \\
X \times Y & \Delta(X \times Y) \arrow[r, "(\pi_X{,}\pi_Y)"'] & (X, Y)
\end{tikzcd}$$

in **rCatl** and so

$$\varepsilon_{(Y,Z)}\Delta((f,g)^\sharp h) = \varepsilon_{(Y,Z)}\Delta((f,g)^\sharp)\Delta(h) \leq (f,g)\Delta(h) = (fh, gh).$$

Hence $(fh, gh)^\sharp = (f, g)^\sharp h$.

(iv) Proposition 4.3(i) in [16].

(v) Proposition 4.3(ii) in [16].

□

For the binary partial products, we have:

Lemma 2.1.5 *Given a restriction category* **X**, *the following are equivalent:*

(i) **X** *has binary partial products;*

(*ii*) *for each pair of* $\mathbf{X}$*-objects* X *and* Y*, there is a pair of total maps* $\pi_X : X \times Y \to X$ *and* $\pi_Y : X \times Y \to Y$ *such that for each pair of* $\mathbf{X}$*-maps* $f : Z \to X$ *and* $g : Z \to Y$ *there exists a unique* $\mathbf{X}$*-map* $\langle f, g\rangle : Z \to X \times Y$ *such that*

$$\pi_X\langle f, g\rangle = f\overline{g} \text{ and } \pi_Y\langle f, g\rangle = g\overline{f};$$

(*iii*) *for each pair of* $\mathbf{X}$*-objects* X *and* Y*, there is a pair of total maps* $\pi_X : X \times Y \to X$ *and* $\pi_Y : X \times Y \to Y$ *such that for each pair of* $\mathbf{X}$*-maps* $f : Z \to X$ *and* $g : Z \to Y$ *there exists a unique* $\mathbf{X}$*-map* $\langle f, g\rangle : Z \to X \times Y$ *such that*

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[d, dotted, "\langle f,g\rangle"] \arrow[dr, "g"] \arrow[dl, phantom, "\geq", shift left=3ex] \arrow[dr, phantom, "\leq"', shift right=3ex] & \\
X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y
\end{tikzcd}$$

(*iv*) $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ *has a right adjoint* $U : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ *in* **rCatl**.

Proof: (*i*) ⇒ (*ii*): By (*i*), there is a unique $\langle f, g\rangle : Z \to X \times Y$ such that

$$\pi_X\langle f, g\rangle \leq f,\ \pi_Y\langle f, g\rangle \leq g, \text{ and } \overline{\langle f, g\rangle} = \overline{f}\overline{g}.$$

Hence

$$\pi_X\langle f, g\rangle = f\overline{\pi_X\langle f, g\rangle} = f\overline{\overline{\pi_X}\langle f, g\rangle} = f\overline{\langle f, g\rangle} = f\overline{f}\overline{g} = f\overline{g}.$$

Similarly, we have $\pi_Y\langle f, g\rangle = g\overline{f}$.

(*ii*) ⇒ (*iii*): Clear.

(*iii*) ⇒ (*iv*): Define $U : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ by sending $(f, g) : (X, Y) \to (X', Y')$ to the unique $\mathbf{X}$-map $f \times g = \langle f\pi_X, g\pi_Y\rangle : X \times Y \to X' \times Y'$ satisfying

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, phantom, "\geq"] & X \times Y \arrow[l, "\pi_X"'] \arrow[r, "\pi_Y"] \arrow[d, "f\times g"] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "g"] \\
X' & X' \times Y' \arrow[l, "\pi_{X'}"] \arrow[r, "\pi_{Y'}"'] & Y'
\end{tikzcd}$$

Let $\varepsilon_{(X,Y)} = (\pi_X, \pi_Y) : \Delta U(X,Y) \to (X,Y)$. Clearly, the diagram in (iii) gives rise to the following commutative diagram

$$\begin{tikzcd}
\Delta U(X,Y) \arrow[r, "\varepsilon_{(X,Y)}"] \arrow[d, "\Delta U(f,g)"'] & (X,Y) \arrow[d, "(f,g)"] \\
\Delta U(X',Y') \arrow[r, "\varepsilon_{(X',Y')}"] & (X',Y')
\end{tikzcd}$$

in **rCatl**. Hence $\varepsilon : \Delta U \to 1$ is a lax restriction natural transformation. Obviously, by the diagram in (iii), $\varepsilon_{(X,Y)} = (\pi_X, \pi_Y) : \Delta U(X,Y) \to (X,Y)$ is a universal arrow from Δ to (X,Y). Thus, $\Delta \dashv U$ in **rCatl**.

$(iv) \Rightarrow (i)$: The diagonal $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ has a right adjoint if and only if there is a universal arrow from Δ to each given (X,Y), namely, for each $\mathbf{X}\times\mathbf{X}$-map $\Delta Z \to (X,Y)$ there is a unique $\mathbf{X}$-map $\langle f,g\rangle : Z \to X \times Y$ such that

$$\begin{tikzcd}
Z \arrow[d, dotted, "\exists!\langle f,g\rangle"'] & \Delta(Z) \arrow[d, "\Delta(\langle f,g\rangle)"'] \arrow[dr, "(f,g)"] & \\
X \times Y & \Delta(X \times Y) \arrow[r, "\varepsilon_{(X,Y)}"] & (X,Y)
\end{tikzcd}$$

commutes in **rCatl**. This is equivalent to the diagram (i) above with $\varepsilon_{(X,Y)} = (\pi_X, \pi_Y)$. On the other hand, by Lemma 2.1.4,

$$\overline{\langle f,g\rangle} = \overline{(f,g)^\sharp} = \overline{f}\,\overline{g}.$$

Thus, $(iv) \Rightarrow (i)$. $\square$

By Lemmas 2.1.3 and 2.1.5, obviously we have:

Proposition 2.1.6 *Let* $\mathbf{X}$ *be a restriction category. Then* $\mathbf{X}$ *has finite partial products if and only if* $\mathbf{X}$ *is a cartesian object in* **rCatl**.

By Lemma 2.1.2 and Proposition 2.1.6, immediately one has:

Proposition 2.1.7 *If a restriction category* $\mathbf{X}$ *is a cartesian object in* **rCat**, *then* $\mathbf{X}$ *is a cartesian object in* **rCatl** *and* $\mathbf{X}$ *has the trivial restriction.*

Proof: By Lemma 2.1.2, $\mathbf{X}$ has finite products and has the trivial restriction. By Proposition 2.1.6, it suffices to prove that $\mathbf{X}$ has a partial terminal object and binary partial products.

Let 1 be a terminal object in $\mathbf{X}$. Then there is a unique map $!_X : X \to 1$ that is total since $\mathbf{X}$ has the trivial restriction. For any $\mathbf{X}$-map $f : X \to Y$, since $\overline{f} = 1_X$, we have

$$!_Y f = !_X = !_X \overline{f}.$$

Hence 1 is a partial terminal object in $\mathbf{X}$.

For each pair of $\mathbf{X}$-objects X and Y, assume that $(X \times Y, \pi_X, \pi_Y)$ is a binary product of X and Y. Then, for each pair of $\mathbf{X}$-maps $f : Z \to X$ and $g : Z \to Y$ there is a unique map $\langle f, g\rangle : Z \to X \times Y$ such that

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[d, dotted, "\langle f,g\rangle"] \arrow[dr, "g"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

commutes. Since $\mathbf{X}$ has the trivial restriction, we have $\overline{\pi_1} = \overline{\pi_2} = 1_{X\times Y}$ and $\overline{\langle f, g\rangle} = 1_Z = \overline{f}\overline{g}$. Hence $(X \times Y, \pi_1, \pi_2)$ is a binary partial product of X and Y in $\mathbf{X}$. ☐

Recall that there are 2-functors $\mathsf{Total} : \mathbf{rCatl} \to \mathbf{Cat}$, $\mathsf{Total} : \mathbf{rCat} \to \mathbf{Cat}$, $\mathsf{Split} : \mathbf{rCat} \to \mathbf{rCat}$, and $\mathsf{Split} : \mathbf{rCatl} \to \mathbf{rCatl}$ and each 2-functor preserves adjunctions. So immediately one has:

Proposition 2.1.8 *Let $\mathbf{X}$ be a restriction category.*

- *(i) If $\mathbf{X}$ is a cartesian object in $\mathbf{rCat}$, then $\mathsf{Split}(\mathbf{X})$ is a cartesian object in $\mathbf{rCat}$ and both $\mathsf{Total}(\mathsf{Split}(\mathbf{X}))$ and $\mathsf{Total}(\mathbf{X})$ are cartesian objects in $\mathbf{Cat}$;*
- *(ii) If $\mathbf{X}$ is a cartesian object in $\mathbf{rCatl}$, then $\mathsf{Split}(\mathbf{X})$ is a cartesian object in $\mathbf{rCatl}$ and both $\mathsf{Total}(\mathsf{Split}(\mathbf{X}))$ and $\mathsf{Total}(\mathbf{X})$ are cartesian objects in $\mathbf{Cat}$.*

Proof: Clearly, $\mathsf{Total}(\mathbf{1}) \cong \mathbf{1}$, $\mathsf{Split}(\mathbf{1}) \cong \mathbf{1}$, $\mathsf{Total}(\mathbf{X} \times \mathbf{X}) \cong \mathsf{Total}(\mathbf{X}) \times \mathsf{Total}(\mathbf{X})$, $\mathsf{Split}(\mathbf{X}\times\mathbf{X}) \cong \mathsf{Split}(\mathbf{X})\times\mathsf{Split}(\mathbf{X})$, $\mathbf{Total}(!) \cong !$, and $\mathsf{Split}(!) \cong !$. The lemma immediately follows as 2-functors $\mathsf{Total} : \mathbf{rCatl} \to \mathbf{Cat}$, $\mathsf{Total} : \mathbf{rCat} \to \mathbf{Cat}$, $\mathsf{Split} : \mathbf{rCat} \to \mathbf{rCat}$, and $\mathsf{Split} : \mathbf{rCatl} \to \mathbf{rCatl}$ preserve adjoints. ☐

2.1.2 Partial Map Categories as Cartesian Objects

Given an $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, we have a restriction category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ that is a object of **Cat**, **rCat**, and **rCatl**. This subsection is intended to study when a partial map category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a cartesian object in **Cat**, **rCat**, and **rCatl**, respectively.

First, let's study when a partial map category is a cartesian object in **Cat**. Given an $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, by Proposition 2.1.1, $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a cartesian object in **Cat** if and only if $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ has finite products. The following lemma considers when a partial map category has terminal objects.

Lemma 2.1.9 *Given an $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, if 0 is a $\mathbf{C}$-object such that for each $\mathbf{C}$-object X*

$$\emptyset \neq \mathrm{map}_{\mathbf{C}}(0, X) \subseteq \mathcal{M},$$

then 0 is a terminal object in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ if and only if 0 is a strict initial object in $\mathbf{C}$.

Proof: If 0 is a terminal object in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, then for each $\mathbf{C}$-object X we have $m \in \mathrm{map}_{\mathbf{C}}(0, X) \subseteq \mathcal{M}$ and so the partial map $(m, 1_0) : X \to 0$ must be the unique map from X to 0 in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$. Clearly, 0 is initial in $\mathbf{C}$. For each map $t_X : X \to 0$, the partial map $(1_X, t_X) : X \to 0$ must be $(m, 1_0)$. Hence $t_X : X \to 0$ is an isomorphism and therefore 0 is strict.

Conversely, If 0 is a strict initial object in $\mathbf{C}$, then for each $\mathbf{C}$-object X we have a partial map $(m_X, 1_0) : X \to 0$, where $m_X : 0 \to X \in \mathrm{map}_{\mathbf{C}}(0, X) \subseteq \mathcal{M}$. For any partial map $(n, f) : X \to 0$,

$$\begin{tikzcd}
& X' \arrow[dl, "n"'] \arrow[dr, "f"] \arrow[dd, dotted, "f"] & \\
X & & 0 \\
& 0 \arrow[ul, "m_X"] \arrow[ur, "1_0"'] &
\end{tikzcd}$$

since 0 is strict, f is an isomorphism and X' is a strict initial object. It follows that $m_X f = n$. Hence $(n, f) = (m_X, 1_0)$ and so 0 is a terminal object in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$. $\square$

Now we consider when a partial map category can have binary products by providing an example that $\mathsf{Par}(\mathbf{Set}, \text{Monics}_{\mathbf{Set}})$ has binary products as shown in the following lemma.

Lemma 2.1.10 $\mathsf{Par}(\mathbf{Set}, \text{Monics}_{\mathbf{Set}})$ *has binary products,*

Proof: Given two sets X and Y, we claim that

$$(X + X \times Y + Y, (\iota_1, \langle 1_X | \pi_X \rangle), (\iota_2, \langle \pi_Y | 1_Y \rangle))$$

is a product of X and Y in $\mathsf{Par}(\mathbf{Set}, \text{Monics}_{\mathbf{Set}})$, where $\iota_1 : X + X \times Y \hookrightarrow X + X \times Y + Y$ and $\iota_2 : X \times Y + Y \hookrightarrow X + X \times Y + Y$ are the coproduct injections.

For partial maps $(m, f) : Z \to X$ and $(n, g) : Z \to Y$, without loss of generality, we assume that both m and n are set-inclusions. Define the partial function $(i, f \cup g) : Z \to X + X \times Y + Y$ by

$$(f \cup g)(x) = \begin{cases} \iota_X(f(x)) & \text{if } x \in X' \setminus Y'; \\ \iota_{X \times Y}(f(x), g(x)) & \text{if } x \in X' \cap Y'; \\ \iota_Y(g(x)) & \text{if } x \in Y' \setminus X'. \end{cases}$$

Here

$$\iota_X : X \hookrightarrow X + X \times Y + Y,$$

$$\iota_{X \times Y} : X \times Y \hookrightarrow X + X \times Y + Y,$$

and

$$\iota_Y : Y \hookrightarrow X + X \times Y + Y$$

are the coproduct injections. Now it is easy to show that

$$\begin{tikzcd}
 & Z \arrow[dl, "{(m,f)}"'] \arrow[d, "{(i,f\cup g)}"] \arrow[dr, "{(n,g)}"] & \\
X & X + X \times Y + Y \arrow[l, "{(\iota_1,\langle 1_X|\pi_X\rangle)}"] \arrow[r, "{(\iota_2,\langle \pi_Y|1_Y\rangle)}"'] & Y
\end{tikzcd}$$

commutes in $\mathsf{Par}(\mathbf{Set}, \mathrm{Monics}_{\mathbf{Set}})$ by looking at the following diagrams

$$\begin{tikzcd}
 & & X' \arrow[dl, hook] \arrow[dr, "f\cup g|_{X'}"] & & \\
 & X' \cup Y' \arrow[dl, hook, "i"'] \arrow[dr, "f\cup g"] & & X + X \times Y \arrow[dl, hook, "\iota_1"'] \arrow[dr, "\langle 1_X|\pi_X\rangle"] & \\
Z & & X + X \times Y + Y & & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & Y' \arrow[dl, hook] \arrow[dr, "f\cup g|_{Y'}"] & & \\
 & X' \cup Y' \arrow[dl, hook, "i"'] \arrow[dr, "f\cup g"] & & X \times Y + Y \arrow[dl, hook, "\iota_2"'] \arrow[dr, "\langle \pi_Y|1_Y\rangle"] & \\
Z & & X + X \times Y + Y & & Y
\end{tikzcd}$$

To prove the uniqueness of $(i, f \cup g) : Z \to X + X \times Y + Y$, let $(j, t) : Z \to X + X \times Y + Y$ be a partial function such that

$$\begin{tikzcd}
 & Z \arrow[dl, "{(m,f)}"'] \arrow[d, "{(j,t)}"] \arrow[dr, "{(n,g)}"] & \\
X & X + X \times Y + Y \arrow[l, "{(\iota_1,\langle 1_X|\pi_X\rangle)}"] \arrow[r, "{(\iota_2,\langle \pi_Y|1_Y\rangle)}"'] & Y
\end{tikzcd}$$

commutes in $\mathsf{Par}(\mathbf{Set}, \mathrm{Monics}_{\mathbf{Set}})$, with a set-inclusion j. Clearly, $t^{-1}(\iota_1(X+X\times Y)) = X'$ and $t^{-1}(\iota_2(X \times Y + Y)) = Y'$ imply $t(X' \cap Y') \subseteq X \times Y$ as $t(a) \in X$ or Y for some $a \in X' \cap Y'$ gives rise to $t^{-1}(\iota_1(X + X \times Y)) \neq X'$ or $t^{-1}(\iota_2(X \times Y + Y)) \neq Y'$. Hence

$t(X' \setminus Y') \subseteq X$. Similarly, $t(Y' \setminus X') \subseteq Y$. Therefore, in order to have

$$\begin{tikzcd}
 & & & & & & X' \arrow[dl, hook'] \arrow[dr, "t|_{X'}"] & & \\
 & X' \arrow[dl, "m"'] \arrow[dr, "f"] & & & & X' \cup Y' \arrow[dl, hook', "i"'] \arrow[dr, "t"] & & X + X \times Y \arrow[dl, hook', "\iota_1"'] \arrow[dr, "\langle 1_X | \pi_X \rangle"] & \\
Z & & X & = & Z & & X + X \times Y + Y & & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & & & & & Y' \arrow[dl, hook'] \arrow[dr, "t|_{X'}"] & & \\
 & Y' \arrow[dl, "n"'] \arrow[dr, "g"] & & & & X' \cup Y' \arrow[dl, hook', "i"'] \arrow[dr, "t"] & & X \times Y + Y \arrow[dl, hook', "\iota_2"'] \arrow[dr, "\langle \pi_Y | 1_Y \rangle"] & \\
Z & & Y & = & Z & & X + X \times Y + Y & & Y
\end{tikzcd}$$

$t = f \cup g$ and the uniqueness of $(i, f \cup g) : Z \to X + X \times Y + Y$ follows. ☐

By Lemmas 2.1.9 and 2.1.10, we have immediately

Corollary 2.1.11 $\mathsf{Par}(\mathbf{Set}, \mathrm{Monics}_{\mathbf{Set}})$ *has finite products so that it is a cartesian object in* **Cat**.

But $\mathsf{Par}(\mathbf{Set}_{\mathrm{ffib}}, \mathcal{M})$, given in Example 1.6.16, is not cartesian as it does not have partial terminal objects, where $\mathcal{M} = \{\text{injections } i : A \hookrightarrow B \mid |B \setminus i(A)| < +\infty\}$.

Now we turn to study when a partial map category is a cartesian object in **rCat**. Given an $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, by Proposition 2.1.1 and Lemma 2.1.2, $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a cartesian object in **rCat** if and only if $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ has finite products and $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ has the trivial restriction. But $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ has the trivial restriction if and only if $\mathcal{M} = \{\text{isomorphisms in } \mathbf{C}\}$. We have:

Proposition 2.1.12 $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ *is a cartesian object in* **rCat** *if and only if*

$$\mathcal{M} = \{\textit{isomorphisms in } \mathbf{C}\}$$

and $\mathbf{C}$ *has finite products.*

Proof: If $\mathcal{M} = \{\text{isomorphisms in } \mathbf{C}\}$, then $\mathsf{Par}(\mathbf{C}, \mathcal{M}) \cong \mathbf{C}$. By Proposition 2.1.7, $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a cartesian object in **rCat** if and only if $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ has finite products and $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ has the trivial restriction if and only if $\mathcal{M} = \{\text{isomorphisms in } \mathbf{C}\}$ and $\mathbf{C}$ has finite products as $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ has the trivial restriction if and only if $\mathcal{M} = \{\text{isomorphisms in } \mathbf{C}\}$ clearly. ☐

Finally, we study when a partial map category is a cartesian object in **rCatl**.

Proposition 2.1.13 *For a given $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, $\mathbf{C}$ is a cartesian object in* **Cat** *if and only if $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a cartesian object in* **rCatl***.*

Proof: If 1 is a terminal object of $\mathbf{C}$, then, for each $\mathbf{C}$-object X, there is a unique map $!_X : X \to 1$. Consider the total $\mathsf{Par}(\mathbf{C}, \mathcal{M})$-map $t_X = (1_X, !_X) : X \to 1$. For each $\mathsf{Par}(\mathbf{C}, \mathcal{M})$-map $(m, f) : X \to 1$, we have

$$t_X\overline{(m, f)} = (1_X, !_X)(m, m) = (m, !_X m) = (m, f)$$

since $!_X m =!_{X'} = f$ and $(*)$ is a pullback in the following diagram

$$\begin{tikzcd}
& & X' \arrow[dl, "1_{X'}"'] \arrow[dr, "m"] \arrow[ddrr, bend left, "f"] & & \\
& X' \arrow[dl, "m"'] \arrow[dr, "m"] & (*) & X \arrow[dl, "1_X"'] \arrow[dr, "!_X"] & \\
X & & X & & 1
\end{tikzcd}$$

Hence 1 is a partial terminal object in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ with $t_X = (1_X, !_X) : X \to 1$.

For each pair of $(\mathbf{C}, \mathcal{M})$-objects X and Y, assume that $(X \times Y, \pi_X, \pi_Y)$ is a product of X and Y in $\mathbf{C}$. We claim that $(X \times Y, (1_{X \times Y}, \pi_X), (1_{X \times Y}, \pi_Y))$ is a partial product of X and Y in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$.

In fact, for each pair of $(\mathbf{C}, \mathcal{M})$-maps $(m, f) : Z \to X$ and $(n, g) : Z \to Y$, assume the square (2) is a pullback in the following diagram:

$$\begin{tikzcd}
& & & P \arrow[dl, equal] \arrow[dr, equal] & & & \\
& & P \arrow[dl, "n'"'] \arrow[dr, "m'"] & (1) & P \arrow[dl, "m'"'] \arrow[dr, "n'"] & & \\
& X' \arrow[dl, "m"'] & & Y' \arrow[dr, "n"] & (2) & X' \arrow[dl, "m"'] \arrow[dr, "f"] & \\
Z & & & & Z & & X
\end{tikzcd}$$

Since (1) is a pullback square, we have

$$(m, f)(nm', mn') = (mn', fn').$$

Hence

$$\begin{aligned}
(1_{X\times Y}, \pi_X)(mn', \langle fn', gm'\rangle) &= (mn', \pi_X\langle fn', gm'\rangle) \\
&= (mn', fn') \\
&= (m, f)(n'm, mn') \\
&= (m, f)\overline{(mn', fn')} \\
&\leq (m, f).
\end{aligned}$$

Symmetrically,

$$(1_{X\times Y}, \pi_Y)(mn', \langle fn', gm'\rangle) \leq (n, g).$$

So we have

$$\begin{tikzcd}
& Z \arrow[dl, "{(m,f)}"'] \arrow[d, dotted, "{(mn',\langle fn',gm'\rangle)}"] \arrow[dr, "{(n,g)}"] & \\
X & X \times Y \arrow[l, "{(1_{X\times Y},\pi_X)}"] \arrow[r, "{(1_{X\times Y},\pi_Y)}"'] \arrow[from=1-2, to=2-1, phantom, "\geq" near end] \arrow[from=1-2, to=2-3, phantom, "\leq" near end] & Y
\end{tikzcd}$$

Clearly, both $(1_{X\times Y}, \pi_1)$ and $(1_{X\times Y}, \pi_2)$ are total and

$$\overline{(mn', \langle fn', gm'\rangle)} = (mn', mn') = (m, m)(n, n) = \overline{(m, f)}\ \overline{(n, g)}.$$

If $(k,h) : Z \to X \times Y$ is a partial map such that

$$(1_{X\times Y},\pi_X)(k,h) = (m,f)\overline{(n,g)} \text{ and } (1_{X\times Y},\pi_Y)(k,h) = (n,g)\overline{(m,f)} :$$

$$\begin{tikzcd}
& Z \arrow[dl, "{(m,f)}"', "\geq"] \arrow[d, dotted, "{(k,h)}"] \arrow[dr, "{(n,g)}", "\leq"'] & \\
X & X \times Y \arrow[l, "{(1_{X\times Y},\pi_X)}"] \arrow[r, "{(1_{X\times Y},\pi_Y)}"'] & Y
\end{tikzcd}$$

then

$$(k,k) = \overline{(k,h)} = \overline{(m,f)}\ \overline{(n,g)} = (mn', mn'),$$

$$(mn', \pi_X h) = (mn', fn'),$$

and

$$(mn', \pi_Y h) = (mn', gm')$$

and so $(k,h) = (mn', \langle fn', gm'\rangle)$, namely, the uniqueness of $(mn', \langle fn', gm'\rangle)$ follows. Thus, $(X \times Y, (1_{X\times Y}, \pi_1), (1_{X\times Y}, \pi_2))$ is a partial product of X and Y in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$.

Conversely, suppose now that $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ has partial terminal object 1 and partial binary product $(X \times Y, (m, p_1), (n, p_2))$ for objects X and Y.

For each $\mathbf{C}$-object X, there is unique total map $(1_X, t_X) : X \to 1$ such that

$$(1_X, t_X)\overline{(m,f)} = (1_X, t_X)(m,m) = (m, t_X m) = (m, f)$$

for any map $(m,f) : X \to 1$ in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$. In particular, for each $\mathbf{C}$-map $f : X \to Y$, $(1_X, f) : X \to Y$ is a $\mathsf{Par}(\mathbf{C}, \mathcal{M})$-map. Then $(1_X, t_X)\overline{(1_X, f)} = (1_X, f)$ and so $f \cong t_X$. Thus, 1 is a terminal object with the unique map $!_X = t_X : X \to 1$ in $\mathbf{C}$.

If $(X \times Y, (m, p_1), (n, p_2))$ is a binary partial product of X and Y in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, then $m \cong 1_{X\times Y}$ and $n \cong 1_{X\times Y}$ since projections must be total. For each pair of $\mathbf{C}$-maps $f : Z \to Y$ and $g : Z \to Y$, there is a unique $\langle (1_Z, f), (1_Z, g)\rangle = (i, h)$ such that

$$\overline{\langle (1_Z, f), (1_Z, g)\rangle} = \overline{(i,h)} = (i,i) = \overline{(1_Z, f)}\ \overline{(1_Z, g)} = (1_Z, 1_Z)$$

and

$$\begin{tikzcd}
& Z \arrow[dl, "{(1_Z,f)}"', "\geq"] \arrow[dr, "{(1_Z,g)}", "\leq"'] \arrow[d, dotted, "{(i,h)}" description] & \\
X & X \times Y \arrow[l, "{(1_{X\times Y},p_1)}"] \arrow[r, "{(1_{X\times Y},p_2)}"'] & Y
\end{tikzcd}$$

It follows that

$$\begin{aligned}
(1_{X\times Y}, p_1)(i,h) &= (1_{X\times Y}, p_1)(1_Z, h) \\
&= (1_Z, p_1 h) \\
&= (1_Z, f)\overline{(1_Z, p_1 h)} \\
&= (1_Z, f)(1_Z, 1_Z) \\
&= (1_Z, f).
\end{aligned}$$

Then $p_1 h \cong f$. Similarly, $p_2 h \cong g$ and so

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dotted, "{\langle f,g\rangle = h}" description] & \\
X & X \times Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y
\end{tikzcd}$$

commutes. If $h' : Z \to X \times Y$ is such that

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dotted, "h'" description] & \\
X & X \times Y \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y
\end{tikzcd}$$

commutes, then

$$\begin{tikzcd}
& Z \arrow[dl, "{(1_Z,f)}"', "\geq"] \arrow[dr, "{(1_Z,g)}", "\leq"'] \arrow[d, dotted, "{(1_Z,h')}" description] & \\
X & X \times Y \arrow[l, "{(1_{X\times Y},p_1)}"] \arrow[r, "{(1_{X\times Y},p_2)}"'] & Y
\end{tikzcd}$$

and so $h' \cong h$. Thus, $(X \times Y, p_1, p_2)$ is a product of X and Y in $\mathbf{C}$. $\square$

2.2 Cartesian Completion for Restriction Categories

Recall that the *product completion* $\prod \mathbf{X}$ of a category $\mathbf{X}$ has small-indexed $\mathbf{X}$-objects families $(X_i)_{i\in I}$ as objects and a map $f : (X_i)_{i\in I} \to (Y_j)_{j\in J}$ of $\prod \mathbf{X}$ is specified by a function $\phi : J \to I$ and a family $(f_j : X_{\phi(j)} \to B_j)_{j\in J}$ of $\mathbf{X}$-maps [18]. As in [4], $\prod \mathbf{X} = (\mathrm{Fam}(\mathbf{X}^{\mathrm{op}}))^{\mathrm{op}}$. To form free cartesian restriction categories over restriction categories, we need to keep restrictions working with free products.

2.2.1 The Construction $\pi(-)$

Given a restriction category $\mathbf{X}$, define the following construction $\pi(\mathbf{X})$ with

objects: functions $F : I \to \mathbf{X}_0$ where I is a finite set;

maps: a map from object $F : I \to \mathbf{X}_0$ to object $G : J \to \mathbf{X}_0$ is a triple (e, γ, g) such that

$$\begin{tikzcd}
I \arrow[dd, "F"'] & & J \arrow[ll, "g"'] \arrow[dl, "\gamma"'] \arrow[dd, "G"] \\
& \mathbf{X}_1 \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\
\mathbf{X}_0 & & \mathbf{X}_0
\end{tikzcd}$$

and

$$\begin{tikzcd}
I \arrow[dd, "F"'] \arrow[dr, "e"] & & J \arrow[ll, "g"'] \arrow[dl, "\overline{\gamma}"'] \\
& \mathbf{X}_1 \arrow[dl, shift right, "\partial_0"'] \arrow[dl, shift left, "\partial_1"] & \\
\mathbf{X}_0 & &
\end{tikzcd}$$

commute, where $e : F \to F$ satisfies $\overline{e(i)} = e(i)$ for each $i \in I$.

Obviously, $e : F \to F$ and $\gamma : Fg \to G$ in (e, γ, g) can be characterized as

$$\Big(e(i) : F(i) \to F(i)\Big)_{i\in I} \quad \text{and} \quad \Big(\gamma(j) : F(g(j)) \to G(j)\Big)_{j\in J}$$

in which $e(i) = \overline{e(i)}$ and $\overline{\gamma(j)} = e(g(j))$ for each $i \in I$ and $j \in J$;

identities: For each object $F : I \to \mathbf{X}_0$, $1_{(F:I\to\mathbf{X}_0)} = (F1, F1, 1)$:

$$\begin{tikzcd}
I \arrow[d, "F"'] & & I \arrow[ll, "1"'] \arrow[dl, "F1"'] \arrow[d, "F"] \\
& \mathbf{X}_1 \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\
\mathbf{X}_0 & & \mathbf{X}_0
\end{tikzcd}$$

where $\Big(F1(i) = 1_{F(i)} : F(i) \to F(i)\Big)_{i\in I}$;

composition: For maps $(e, \gamma, g) : (F : I \to \mathbf{X}_0) \to (G : J \to \mathbf{X}_0)$ and $(f, \varphi, h) : (G : J \to \mathbf{X}_0) \to (H : K \to \mathbf{X}_0)$, define

$$(f, \varphi, h)(e, \gamma, g) = (d, \psi, gh)$$

where $d : I \to \mathbf{X}_1$ and $\psi : K \to \mathbf{X}_1$ are given by

$$d = \Big(e(i) \prod_{g(j)=i} \overline{f(j)\gamma(j)} : F(i) \to F(i)\Big)_{i\in I}$$

and

$$\begin{aligned}
\psi &= \Big(\varphi(k)\gamma(h(k))d(g(h(k))) : F(g(h(k))) \to H(k)\Big)_{k\in K} \\
&= \Big(\varphi(k)\gamma(h(k)) \prod_{g(j)=g(h(k))} \overline{f(j)\gamma(j)} : F(g(h(k))) \to H(k)\Big)_{k\in K} :
\end{aligned}$$

$$\begin{tikzcd}
I \arrow[d, "F"'] & & J \arrow[ll, "g"'] \arrow[dl, "\gamma"'] \arrow[d, "G"] & & K \arrow[ll, "h"'] \arrow[dl, "\varphi"'] \arrow[d, "H"] \\
& \mathbf{X}_1 \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & & \mathbf{X}_1 \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\
\mathbf{X}_0 & & \mathbf{X}_0 & & \mathbf{X}_0
\end{tikzcd}$$

restriction: $\overline{(e, \gamma, g)} = (e, e, 1)$.

2.2.2 $\pi(\mathbf{X})$ is a Cartesian Restriction Category

First, we verify that composition is well-defined and identity and associative laws hold true.

(1) Obviously, each $d(i) = e(i)\prod_{g(j)=i}\overline{f(j)\gamma(j)}$ is a restriction idempotent. Since $\overline{\gamma(j)} = e(g(j))$ and $\overline{\varphi(k)} = f(h(k))$, we have

$$\begin{aligned}
\overline{\psi(k)} &= \overline{\varphi(k)\gamma(h(k)) \cdot \prod_{g(j)=g(h(k))} \overline{f(j)\gamma(j)}} \\
&= \overline{\overline{\varphi(k)}\gamma(h(k))} \cdot \prod_{g(j)=g(h(k))} \overline{f(j)\gamma(j)} \\
&= \overline{f(h(k))\gamma(h(k))} \cdot \prod_{g(j)=g(h(k))} \overline{f(j)\gamma(j)} \\
&= \overline{\gamma(h(k))}\;\overline{f(h(k))\gamma(h(k))} \cdot \prod_{g(j)=g(h(k))} \overline{f(j)\gamma(j)} \\
&= e(g(h(k)))\;\overline{f(h(k))\gamma(h(k))} \cdot \prod_{g(j)=g(h(k))} \overline{f(j)\gamma(j)} \\
&= e(g(h(k))) \prod_{g(j)=g(h(k))} \overline{f(j)\gamma(j)} \\
&= d(g(h(k))).
\end{aligned}$$

So the composition is well-defined.

(2) **Identity Law:** For any map $(e, \gamma, g) : (F : I \to \mathbf{X}_0) \to (G : J \to \mathbf{X}_0)$, assume

$$(G1, G1, 1)(e, \gamma, g) = (f, \psi, g).$$

Then, for each $i \in I$ and each $j \in J$,

$$f(i) = e(i)\prod_{g(j)=i}\overline{G1(j)\gamma(j)} = e(i)\prod_{g(j)=i}\overline{\gamma(j)} = e(i)\prod_{g(j)=i}e(g(j)) = e(i)$$

and

$$\psi(j) = G1(j)\gamma(j)f(g(j)) = \gamma(j)e(g(j)) = \gamma(j)\overline{\gamma(j)} = \gamma(j).$$

Hence $(G1, G1, 1)(e, \gamma, g) = (e, \gamma, g)$. Assume now that $(e, \gamma, g)(F1, F1, 1) = (f, \psi, g)$. Then for each $i \in I$ and $j \in J$,

$$f(i) = F1(i) \prod_{1(j)=i} \overline{e(j)F1(j)} = \prod_{1(j)=i} \overline{e(j)} = e(i)$$

and

$$\psi(j) = \gamma(j)F1(j) = \gamma(j)f(g(j)) = \gamma(j)e(g(j)) = \gamma(j)\overline{\gamma(j)} = \gamma(j).$$

Hence $(e, \gamma, g)(F1, F1, 1) = (e, \gamma, g)$.

(3) **Association Law:** For maps $(e, \gamma, g) : (F : I \to \mathbf{X}_0) \to (G : J \to \mathbf{X}_0)$, $(d, \varphi, h) : (G : J \to \mathbf{X}_0) \to (H : K \to \mathbf{X}_0)$, and $(c, \psi, m) : (H : K \to \mathbf{X}_0) \to (M : L \to \mathbf{X}_0)$, we have

$$\begin{aligned}
& (c, \psi, m)\Big((d, \varphi, h)(e, \gamma, g)\Big) \\
= \;& (c, \psi, m)\Big(\Big(e(i) \prod_{g(j)=i} \overline{d(j)\gamma(j)}\Big)_{i \in I}, \\
& \Big(\varphi(k)\gamma(h(k)) \cdot \prod_{g(j)=g(h(k))} \overline{d(j)\gamma(j)}\Big)_{k \in K}, gh\Big) \\
= \;& \Big(\Big(e(i) \prod_{g(j)=i} \overline{d(j)\gamma(j)} \cdot \prod_{g(h(k))=i} \overline{c(k)\varphi(k)\gamma(h(k)) \prod_{g(j)=g(h(k))} \overline{d(j)\gamma(j)}}\Big)_{i \in I}, \\
& \Big(\psi(l)\varphi(m(l))\gamma(h(m(l))) \prod_{g(j)=g(h(m(l)))} \overline{d(j)\gamma(j)} \cdot \\
& \prod_{g(h(k))=g(h(m(l)))} \overline{c(k)\varphi(k)\gamma(h(k)) \prod_{g(j)=g(h(k))} \overline{d(j)\gamma(j)}}\Big)_{l \in L}, ghm\Big) \\
= \;& \Big(\Big(e(i) \prod_{g(j)=i} \overline{d(j)\gamma(j)} \cdot \prod_{g(h(k))=i} \Big(\overline{c(k)\varphi(k)\gamma(h(k))} \prod_{g(j')=g(h(k))} \overline{d(j')\gamma(j')}\Big)\Big)_{i \in I}, \\
& \Big(\psi(l)\varphi(m(l))\gamma(h(m(l))) \prod_{g(j)=g(h(m(l)))} \overline{d(j)\gamma(j)} \cdot \\
& \prod_{g(h(k))=g(h(m(l)))} \Big(\overline{c(k)\varphi(k)\gamma(h(k))} \prod_{g(j)=g(h(k))} \overline{d(j)\gamma(j)}\Big)\Big)_{l \in L}, ghm\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\Big(e(i)\prod_{g(j)=i}\overline{d(j)\gamma(j)}\cdot\prod_{g(h(k))=i}\overline{c(k)\varphi(k)\gamma(h(k))}\cdot\prod_{g(j')=i}\overline{d(j')\gamma(j')}\Big)_{i\in I},\\
&\quad \Big(\psi(l)\varphi(m(l))\gamma(h(m(l)))\prod_{g(j)=g(h(m(l)))}\overline{d(j)\gamma(j)}\cdot\\
&\quad \prod_{g(h(k))=g(h(m(l)))}\Big(\overline{c(k)\varphi(k)\gamma(h(k))}\prod_{g(j)=g(h(m(l)))}\overline{d(j)\gamma(j)}\Big)\Big)_{l\in L}, ghm\Big)\\
&= \Big(\Big(e(i)\prod_{g(j)=i}\overline{d(j)\gamma(j)}\cdot\prod_{g(h(k))=i}\overline{c(k)\varphi(k)\gamma(h(k))}\Big)_{i\in I},\\
&\quad \Big(\psi(l)\varphi(m(l))\gamma(h(m(l)))\prod_{g(j)=g(h(m(l)))}\overline{d(j)\gamma(j)}\cdot\\
&\quad \prod_{g(h(k))=g(h(m(l)))}\overline{c(k)\varphi(k)\gamma(h(k))}\Big)_{l\in L}, ghm\Big)\\
&= \Big(\Big(e(i)\prod_{j\in g^{-1}(i)}\overline{d(j)\gamma(j)}\cdot\prod_{k\in(gh)^{-1}(i)}\overline{c(k)\varphi(k)\gamma(h(k))}\Big)_{i\in I},\\
&\quad \Big(\psi(l)\varphi(m(l))\gamma(h(m(l)))\prod_{j\in g^{-1}(g(h(m(l))))}\overline{d(j)\gamma(j)}\cdot\\
&\quad \prod_{k\in(gh)^{-1}(g(h(m(l))))}\overline{c(k)\varphi(k)\gamma(h(k))}\Big)_{l\in L}, ghm\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big((c,\psi,m)(d,\varphi,h)\Big)(e,\gamma,g)\\
= &\Big(\Big(d(j)\prod_{h(k)=j}\overline{c(k)\varphi(k)}\Big)_{j\in J},\\
&\Big(\psi(l)\varphi(m(l))\cdot\prod_{h(k)=h(m(l))}\overline{c(k)\varphi(k)}\Big)_{l\in L}, hm\Big)(e,\gamma,g)\\
= &\Big(\Big(e(i)\prod_{g(j)=i}\overline{d(j)\prod_{h(k)=j}\overline{c(k)\varphi(k)}\gamma(j)}\Big)_{i\in I},\\
&\Big(\psi(l)\varphi(m(l))\prod_{h(k)=h(m(l))}\overline{c(k)\varphi(k)}\gamma(h(m(l)))\cdot\\
&\prod_{g(j)=g(h(m(l)))}\overline{d(j)\prod_{h(k)=j}\overline{c(k)\varphi(k)}\gamma(j)}\Big)_{l\in L}, ghm\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\Big(e(i) \prod_{g(j)=i} \overline{d(j)\gamma(j) \prod_{h(k)=j} \overline{c(k)\varphi(k)\gamma(j)}}\Big)_{i\in I}, \\
&\quad \Big(\psi(l)\varphi(m(l))\gamma(h(m(l))) \prod_{h(k)=h(m(l))} \overline{c(k)\varphi(k)\gamma(h(m(l)))} \cdot \\
&\quad \prod_{g(j)=g(h(m(l)))} \Big(\overline{d(j)\gamma(j)} \prod_{h(k)=j} \overline{c(k)\varphi(k)\gamma(j)}\Big)\Big)_{l\in L}, ghm\Big) \\
&= \Big(\Big(e(i) \prod_{j\in g^{-1}(i)} \Big(\overline{d(j)\gamma(j)} \prod_{k\in h^{-1}(j)} \overline{c(k)\varphi(k)\gamma(h(k))}\Big)\Big)_{i\in I}, \\
&\quad \Big(\psi(l)\varphi(m(l))\gamma(h(m(l))) \prod_{h(k)=h(m(l))} \overline{c(k)\varphi(k)\gamma(h(k))} \cdot \\
&\quad \prod_{g(j)=g(h(m(l)))} \Big(\overline{d(j)\gamma(j)} \prod_{h(k)=j} \overline{c(k)\varphi(k)\gamma(h(k))}\Big)\Big)_{l\in L}, ghm\Big) \\
&= \Big(\Big(e(i) \prod_{j\in g^{-1}(i)} \overline{d(j)\gamma(j)} \prod_{k\in h^{-1}(g^{-1}(i))} \overline{c(k)\varphi(k)\gamma(h(k))}\Big)_{i\in I}, \\
&\quad \Big(\psi(l)\varphi(m(l))\gamma(h(m(l))) \prod_{k\in h^{-1}(h(m(l)))} \overline{c(k)\varphi(k)\gamma(h(k))} \cdot \\
&\quad \prod_{j\in g^{-1}(g(h(m(l))))} \Big(\overline{d(j)\gamma(j)} \cdot \prod_{k\in h^{-1}(j)} \overline{c(k)\varphi(k)\gamma(j)}\Big)\Big)_{l\in L}, ghm\Big) \\
&= \Big(\Big(e(i) \prod_{j\in g^{-1}(i)} \overline{d(j)\gamma(j)} \prod_{k\in h^{-1}(g^{-1}(i))} \overline{c(k)\varphi(k)\gamma(h(k))}\Big)_{i\in I}, \\
&\quad \Big(\psi(l)\varphi(m(l))\gamma(h(m(l))) \cdot \prod_{j\in g^{-1}(g(h(m(l))))} \overline{d(j)\gamma(j)} \cdot \\
&\quad \prod_{k\in h^{-1}(g^{-1}(g(h(m(l)))))} \overline{c(k)\varphi(k)\gamma(h(k))}\Big)_{l\in L}, ghm\Big) \\
&= \Big(\Big(e(i) \prod_{j\in g^{-1}(i)} \overline{d(j)\gamma(j)} \cdot \prod_{k\in (gh)^{-1}(i)} \overline{c(k)\varphi(k)\gamma(h(k))}\Big)_{i\in I}, \\
&\quad \Big(\psi(l)\varphi(m(l))\gamma(h(m(l))) \prod_{j\in g^{-1}(g(h(m(l))))} \overline{d(j)\gamma(j)} \cdot \\
&\quad \prod_{k\in (gh)^{-1}(g(h(m(l))))} \overline{c(k)\varphi(k)\gamma(h(k))}\Big)_{l\in L}, ghm\Big) \\
&= (c,\psi,m)\Big((d,\varphi,h)(e,\gamma,g)\Big).
\end{aligned}
$$

Hence $\pi(\mathbf{X})$ is a category. Now we verify four restriction axioms as follows.

[**R.1**] For any $\pi(\mathbf{X})$-map $(e,\gamma,g):(F:I\to\mathbf{X}_0)\to(G:J\to\mathbf{X}_0)$,

$$\begin{aligned}
(e,\gamma,g)\overline{(e,\gamma,g)} &= (e,\gamma,g)(e,e,1)\\
&= \Big(\Big(e(i)\overline{e(i)e(i)}\Big)_{i\in I},\Big(\gamma(j)e(g(j))\Big)_{j\in J},g\Big)\\
&= \Big(\Big(e(i)\Big)_{i\in I},\Big(\gamma(j)\Big)_{j\in J},g\Big)\\
&= (e,\gamma,g).
\end{aligned}$$

[**R.2**] For any $\pi(\mathbf{X})$-maps $(e,\gamma,g):(F:I\to\mathbf{X}_0)\to(G:G\to\mathbf{X}_0)$ and $(d,\varphi,h):(F:I\to\mathbf{X}_0)\to(H:K\to\mathbf{X}_0)$,

$$\begin{aligned}
\overline{(e,\gamma,g)}\;\overline{(d,\varphi,h)} &= (e,e,1)(d,d,1)\\
&= \Big(\Big(d(i)\overline{e(i)e(i)}\Big)_{i\in I},\Big(e(i)d(i)\cdot e(i)d(i)\Big)_{i\in I},1\Big)\\
&= \Big(\Big(e(i)d(i)\Big)_{i\in I},\Big(e(i)d(i)\Big)_{i\in I},1\Big)\\
&= (d,d,1)(e,e,1)\\
&= \overline{(d,\varphi,h)}\;\overline{(e,\gamma,g)}.
\end{aligned}$$

[**R.3**] For any $\pi(\mathbf{X})$-maps $(e,\gamma,g):(F:I\to\mathbf{X}_0)\to(G:J\to\mathbf{X}_0)$ and $(d,\varphi,h):(F:I\to\mathbf{X}_0)\to(H:K\to\mathbf{X}_0)$,

$$\begin{aligned}
\overline{(d,\varphi,h)\overline{(e,\gamma,g)}} &= \overline{(d,\varphi,h)(e,e,1)}\\
&= \Big(\Big(e(i)\overline{d(i)e(i)}\Big)_{i\in I},\Big(e(i)\overline{d(i)e(i)}\Big)_{i\in I},1\Big)\\
&= \Big(\Big(e(i)d(i)\Big)_{i\in I},\Big(e(i)d(i)\Big)_{i\in I},1\Big)\\
&= (d,d,1)(e,e,1)\\
&= \overline{(d,\varphi,h)}\;\overline{(e,\gamma,g)}.
\end{aligned}$$

[**R.4**] For any $\pi(\mathbf{X})$-maps $(e,\gamma,g):(F:I\to\mathbf{X}_0)\to(G:J\to\mathbf{X}_0)$ and $(d,\varphi,h):(G:$

$J \to \mathbf{X}_0) \to (H : K \to \mathbf{X}_0)$,

$$
\begin{aligned}
& \overline{(d,\varphi,h)}(e,\gamma,g) \\
= \ & (d,d,1)(e,\gamma,g) \\
= \ & \Big(\Big(e(i)\prod_{g(j)=i}\overline{d(j)\gamma(j)}\Big)_{i\in I}, \Big(d(j)\gamma(j)\prod_{g(j')=g(j)}\overline{d(j')\gamma(j')}\Big)_{j\in J}, g\Big) \\
= \ & \Big(\Big(e(i)\prod_{g(j)=i}\overline{d(j)\gamma(j)}\Big)_{i\in I}, \Big(\gamma(j)\overline{d(j)\gamma(j)}\prod_{g(j')=g(j)}\overline{d(j')\gamma(j')}\Big)_{j\in J}, g\Big) \\
= \ & \Big(\Big(e(i)\prod_{g(j)=i}\overline{d(j)\gamma(j)}\Big)_{i\in I}, \Big(\gamma(j)e(g(j))\prod_{g(j')=g(j)}\overline{d(j')\gamma(j')}\Big)_{j\in J}, g\Big) \\
= \ & (e,\gamma,g)\Big(\Big(e(i)\prod_{g(j)=i}\overline{d(j)\gamma(j)}\Big)_{i\in I}, \Big(e(i)\prod_{g(j)=i}\overline{d(j)\gamma(j)}\Big)_{i\in I}, 1\Big) \\
= \ & (e,\gamma,g)\overline{(d,\varphi,h)(e,\gamma,g)}.
\end{aligned}
$$

Hence $\pi(\mathbf{X})$ is a restriction category.

To prove that $\pi(\mathbf{X})$ is cartesian, it suffices to show that $\pi(\mathbf{X})$ has both a partial terminal object and binary partial products.

First, the partial terminal object in $\pi(\mathbf{X})$ is given by $\emptyset : \emptyset \to \mathbf{X}_0$. In fact, for a given object $F : I \to \mathbf{X}_0$ in $\pi(\mathbf{X})$, there is a (unique) total map $!_{(F:I\to\mathbf{X}_0)} = (1_{F(I)}, \emptyset, \emptyset)$ from $(F : I \to \mathbf{X}_0)$ to $(\emptyset : \emptyset \to \mathbf{X}_0)$:

$$\begin{tikzcd}
I \arrow[dd, "F"'] & & \emptyset \arrow[ll, "\emptyset"'] \arrow[dl, "\emptyset"'] \arrow[dd, "\emptyset"] \\
& \mathbf{X}_1 \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\
\mathbf{X}_0 & & \mathbf{X}_0
\end{tikzcd}$$

where $1_{F(I)}$ is given by $1_{F(I)}(i) = 1_{F(i)} : F(i) \to F(i)$. We verify the following:

(i) Obviously, $(1_{F(I)}, \emptyset, \emptyset)$ is a map in $\pi(\mathbf{X})$.

(ii) Since $\overline{(1_{F(I)}, \emptyset, \emptyset)} = (1_{F(I)}, 1_{F(I)}, 1) = 1_{(F:I\to\mathbf{X}_0)}$, $(1_{F(I)}, \emptyset, \emptyset)$ is total.

(iii) For any map $(e, \gamma, g) : (F : I \to \mathbf{X}_0) \to (\emptyset : \emptyset \to \mathbf{X}_0)$, $g : \emptyset \to I$ and $\gamma : \emptyset \to \mathbf{X}_1$ must be the unique maps since $\emptyset$ is initial in **Sets**. Note that

$$\begin{aligned} (1_{F(I)}, \emptyset, \emptyset)\overline{(e, \gamma, g)} &= (1_{F(I)}, \emptyset, \emptyset)(e, e, 1) \\ &= (e, \emptyset, \emptyset) \\ &= (e, \gamma, g). \end{aligned}$$

Thus, $\emptyset : \emptyset \to \mathbf{X}_0$ is the partial final object in $\pi(\mathbf{X})$.

Given two objects $F : I \to \mathbf{X}_0$ and $G : J \to \mathbf{X}_0$ in $\pi(\mathbf{X})$, their binary partial product is given by $\langle F|G\rangle : I + J \to \mathbf{X}_0$ with two total projections $\pi_1 = (e, \gamma, \iota_1)$ and $\pi_2 = (f, \varphi, \iota_2)$:

$$\begin{tikzcd}
I \arrow[rr, "\iota_1"] \arrow[dr, "\gamma"] \arrow[dd, "F"'] & & I + J \arrow[dd, "\langle F|G\rangle"] & & J \arrow[ll, "\iota_2"'] \arrow[dl, "\varphi"'] \arrow[dd, "G"] \\
& \mathbf{X}_1 \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & & \mathbf{X}_1 \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\
\mathbf{X}_0 & & \mathbf{X}_0 & & \mathbf{X}_0
\end{tikzcd}$$

where, for each $l \in I + J$,

$$e(l) = f(l) = 1_{\langle F|G\rangle(l)} : \langle F|G\rangle(l) \to \langle F|G\rangle(l),$$

$\gamma : I \to \mathbf{X}_1$ is given by

$$\gamma(i) = 1_{F(i)} : \langle F|G\rangle(\iota_1(i)) \to F(i),$$

and $\varphi : J \to \mathbf{X}_1$ is given by

$$\varphi(j) = 1_{G(j)} : \langle F|G\rangle(\iota_2(j)) \to G(j).$$

Now we check the following:

(i) (e, γ, ι_1) and (f, φ, ι_2) are total maps in $\pi(\mathbf{X})$.

Since $\overline{\gamma(i)} = \overline{1_{F(i)}} = 1_{F(i)} = e(\iota_1(i))$, (e, γ, ι_1) is a map in $\pi(\mathbf{X})$. Similarly, one can easily check that (f, φ, ι_2) is a map in $\pi(\mathbf{X})$ too. Since $\overline{(e, \gamma, \iota_1)} = (e, e, 1) = 1_{(\langle F|G\rangle : I+J \to \mathbf{X}_0)}$ and $\overline{(f, \varphi, \iota_2)} = (f, f, 1) = 1_{(\langle F|G\rangle : I+J \to \mathbf{X}_0)}$, (e, γ, ι_1) and (f, φ, ι_2) are total.

(ii) For any $\pi(\mathbf{X})$-maps $(c_1, \psi_1, m_1) : (X : T \to \mathbf{X}_0) \to (F : I \to \mathbf{X}_0)$ and $(c_2, \psi_2, m_2) : (X : T \to \mathbf{X}_0) \to (G : J \to \mathbf{X}_0)$:

$$\begin{tikzcd}
T \arrow[dd, "X"'] & & I \arrow[ll, "m_1"'] \arrow[dl, "\psi_1"'] \arrow[dd, "F"] \\
& \mathbf{X}_1 \arrow[dl, "\partial_0"] \arrow[dr, "\partial_1"'] & \\
\mathbf{X}_0 & & \mathbf{X}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[dd, "X"'] & & J \arrow[ll, "m_2"'] \arrow[dl, "\psi_2"'] \arrow[dd, "G"] \\
& \mathbf{X}_1 \arrow[dl, "\partial_0"] \arrow[dr, "\partial_1"'] & \\
\mathbf{X}_0 & & \mathbf{X}_0
\end{tikzcd}$$

there is a $\pi(\mathbf{X})$-map

$$\Big\langle (c_1, \psi_1, m_1), (c_2, \psi_2, m_2) \Big\rangle = (c, \psi, m) : (X : T \to \mathbf{X}_0) \to (\langle F|G\rangle : I + J \to \mathbf{X}_0) :$$

$$\begin{tikzcd}
T \arrow[dd, "X"'] & & I+J \arrow[ll, "m"'] \arrow[dl, "\psi"'] \arrow[dd, "\langle F|G\rangle"] \\
& \mathbf{X}_1 \arrow[dl, "\partial_0"] \arrow[dr, "\partial_1"'] & \\
\mathbf{X}_0 & & \mathbf{X}_0
\end{tikzcd}$$

where

$$m = \langle m_1 | m_2 \rangle : I + J \to T,$$

$$c(t) = c_1(t)c_2(t) : X(t) \to X(t)$$

for each $t \in T$, and

$$\psi = \langle \psi_1 \cdot c_2 m_1 | \psi_2 \cdot c_1 m_2 \rangle : I + J \to \langle F|G\rangle.$$

Recalling that coproducts are given by disjoint unions in **Set**, we have

$$\psi(l) \quad = \quad \begin{cases} \psi_1(l)c_2(m_1(l)) & \text{if } l \in I; \\ \psi_2(l)c_1(m_2(l)) & \text{if } l \in J. \end{cases}$$

Since, for each $t \in T$, both both $c_1(t)$ and $c_2(t)$ are restriction idempotents, $c(t)$ is a restriction idempotent too. For each $i \in I$ and $j \in J$, we have

$$\overline{\psi_1(i)} = c_1(m_1(i)) \text{ and } \overline{\psi_2(j)} = c_2(m_2(j)).$$

Then, for each $l \in I + J$,

$$\begin{aligned} \overline{\psi(l)} &= \begin{cases} \overline{\psi_1(l)c_2(m_1(l))} = \overline{\psi_1(l)}c_2(m_1(l)) = c_1(m_1(l))c_2(m_1(l)) & \text{if } l \in I \\ \overline{\psi_2(l)c_1(m_2(l))} = \overline{\psi_2(l)}c_1(m_2(l)) = c_1(m_2(l))c_2(m_2(l)) & \text{if } l \in J \end{cases} \\ &= c(m(l)). \end{aligned}$$

Hence $(c, \psi, m) : \langle F|G\rangle \to X$ is a well-defined map in $\pi(\mathbf{X})$.

(iii)

$$\begin{aligned} (e, \gamma, \iota_1)(c, \psi, m) &= \Big(\big(c(t) \prod_{m(l)=t} \overline{e(l)\psi(l)}\big)_{t\in T}, \\ &\quad \big(\gamma(i)\psi(\iota_1(i)) \prod_{m(l)=m(\iota_1(i))} \overline{e(l)\psi(l)}\big)_{i\in I}, m_1\Big) \\ &= \Big(\big(c_1(t)c_2(t) \prod_{m(l)=t} \overline{\psi(l)}\big)_{t\in T}, \\ &\quad \big(\psi(\iota_1(i)) \prod_{m(l)=m(\iota_1(i))} \overline{\psi(l)}\big)_{i\in I}, m_1\Big) \\ &= \Big(\big(c_1(t)c_2(t) \prod_{m(l)=t} c(m(l))\big)_{t\in T}, \\ &\quad \big(\psi(\iota_1(i)) \prod_{m(l)=m(\iota_1(i))} c(m(l))\big)_{i\in I}, m_1\Big) \end{aligned}$$

$$
\begin{aligned}
&= \Big(\big(c_1(t)c_2(t) \prod_{m(l)=t} c(t)\big)_{t\in T}, \\
&\qquad \big(\psi(\iota_1(i)) \prod_{m(l)=m(\iota_1(i))} c(m_1(i))\big)_{i\in I}, m_1\Big) \\
&= \Big(\big(c_1(t)c_2(t)\big)_{t\in T}, \big(\psi_1(i)c_1(m_1(i))c_2(m_1(i))\big)_{i\in I}, m_1\Big) \\
&= \Big(\big(c_2(t) \prod_{1_T(t')=t} \overline{c_1(t')c_2(t')}\big)_{t\in T}, \\
&\qquad \big(\psi_1(i)c_2(m_1(i)) \prod_{1_T(t)=1_T(m_1(i))} \overline{c_1(t)c_2(t)}\big)_{i\in I}, m_1\Big) \\
&= (c_1, \psi_1, m_1)(c_2, c_2, 1) \\
&= (c_1, \psi_1, m_1)\overline{(c_2, \psi_2, m_2)}.
\end{aligned}
$$

(iv)

$$
\begin{aligned}
(f, \varphi, \iota_2)(c, \psi, m) &= \Big(\big(c(t) \prod_{m(l)=t} \overline{f(l)\psi(l)}\big)_{t\in T}, \\
&\qquad \big(\varphi(j)\psi(\iota_2(j)) \prod_{m(l)=m(\iota_2(j))} \overline{f(l)\psi(l)}\big)_{j\in J}, m_2\Big) \\
&= \Big(\big(c_1(t)c_2(t) \prod_{m(l)=t} \overline{\psi(l)}\big)_{t\in T}, \\
&\qquad \big(\psi(\iota_2(j)) \prod_{m(l)=m(\iota_2(j))} \overline{\psi(l)}\big)_{j\in J}, m_2\Big) \\
&= \Big(\big(c_1(t)c_2(t) \prod_{m(l)=t} c(t)\big)_{t\in T}, \\
&\qquad \big(\psi(\iota_2(j)) \prod_{m(l)=m(\iota_2(j))} c(m_2(j))\big)_{j\in J}, m_2\Big) \\
&= \Big(\big(c_1(t)c_2(t)\big)_{t\in T}, \big(\psi_2(j)c_1(m_2(j))c_2(m_2(j))\big)_{j\in J}, m_2\Big) \\
&= \Big(\big(c_1(t) \prod_{1_T(t')=t} \overline{c_2(t')c_1(t')}\big)_{t\in T}, \big(\psi_2(j)c_1(m_2(j)) \\
&\qquad \prod_{1_T(t)=1_T(m_2(j))} \overline{c_2(t)c_1(t)}\big)_{j\in J}, m_2\Big) \\
&= (c_2, \psi_2, m_2)(c_1, c_1, 1) \\
&= (c_2, \psi_2, m_2)\overline{(c_1, \psi_1, m_1)}.
\end{aligned}
$$

Then there is a $\pi(\mathbf{X})$-map $(c, \psi, m) : (X : T \to \mathbf{X}_0) \to (\langle F|G\rangle : I + J \to \mathbf{X}_0)$ such that

$$(e, \gamma, \iota_1)(c, \psi, m) = (c_1, \psi_1, m_1)\overline{(c_2, \psi_2, m_2)}$$

and

$$(f, \varphi, \iota_2)(c, \psi, m) = (c_2, \psi_2, m_2)\overline{(c_1, \psi_1, m_1)} :$$

$$\begin{tikzcd}
& (X : T \to \mathbf{X}_0) \arrow[dl, "{(c_1,\psi_1,m_1)}"'] \arrow[d, dashed, "{(c,\psi,m)}"] \arrow[dr, "{(c_2,\psi_2,m_2)}"] & \\
(F : I \to \mathbf{X}_0) & (\langle F|G\rangle : I + J \to \mathbf{X}_0) \arrow[l, "{(e,\gamma,\iota_1)}"] \arrow[r, "{(f,\varphi,\iota_2)}"'] & (G : J \to \mathbf{X}_0)
\end{tikzcd}$$

To prove the uniqueness of (c, ψ, m), we consider a $\pi(\mathbf{X})$-map $(c', \psi', m') : (X : T \to \mathbf{X}_0) \to (\langle F|G\rangle : I + J \to \mathbf{X}_0)$ such that

$$(e, \gamma, \iota_1)(c', \psi', m') = (c_1, \psi_1, m_1)\overline{(c_2, \psi_2, m_2)}$$

and

$$(f, \varphi, \iota_2)(c', \psi', m') = (c_2, \psi_2, m_2)\overline{(c_1, \psi_1, m_1)}.$$

Since

$$\begin{aligned}
(e, \gamma, \iota_1)(c', \psi', m') &= \Big(\big(c'(t) \prod_{m'(l)=t} \overline{e(l)\psi'(l)}\big)_{t\in T}, \\
&\qquad \big(\gamma(i)\psi'(\iota_1(i)) \prod_{m'(l)=m'(\iota_1(i))} \overline{e(l)\psi'(l)}\big)_{i\in I}, \iota_1 m'\Big) \\
&= \Big(\big(c'(t) \prod_{m'(l)=t} \overline{\psi'(l)}\big)_{t\in T}, \big(\psi'(\iota_1(i)) \prod_{m'(l)=m'(\iota_1(i))} \overline{\psi'(l)}\big)_{i\in I}, \iota_1 m'\Big) \\
&= \Big(\big(c'(t) \prod_{m'(l)=t} c'(m'(l))\big)_{t\in T}, \\
&\qquad \big(\psi'(\iota_1(i)) \prod_{m'(l)=m'(\iota_1(i))} c'(m'(l))\big)_{i\in I}, \iota_1 m'\Big) \\
&= \Big(\big(c'(t)\big)_{t\in T}, \big(\psi'(\iota_1(i)) \prod_{m'(l)=m'(\iota_1(i))} c'(m'(\iota_1(i)))\big)_{i\in I}, \iota_1 m'\Big) \\
&= \Big(\big(c'(t)\big)_{t\in T}, \big(\psi'(\iota_1(i))c'(m'(\iota_1(i)))\big)_{i\in I}, \iota_1 m'\Big) \\
&= \Big(\big(c_1(t)c_2(t)\big)_{t\in T}, \big(\psi_1(i)c_1(m_1(i))c_2(m_1(i))\big)_{i\in I}, m_1\Big)
\end{aligned}$$

and

$$\begin{aligned}
(f,\varphi,\iota_2)(c',\psi',m') &= \Big(\big(c'(t)\prod_{m'(l)=t}\overline{f(l)\psi'(l)}\big)_{t\in T},\\
&\quad \big(\varphi(j)\psi'(\iota_2(j))\prod_{m'(l)=m'(\iota_2(j))}\overline{f(l)\psi'(l)}\big)_{j\in J},\iota_2 m'\Big)\\
&= \Big(\big(c'(t)\prod_{m'(l)=t}\overline{\psi'(l)}\big)_{t\in T},\big(\psi'(\iota_2(j))\prod_{m'(l)=m'(\iota_2(j))}\overline{\psi'(l)}\big)_{j\in J},\iota_2 m'\Big)\\
&= \Big(\big(c'(t)\prod_{m'(l)=t}c'(m'(l))\big)_{t\in T},\\
&\quad \big(\psi'(\iota_2(j))\prod_{m'(l)=m'(\iota_2(j))}c'(m'(l))\big)_{j\in J},\iota_2 m'\Big)\\
&= \Big(\big(c'(t)\big)_{t\in T},\big(\psi'(\iota_2(j))c'(m'(\iota_2(j)))\big)_{j\in J},\iota_2 m'\Big)\\
&= \Big(\big(c_1(t)c_2(t)\big)_{t\in T},\big(\psi_2(j)c_1(m_2(j))c_2(m_2(j))\big)_{j\in I},m_2\Big),
\end{aligned}$$

we have

$$c'(t)=c_1(t)c_2(t) \text{ for all } t\in T,$$

$$\psi'(\iota_1(i))c'(m'(\iota_1(i)))=\psi_1(i)c_1(m_1(i))c_2(m_1(i)) \text{ for all } i\in I,$$

$$\psi'(\iota_2(j))c'(m'(\iota_2(j)))=\psi_2(j)c_1(m_2(j))c_2(m_2(j)) \text{ for all } j\in J,$$

$$\iota_1 m'=m_1, \iota_2 m'=m_2.$$

Then

$$m'=\langle m_1|m_2\rangle=m,$$

$$c'(t)=c_1(t)c_2(t)=c(t),$$

$$\begin{aligned}
\psi'(\iota_1(i)) &= \psi'(\iota_1(i))\overline{\psi'(\iota_1(i))} \\
&= \psi'(\iota_1(i))c'(m'(\iota_1(i))) \\
&= \psi'(\iota_1(i))c(m(\iota_1(i))) \\
&= \psi_1(i)c_1(m_1(i))c_2(m_1(i)) \\
&= \psi(\iota(i))
\end{aligned}$$

for each $i \in I$ and

$$\begin{aligned}
\psi'(\iota_2(j)) &= \psi'(\iota_2(j))\overline{\psi'(\iota_2(j))} \\
&= \psi'(\iota_2(j))c'(m'(\iota_2(j))) \\
&= \psi'(\iota_2(j))c(m(\iota_2(j))) \\
&= \psi_2(j)c_1(m_2(j))c_2(m_2(j)) \\
&= \psi(\iota(i))
\end{aligned}$$

for each $j \in J$ and so $(c', \psi', m') = (c, \psi, m)$. The uniqueness of (c, ψ, m) follows. Hence $\pi(\mathbf{X})$ has binary partial products and therefore $\pi(\mathbf{X})$ is cartesian.

2.2.3 $\pi(\mathbf{X})$ is Free

There is a forgetful functor

$$U : \pi(\mathbf{X})^{\mathrm{op}} \to \mathbf{Sets}$$

sending $(e, \gamma, g) : (F : I \to \mathbf{X}_0) \to (G : J \to \mathbf{X}_0)$ in $\pi(\mathbf{X})$ to $g : J \to I$ and a forgetful functor

$$U_c : \mathbf{crCat}_0 \to \mathbf{rCat}_0$$

which forgets finite partial products. Obviously,

$$\pi : \mathbf{rCat}_0 \to \mathbf{crCat}_0,$$

sending each restriction functor $\mathcal{F} : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{rCat}_0$ to a cartesian restriction functor $\pi(\mathcal{F})$ in $\mathbf{crCat}_0$, is a functor, where $\pi(\mathcal{F})$ is defined by sending $(e, \gamma, g) : (F : I \to \mathbf{X}_0) \to (G : J \to \mathbf{X}_0)$ in $\pi(\mathbf{X})$ to $(\mathcal{F}(e), \mathcal{F}(\gamma), g) : (\mathcal{F}(F) : I \to \mathbf{Y}_0) \to (\mathcal{F}(G) : J \to \mathbf{Y}_0)$ in $\pi(\mathbf{Y})$ by noting

$$\begin{aligned} \overline{\mathcal{F}(\gamma)(j)} &= \mathcal{F}(\overline{\gamma(j)}) \\ &= \mathcal{F}(e(g(j)) \\ &= \mathcal{F}(e)(g(j)). \end{aligned}$$

There is also an canonical embedding

$$\mathcal{J} : \mathbf{X} \to \pi(\mathbf{X})$$

which identities objects of $\mathbf{X}$ with singleton families:

$$\mathcal{J}(f : X \to Y) = (\overline{f}, \gamma_f, [Y \mapsto X]) : (X : \{X\} \to \mathbf{X}_0) \to (Y : \{Y\} \to \mathbf{X}_0) :$$

$$\begin{tikzcd}
\{X\} \arrow[dd, "X"'] & & \{Y\} \arrow[ll, "{[Y\mapsto X]}"'] \arrow[dl, "\gamma_f"'] \arrow[dd, "Y"] \\
& \mathbf{X}_1 \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\
\mathbf{X}_0 & & \mathbf{X}_0
\end{tikzcd}$$

where $\gamma_f(Y) = (f : X \to Y)$. Obviously, $(\overline{f}, \gamma_f, [Y \mapsto X]) : (X : \{X\} \to \mathbf{X}_0) \to (Y : \{Y\} \to \mathbf{X}_0)$ is a well-defined map in $\pi(\mathbf{X})$.

Note that in the cartesian restriction category $\pi(\mathbf{X})$ each object $F : I \to \mathbf{X}_0$ can be written as the finite partial product of singleton families:

$$(F : I \to \mathbf{X}_0) = \prod_{i \in I} (F(i) : \{i\} \to \mathbf{X}_0).$$

For each map $(e, \gamma, g) : (F : I \to \mathbf{X}_0) \to (G : J \to \mathbf{X}_0)$ and each $i \in I$, if $i = g(j)$ for some $j \in J$, we can construct a map $S_i = (e(i), \gamma_{\gamma(j)}, [j \mapsto g(j)]) : (F(i) : \{i\} \to \mathbf{X}_0) \to$

$(G(j) : \{j\} \to \mathbf{X}_0)$ between singleton families:

$$\begin{tikzcd}
\{g(j)\} \arrow[dd, "F(i)"'] & & \{j\} \arrow[ll, "{[j\mapsto g(j)]}"'] \arrow[dl, "\gamma_{\gamma(j)}"'] \arrow[dd, "G(j)"] \\
& \mathbf{X}_1 \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\
\mathbf{X}_0 & & \mathbf{X}_0
\end{tikzcd}$$

where $\gamma_{\gamma(j)} = \gamma(j) : F(g(j)) \to G(j)$ and $\overline{\gamma(j)} = e(i)$. Otherwise, when $i \notin g(J)$, we have the following map between singleton families $S_i = (e(i), \emptyset, \emptyset) : (F(i) : \{i\} \to \mathbf{X}_0) \to (\emptyset : \emptyset \to \mathbf{X}_0)$:

$$\begin{tikzcd}
\{i\} \arrow[dd, "F(i)"'] & & \emptyset \arrow[ll, "\emptyset"'] \arrow[dl, "\emptyset"'] \arrow[dd, "\emptyset"] \\
& \mathbf{X}_1 \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\
\mathbf{X}_0 & & \mathbf{X}_0
\end{tikzcd}$$

It is easy to see that $(e, \gamma, g) : (F : I \to \mathbf{X}_0) \to (G : J \to \mathbf{X}_0)$ can be written as the finite partial product of maps between singleton families:

$$(e, \gamma, g) = \prod_{i \in I} S_i.$$

To summarize, we have

Lemma 2.2.1 *(i) Each object can be written as a finite partial product of singleton families in $\pi(\mathbf{X})$;*

(ii) Each map between singleton families in $\pi(\mathbf{X})$ is the form of

$$(\overline{f}, \gamma_f, [Y \mapsto X]) : (X : \{X\} \to \mathbf{X}_0) \to (Y : \{Y\} \to \mathbf{X}_0);$$

(iii) Each map can be written as a finite partial product of maps between singleton families in $\pi(\mathbf{X})$.

For each Cartesian restriction category $\mathbf{Y}$, every restriction functor $\mathcal{F} : \mathbf{X} \to U_c(\mathbf{Y})$ has a finite partial product-preserving restriction functor extension $\mathcal{F}^* : \pi(\mathbf{X}) \to \mathbf{Y}$ in $\mathbf{crCat}_0$:

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\mathcal{J}"] \arrow[dr, "\mathcal{F}"'] & U_c(\pi(\mathbf{X})) \arrow[d, "U_c(\mathcal{F}^*)"] & \pi(\mathbf{X}) \arrow[d, dotted, "\exists! \mathcal{F}^*"] \\
& U_c(\mathbf{Y}) & \mathbf{Y}
\end{tikzcd}$$

where $\mathcal{F}^* : \pi(\mathbf{X}) \to \mathbf{Y}$ is constructed by

$$\mathcal{F}^*((e_X, \gamma_Y, [Y \mapsto X]) : (X : \{X\} \to \mathbf{X}_0) \to (Y : \{Y\} \to \mathbf{X}_0)) = \mathcal{F}(\gamma_Y),$$

and

$$\mathcal{F}^*((e, \gamma, g) : (F : I \to \mathbf{X}_0) \to (G : J \to \mathbf{X}_0)) = \mathcal{F}^*(\prod_{i \in I} S_i) = \prod_{i \in I} \mathcal{F}^*(S_i).$$

Obviously, $U_c(F^*)\mathcal{J} = \mathcal{F}$. If $\mathcal{G}$ is a finite partial product-preserving restriction functor such that $U_c(\mathcal{G})\mathcal{J} = \mathcal{F}$, then for each map $(e_X, \gamma_Y, [Y \mapsto X])$ between singleton families in $\pi(\mathbf{X})$ we have

$$\mathcal{G}(e_X, \gamma_Y, [Y \mapsto X]) = \mathcal{F}^*(e_X, \gamma_Y, [Y \mapsto X])$$

and so for each map $(e, \gamma, g) = \prod_i S_i$ with maps S_i between singleton families in $\pi(\mathbf{X})$ we have

$$\mathcal{G}(e, \gamma, g) = \mathcal{G}(\prod_i S_i) = \prod_i \mathcal{G}(S_i) = \prod_i \mathcal{F}^*(S_i) = \mathcal{F}^*(\prod_i S_i) = \mathcal{F}^*(e, \gamma, g).$$

That is, $\mathcal{G} = \mathcal{F}^*$. Hence such a finite partial product-preserving restriction functor $\mathcal{F}^*$ is unique up to natural isomorphism. Therefore $\pi(\mathbf{X})$ is free to $U_c : \mathbf{crCat}_0 \to \mathbf{rCat}_0$.

Theorem 2.2.2 $\pi \dashv U_c : \mathbf{crCat}_0 \to \mathbf{rCat}_0$ *is an adjoint pair so that* $\pi(\mathbf{X})$ *is the free cartesian restriction category over a restriction category* $\mathbf{X}$.

Chapter 3

Join Restriction Categories

As each inverse semigroup has a natural partial order given by $s \leq t \Leftrightarrow s = te$ for some idempotent e, join (least upper bound) and meet (greatest lower bound) operations can be introduced to inverse semigroups so that join completion and meet completion theorems can be shown (see [31] or [32] for details). Similarly, as each restriction category is poset enriched with $f \leq g \Leftrightarrow f = g\overline{f}$ by Lemma 1.6.3, join and meet operations may exist in restriction categories. In particular, a join restriction category is a restriction category with a join operation that works well with the restriction and is distributive with respect to composition, introduced to each hom set (see Definition 3.1.7 below). This chapter is devoted to studying join restriction categories.

Not every restriction category is a join restriction category. To form a class of join restriction categories, naturally we ask how a join can be added to a given restriction category freely. After introducing $\smile$-compatible relation, shared by elements having a join, and join restriction categories, we immediately answer this question by providing a construction, called the join completion for restriction categories, using down closed and $\smile$-compatible sets.

Since join completion for inverse semigroups, given by down closed and $\sim$-compatible sets, is well known in inverse semigroup theory, one may wonder how join completion for restriction categories is related to join completion for inverse semigroups. First, as each inverse semigroup is a both restriction and corestriction category, we can talk about $\smile$-compatibility, $\smile^{\mathrm{op}}$-compatibility, and $\underset{\smile}{\smile}$-compatibility, where $f \underset{\smile}{\smile} g$ if $f \smile g$ and $f \smile^{\mathrm{op}} g$. The $\underset{\smile}{\smile}$-compatibility is the same as the $\sim$-compatibility in semigroup theory so that the join completion for restriction categories coincides with the $\underset{\smile}{\smile}$-join completion for inverse semigroups in the setting of inverse categories. Then we provide adjunctions among restriction categories, inverse categories, join restriction categories, and join inverse categories.

Since the partial map category of a given $\mathcal{M}$-category is a restriction category by Proposition 1.6.17, it is very natural to wonder precisely which partial map categories correspond to join restriction categories. The answer, given that join restriction categories are a very natural concept, is much more involved than one might have suspected. It involves the existence of certain colimits which must be stable, and, in addition a significant side-condition on the monics used for the partiality.

Lack and Sobocińsk [28, 29] introduced *adhesive categories* which provide a general setting in which double-pushout (DPO) rewriting (see [19]) could be performed. To our surprise and delight the basic diagrammatic conditions involved in that development matched the diagrammatic conditions needed for our completeness result. In hindsight, of course, this should have been expected. One of the most more significant properties of a join restriction category is that one can formally glue together objects in a join restriction category to form new objects. This, of course, is directly related to the operations required in double pushout rewriting.

The relationship between the join restriction categories and adhesive categories is, unfortunately, not so straightforward. An adhesive category requires all pushouts along monics exist, all pullbacks exist, and all pushouts along monics are "well-behaved" in the sense that they are van Kampen squares (see [28, 29] by Lack and Sobocińsk). However, to form a join for a set of partial maps $\{(m_i, f_i)\}$, which is $\smile$-compatible, in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, we need to have a special colimits for $\mathcal{M}$-maps $\{m_i\}$ that is pullback stable and all "gaps" between such colimits and $\mathcal{M}$-maps must be in $\mathcal{M}$. To express these precisely, we introduce the notions of *van colimits*, $\mathcal{M}$-*adhesive categories*, and $\mathcal{M}$-*gaps*. It turns out that $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a join restriction category if and only if for each $\smile$-compatible set of partial maps $\{(m_i, f_i)\}$, $\{m_i\}$ has a stable colimit and all $\mathcal{M}$-maps between such stable colimits and $\mathcal{M}$-maps are in $\mathcal{M}$, or equivalently, $\mathbf{C}$ is an $\mathcal{M}$-adhesive category and all $\mathcal{M}$-gaps are in $\mathcal{M}$. The second main goal of this chapter is to provide a proof of the completeness of join restriction categories in partial map categories.

3.1 Join Restriction Categories

In this section, we shall introduce join restriction categories and provide a construction that can add a join to eah given restriction category freely, called join completion for restriction category. Then we shall compare join completion for restriction categories with join completion for inverse semigroups by providing adjunctions among restriction categories, inverse categories, join restriction categories, and join inverse categories.

3.1.1 Compatibility and Join

We start with the following lemma.

Lemma 3.1.1 *In a restriction category* $\mathbf{X}$*, let* $\emptyset \neq S \subseteq \mathrm{map}_{\mathbf{X}}(X, Y)$*. If* $\vee_{s \in S} s$ *exists with respect to* $f \leq g \Leftrightarrow f = g\overline{f}$*, then* $s_1\overline{s_2} = s_2\overline{s_1}$ *for all* $s_1, s_2 \in S$*.*

Proof: If $\vee_{s\in S} s$ exists, then $s_1, s_2 \leq \vee_{s\in S} s$ and so

$$s_1 = (\vee_{s\in S} s)\overline{s_1} \text{ and } s_2 = (\vee_{s\in S} s)\overline{s_2}.$$

Hence

$$s_1\overline{s_2} = (\vee_{s\in S} s)\overline{s_1}\overline{(\vee_{s\in S} s)\overline{s_2}} = (\vee_{s\in S} s)\overline{s_2}\ \overline{s_1} = s_2\overline{s_1}.$$

□

In a restriction category, by Lemma 3.1.1 above, if $\vee_{s\in S} s$ exists, then elements in S must have the relationship, described by $s_1\overline{s_2} = s_2\overline{s_1}$, which will be $\smile$-compatible relation defined below.

Definition 3.1.2 *In a restriction category, two maps f and g is called $\smile$-compatible, denoted by $f \smile g$, if $f\overline{g} = g\overline{f}$. A set S of maps is called $\smile$-compatible if $s_1 \smile s_2$ for all $s_1, s_2 \in S$.*

Some properties of $\smile$-compatibility are summarized in the following lemma.

Lemma 3.1.3 *In any restriction category,*

- *(i)* $f \smile f$;
- *(ii)* $f \smile g$ *implies* $g \smile f$;
- *(iii)* *any two restriction idempotents with the same domain are* $\smile$*-compatible;*
- *(iv)* $f \leq g$ *implies* $f \smile g$;
- *(v)* *if* $f \smile f'$ *and* $g \smile g'$*,then* $gf \smile g'f'$;
- *(vi)* *suppose that* $f' \leq f$ *and* $g' \leq g$*. If* $f \smile g$*, then* $f' \smile g'$;
- *(vii)* $f \smile g$ *and* $\overline{f} = \overline{g}$ *imply* $f = g$.

PROOF:

- *(i)* Obvious.
- *(ii)* Clear.
- *(iii)* If maps f and g have the same domain, then $\overline{f}\,\overline{\overline{g}} = \overline{f}\,\overline{g} = \overline{g}\,\overline{f} = \overline{g}\,\overline{\overline{f}}$ and so $\overline{f} \smile \overline{g}$.
- *(iv)* $f \leq g \Rightarrow f = g\overline{f} \Rightarrow f\overline{g} = g\overline{f}\,\overline{g} = g\overline{g}\,\overline{f} = g\overline{f} \Rightarrow f \smile g$.

(v)

$$\begin{aligned} gf\overline{g'f'} &= gf\overline{f'}\,\overline{g'f'} \\ &= gf'\overline{f}\,\overline{g'f'} \\ &= gf'\overline{g'f'}\,\overline{f} \\ &= g\overline{g'}f'\,\overline{f} \\ &= g'\overline{g}f\,\overline{f'} \\ &= g'f\overline{gf}\,\overline{f'} \\ &= g'f\overline{f'}\,\overline{gf} \\ &= g'f'\overline{f}\,\overline{gf} \\ &= g'f'\overline{gf}. \end{aligned}$$

(vi) Since $f' \leq f$, $g' \leq g$, and $f \smile g$, we have

$$f\overline{f'} = f', g\overline{g'} = g' \ \text{ and } \ g\overline{f} = f\overline{g}.$$

Hence

$$\begin{aligned} g'\overline{f'} &= g\overline{g'}\overline{f\overline{f'}} \\ &= g\overline{g'}\,\overline{f}\,\overline{f'} \\ &= g\overline{f}\,\overline{g'}\,\overline{f'} \\ &= f\overline{g}\,\overline{g'}\,\overline{f'} \\ &= f\overline{f'}\,\overline{g\overline{g'}} \\ &= f'\overline{g'} \end{aligned}$$

and therefore $f' \smile g'$.

(vii) $f \smile g$ implies $f\overline{g} = g\overline{f}$ and so, $\overline{f} = \overline{g}$ gives

$$f = f\overline{f} = f\overline{g} = g\overline{f} = g\overline{g} = g.$$

□

Now, let us look at the compatibility $\smile$ in a partial map category.

Lemma 3.1.4 *In a partial map category* $\mathsf{Par}(\mathbf{C},\mathcal{M})$*,* $\{(m_i,f_i)\}_{i\in\Gamma}$ *is compatible if and only if for any* $i,j\in\Gamma$*,* $f_i\pi_j = f_j\pi_i$*, where* (π_i,π_j) *is the pullback of* (m_i,m_j)*:*

$$\begin{tikzcd}
\bullet \arrow[r, "\pi_i"] \arrow[d, "\pi_j"'] & \bullet \arrow[d, "m_j"] \\
\bullet \arrow[r, "m_i"] & \bullet
\end{tikzcd}$$

Proof: Note that

$$\begin{aligned}
(m_i,f_i)\smile(m_j,f_j) &\Leftrightarrow (m_j,f_j)\overline{(m_i,f_i)} = (m_i,f_i)\overline{(m_j,f_j)} \\
&\Leftrightarrow (m_j,f_j)(m_i,m_i) = (m_i,f_i)(m_j,m_j) \\
&\Leftrightarrow (m_i\pi_j, f_j\pi_i) = (m_j\pi_i, f_i\pi_j) \\
&\Leftrightarrow \text{there is an isomorphism } \alpha \text{ such that} \\
&\ m_i\pi_j\alpha = m_j\pi_i \text{ and } f_j\pi_i\alpha = f_i\pi_j.
\end{aligned}$$

Since $m_i\pi_j = m_j\pi_i$ and $m_i,\ m_j$ are monics, $m_i\pi_j\alpha = m_j\pi_i$ implies $\pi_i\alpha = \pi_i$ and $\pi_j\alpha = \pi_j$. Hence $\alpha = 1$. Therefore $(m_i,f_i)\smile(m_j,f_j)\Leftrightarrow f_i\pi_j = f_j\pi_i$. □

For any subset S of a poset $(X,\leq)$, write

$$\downarrow S = \{x\in X \mid \exists s\in S \text{ such that } x\leq s\}.$$

We say a subset S of a poset $(X,\leq)$ is *down closed* if $\downarrow S = S$. The operator $\downarrow(\ )$ is a closure operator that is for any subsets S,T of a poset $(X,\leq)$,

$$\downarrow\emptyset = \emptyset,\ \ \downarrow(\downarrow S) = \downarrow S,\ \ S\subseteq\downarrow S,\ \ \downarrow(S\cup T) = (\downarrow S)\cup(\downarrow T).$$

Lemma 3.1.5 *In a restriction category* $\mathbf{C}$*, if* $S\subseteq\mathbf{C}(B,C)$ *and* $T\subseteq\mathbf{C}(A,B)$*, then*

(i) $\downarrow((\downarrow S)T) = \downarrow(ST) = \downarrow(S(\downarrow T)) = \downarrow((\downarrow S)(\downarrow T)) = (\downarrow S)(\downarrow T)$*;*

(ii) $\downarrow(\overline{S}) = \overline{(\downarrow S)}$. *In particular, if* S *is down closed then so is* $\overline{S}$.

Proof: *(i)* Clearly, $\downarrow(ST) \subseteq \downarrow((\downarrow S)T)$ since $S \subseteq \downarrow S$. For any $x \in \downarrow((\downarrow S)T)$, $x \leq yt$ for some $y \in \downarrow S$ and $t \in T$ and so $y \leq s$ for some $s \in S$. Hence $x \leq yt \leq st$ and therefore $x \in \downarrow(ST)$. It follows that $\downarrow(S(\downarrow T)) \subseteq \downarrow(ST)$. Then $\downarrow(ST) = \downarrow((\downarrow S)T)$. Similarly, $\downarrow(ST) = \downarrow(S(\downarrow T))$. Now, applying $\downarrow(ST) = \downarrow(\downarrow(S)T)$ to $\downarrow(S(\downarrow(T)))$, clearly we have $\downarrow(ST) = \downarrow(S(\downarrow(T))) = \downarrow((\downarrow S)(\downarrow T))$.

For each $f \in \downarrow((\downarrow S)(\downarrow T))$, $f \leq uv$ for some $u \in \downarrow S$ and $v \in \downarrow T$ so that $u \leq s$ and $v \leq t$ for some $s \in S$ and $t \in T$. Hence $f \leq uv \leq st$ and therefore $f = s \cdot t\overline{f} \in (\downarrow S)(\downarrow T)$ as $s \in \downarrow S$ and $t\overline{f} \in \downarrow T$. Thus, $\downarrow((\downarrow S)(\downarrow T)) = (\downarrow S)(\downarrow T)$ as $(\downarrow S)(\downarrow T) \subseteq \downarrow((\downarrow S)(\downarrow T))$ obviously.

(ii) For any $x \in \downarrow(\overline{S})$, $x \leq \overline{s}$ for some $s \in S$ and so $x = \overline{s}\,\overline{x} = \overline{s\overline{x}} \in \overline{(\downarrow S)}$ since $s\overline{x} \leq s$. Hence $\downarrow(\overline{S}) \subseteq \overline{(\downarrow S)}$. Conversely, for any $x \in \downarrow S$, $x \leq s$ for some $s \in S$ and so $s\overline{x} = x$. Then $\overline{x} = \overline{s\overline{x}} = \overline{s}\,\overline{x} \in \downarrow\overline{S}$ since $\overline{s}\,\overline{x} \leq \overline{s}$. Hence $\overline{(\downarrow S)} \subseteq \downarrow\overline{S}$. Therefore $\downarrow(\overline{S}) = \overline{(\downarrow S)}$. ☐

A set S of maps is $\smile$*-compatible* if for any $s, s' \in S$, $s \smile s'$. Clearly, each subset of a $\smile$-compatible set is also $\smile$-compatible.

Lemma 3.1.6 *Let* $F : \mathbf{C} \to \mathbf{D}$ *be a restriction functor. Then, for any* $\smile$*-compatible set* $S \subseteq \mathbf{C}(A, B)$,

(i) *both* $\downarrow S$ *and* $F(S)$ *are also a* $\smile$*-compatible set;*

(ii) *for any map* $f : X \to A$, $Sf = \{sf \mid s \in S\}$ *is* $\smile$*-compatible;*

(iii) $\downarrow F(\downarrow S) = \downarrow F(S)$.

Proof: *(i)* By Lemma 3.1.3 *(vi)*, $\downarrow S$ is $\smile$-compatible.

For any $s_1, s_2 \in S$, if $s_1 \smile s_2$, then $s_1\overline{s_2} = s_2\overline{s_1}$ and so $F(s_1)\overline{F(s_2)} = F(s_2)\overline{F(s_1)}$. Hence $F(s_1) \smile F(s_2)$ and therefore $F(S)$ is $\smile$-compatible.

(ii) For any $sf, s'f \in Sf$, since $s \smile s'$, $s\overline{s'} = s'\overline{s}$. Then $s\overline{s'}f = s'\overline{s}f$ and so $sf\overline{s'f} = s'f\overline{sf}$. Hence $sf \smile s'f$.

(iii) Clearly, $\downarrow F(S) \subseteq \downarrow F(\downarrow S)$ since $S \subseteq \downarrow S$. Conversely, for any $x \in \downarrow F(\downarrow S)$, there is $y \in \downarrow S$ such that $x \leq F(y)$. But $y \leq s$ for some $s \in S$. Then $x \leq F(y) \leq F(s)$. Hence $x \in \downarrow F(S)$ and therefore $\downarrow F(\downarrow S) \subseteq \downarrow F(S)$. So $\downarrow F(\downarrow S) = \downarrow F(S)$. □

Definition 3.1.7 *A restriction category* $\mathbf{C}$ *is called a* (finite) join restriction category *if for each pair of objects* A *and* B *and each (finite)* $\smile$*-compatible subset* $S \subseteq \mathbf{C}(A, B)$ *there is* $\vee_{s \in S} s \in \mathbf{C}(A, B)$ *such that*

[J.1] $\vee_{s \in S} s$ *is the join with respect to the partial order* $\leq$ *on* $\mathbf{C}(A, B)$*,*

[J.2] $\overline{\vee_{s \in S} s} = \vee_{s \in S} \overline{s}$*, and for any* $f \in \mathbf{C}(B, Y)$ *and* $g \in \mathbf{C}(X, A)$*,*

[J.3] $(\vee_{s \in S} s)g = \vee_{s \in S}(sg)$*,*

[J.4] $f(\vee_{s \in S} s) = \vee_{s \in S}(fs)$*.*

The last condition **[J.4]** is, in fact, implied by the other conditions as shown in the following lemma:

Lemma 3.1.8 *In a join restriction category* $\mathbf{C}$*,*

(i) *for each (finite)* $\smile$*-compatible subset* $S \subseteq \mathbf{C}(A, B)$ *and each* $s \in S$*,* $s \smile \vee_{s \in S} s$*;*

(ii) *for each (finite)* $\smile$*-compatible subset* $S \subseteq \mathbf{C}(A, B)$ *with* $t \in S$*,* $(\vee_{s \in S} s)\overline{t} = t$*;*

(iii) *the condition* **[J.4]** *is redundant in Definition 3.1.7;*

(iv) *for any* $\smile$*-compatible sets* $S \subseteq \mathbf{C}(B, C)$ *and* $T \subseteq \mathbf{C}(A, B)$*,* $\big(\vee_{s \in S}\, s\big)\big(\vee_{t \in T}\, t\big) = \vee_{s \in S, t \in T}(st)$*;*

(v) *for any* $\smile$*-compatible set* $T \subseteq \mathbf{C}(A, B)$*,* $\vee_{x \in \downarrow T} x = \vee_{t \in T} t$*. In particular, for any map* f*,* $\vee_{x \in \downarrow \{f\}} x = f$*.*

Proof:

(i) Since $s \leq \vee_{s \in S} s$, by Lemma 3.1.3(iv) we have $s \smile \vee_{s \in S} s$.

(ii) Since S is $\smile$-compatible, for each $s \in S \setminus \{t\}$ we have

$$s\bar{t} = t\bar{s} \leq t.$$

Hence

$$\begin{aligned}
(\vee_{s \in S} s)\bar{t} &= \Big((\vee_{s' \in S \setminus \{s\}} s') \vee t\Big)\bar{t} \\
&= (\vee_{s' \in S \setminus \{s\}} s')\bar{t} \vee t \quad ([\mathbf{J.3}]) \\
&= (\vee_{s' \in S \setminus \{s\}} s'\bar{t}) \vee t \\
&= t.
\end{aligned}$$

(iii) We observe first that $\overline{f}(\vee_{s \in S} s) = \vee_{s \in S}(\overline{f}s)$ since

$$\begin{aligned}
\overline{f}(\vee_{s \in S} s) &= \Big(\vee_{s \in S}\, s\Big)\overline{f(\vee_{s \in S} s)} \\
&= \vee_{s \in S}\Big(s\overline{f(\vee_{s \in S} s)}\Big) \quad ([\mathbf{J.3}]) \\
&= \vee_{s \in S}\Big(s\overline{f(\vee_{s \in S} s)\overline{s}}\Big) \\
&= \vee_{s \in S}\Big(s\overline{fs}\Big) \\
&= \vee_{s \in S}\Big(\overline{f}s\Big).
\end{aligned}$$

It follows that

$$\begin{aligned} \overline{f(\vee_{s \in S} s)} &= \overline{\overline{f}(\vee_{s \in S} s)} \\ &= \overline{\vee_{s \in S} \overline{f} s} \\ &= \vee_{s \in S} \overline{\overline{f} s} \;\; ([\mathbf{J.2}]) \\ &= \vee_{s \in S} \overline{f s} \\ &= \overline{\vee_{s \in S} f s} \;\; ([\mathbf{J.2}]). \end{aligned}$$

Since, for each $t \in S$, $t \leq \vee_{s \in S} s$, we have

$$ft \leq f \vee_{s \in S} s.$$

Hence

$$\vee_{s \in S} f s \leq f \vee_{s \in S} s$$

and therefore

$$\vee_{s \in S} f s \smile f \vee_{s \in S} s$$

Thus, by Lemma 3.1.3(vii),

$$f\big(\vee_{s \in S} s \big) = \vee_{s \in S} (f s).$$

That is, the condition [**J.4**] is redundant.

(iv) By [**J.3**] and [**J.4**].

(v) Since $T \subseteq \downarrow T$, we have $\vee_{t \in T} t \leq \vee_{x \in \downarrow T} x$.

For each $x \in \downarrow T$, there is $t \in T$ such that $x \leq t$. Then $x \leq t \leq \vee_{t \in T} t$ and so $\vee_{x \in \downarrow T} x \leq \vee_{t \in T} t$. Thus, $\vee_{x \in \downarrow T} x = \vee_{t \in T} t$.

□

3.1.2 Join Completion for Restriction Categories

A restriction functor $F : \mathbf{C} \to \mathbf{D}$ between two join restriction categories $\mathbf{C}$ and $\mathbf{D}$ is called a *join restriction functor* if for any $\smile$-compatible subset S, $F(\vee_{s\in S}s) = \vee_{s\in S}F(s)$. Join restriction categories and join restriction functors form a category, denoted by $\mathbf{jrCat}_0$, which is a subcategory of $\mathbf{rCat}_0$. Clearly, we have the inclusion functor $\mathcal{I}_j : \mathbf{jrCat}_0 \to \mathbf{rCat}_0$.

Given any restriction category $\mathbf{X}$, we construct a join restriction category $\mathbf{j}(\mathbf{X})$, called *the join completion* of $\mathbf{X}$, with

objects: $X \in \mathbf{X}$;

maps: a map $S : A \to B$ is given by a down closed and $\smile$-compatible set $S \subseteq \mathbf{X}(A, B)$;

identities: $1_A = \downarrow\{1_A\} = \{e | e = \overline{e} : A \to A \text{ in } \mathbf{X}\}$;

composition: for any maps $S : A \to B$ and $T : B \to C$ in $\mathbf{j}(\mathbf{X})$, $TS = \{ts | s \in S,\ t \in T\}$;

restriction: $\overline{S} = \{\overline{s} | s \in S\}$;

join: $\vee_{i\in\Gamma}S_i = \bigcup_{i\in\Gamma} S_i$, where each S_i is a down closed and $\smile$-compatible set in $\mathbf{X}$ and $\{S_i\}_{i\in\Gamma}$ is a $\smile$-compatible set in $\mathbf{j}(\mathbf{X})$.

By Lemma 3.1.5, it is easy to see that $\mathbf{j}(\mathbf{X})$ is a restriction category as shown in the following lemma.

Lemma 3.1.9 $\mathbf{j}(\mathbf{X})$ *is a restriction category.*

Proof: By Lemma 3.1.5, clearly, identities, composition, and restriction are well-defined. For all $\mathbf{j}(\mathbf{X})$-maps $S : A \to B$, $T : B \to C$, and $U : C \to D$, since

$$1_B S = \downarrow\{1_B\}S = S = S \downarrow\{1_A\} = S1_A$$

and

$$U(TS) = (UTS) = (UT)S,$$

$\mathbf{j}(\mathbf{X})$ is a category. Now the four restriction axioms are verified as follows.

[R.1] $S\overline{S} = \{s_1\overline{s_2} | s_1, s_2 \in S\} = S$ as S is down closed and each $s_1\overline{s_2} \leq s_1 \in S$.

[R.2] $\overline{S_1}\,\overline{S_2} = \{\overline{s_1}\,\overline{s_2} | s_1 \in S_1, s_2 \in S_2\} = \{\overline{s_2}\,\overline{s_1} | s_1 \in S_1, s_2 \in S_2\} = \overline{S_2}\,\overline{S_1}$.

[R.3] $\overline{S_2\overline{S_1}} = \{\overline{s_2\overline{s_1}} | s_1 \in S_1, s_2 \in S_2\} = \{\overline{s_2}\,\overline{s_1} | s_1 \in S_1, s_2 \in S_2\} = \overline{S_2}\,\overline{S_1}$.

[R.4] Clearly, $\overline{T}S = \{\overline{t}s | s \in S, t \in T\} = \{s\overline{ts} | s \in S, t \in T\} \subseteq S\overline{TS}$. On the other hand, for all $s_1, s_2 \in S$ and $t \in T$, since S is $\smile$-compatible, $s_1\overline{ts_2} = s_1\overline{s_2}\,\overline{ts_2} = s_2\overline{s_1}\,\overline{ts_2} \leq s_2\overline{ts_2} = \overline{t}s_2$. Hence $S\overline{TS} \subseteq \overline{T}S$. Thus, $\overline{T}S = S\overline{TS}$.

$\square$

The relations $\smile$ and $\leq$ in $\mathbf{j}(\mathbf{X})$ are characterized in the following lemma.

Lemma 3.1.10 *For any map $S, T : A \to B$ in $\mathbf{j}(\mathbf{X})$:*

(i) *$S \smile T$ in $\mathbf{j}(\mathbf{X})$ if and only if $s \smile t$ in $\mathbf{X}$ for any $s \in S$ and $t \in T$.*

(ii) *If $S \smile T$, then $S \leq S \cup T$ and $T \leq S \cup T$.*

(iii) *$S \leq T$ in $\mathbf{j}(\mathbf{X})$ if and only if $S \subseteq T$.*

Proof: *(i)* If $S \smile T$ in $\mathbf{j}(\mathbf{X})$, then $S\overline{T} = T\overline{S}$. For any $s \in S$ and $t \in T$, since $s\overline{t} \in S\overline{T} = T\overline{S}$, there are $s' \in S, t' \in T$ such that $s\overline{t} \leq t'\overline{s'}$. Hence, by noticing $t \smile t'$ implies $t\overline{t'} = t'\overline{t}$,

$$s\overline{t} = t'\overline{s'}\,\overline{s\overline{t}} = t'\overline{s'}\,\overline{s}\overline{t} = t'\overline{t}\;\overline{s'}\overline{s} = t\overline{t'}\;\overline{s'}\overline{s} = t\overline{s}(\overline{t'}\;\overline{s'}) \leq t\overline{s}.$$

Symmetrically, we have $t\overline{s} \leq s\overline{t}$. Then $t\overline{s} = s\overline{t}$ and so $s \smile t$.

Conversely, if for any $s \in S, t \in T$, $s \smile t$, then $s\bar{t} = \bar{s}t$ and so

$$S\overline{T} = \{s\bar{t} \mid s \in S, t \in T\} = \{t\bar{s} \mid s \in S, t \in T\} = T\overline{S}.$$

Thus, $S \smile T$ in $\mathbf{j}(\mathbf{X})$, as desired.

(ii) Clearly, $S \subseteq (S \cup T)\overline{S}$ since $S \subseteq S \cup T$. For any $x \in (S \cup T)\overline{S}$, $x \leq w\bar{s}$ for some $w \in S \cup T$ and $s \in S$. If $w \in S$, then $x \leq w\bar{s} \leq w \in S$ implies $x \in S$ since S is down closed. If $w \in T$, then $w\bar{s} = s\bar{w} \leq s \in S$ since $S \smile T$ and so $x \in S$ too. Hence $(S \cup T)\overline{S} \subseteq S$ and therefore $(S \cup T)\overline{S} = S$. Then $S \leq S \cup T$. Similarly, $T \leq S \cup T$.

(iii) If $S = \emptyset$, then $T\overline{S} = \emptyset = S$ and so $S \leq T$. If $\emptyset \neq S \subseteq T$, then, for any $t \in T$ and $s \in S$, $t\bar{s} = s\bar{t} \leq s$ since $s, t \in T$ and T is $\smile$-compatible. Hence $t\bar{s} \in S$ and therefore $T\overline{S} \subseteq S$. Clearly, for any $s \in S$, $s = s\bar{s} \in T\overline{S}$. Thus, $S \subseteq T\overline{S}$. Then $S = T\overline{S}$ and so $S \leq T$.

Conversely, if $S \leq T$, then $S = T\overline{S}$. For any $s \in S$, we have $s \leq t\overline{s'} \leq t$ for some $s' \in S, t \in T$ and so $s \in T$. Hence $S \subseteq T$, as desired. $\square$

Now we are ready to show that $\mathbf{j}(\mathbf{X})$ is a join restriction category.

Lemma 3.1.11 $\mathbf{j}(\mathbf{X})$ *is a join restriction category.*

Proof: By Lemma 3.1.10 *(i)*, $\bigcup_{i\in\Gamma} S_i$ is down closed and $\smile$-compatible and so the join $\vee_{i\in\Gamma} S_i = \bigcup_{i\in\Gamma} S_i$ is well-defined.

For any $\smile$-compatible set $\{S_i\}_{i\in\Gamma} \subseteq \mathbf{j}(\mathbf{X})(A, B)$, each $S_i \leq \vee_{i\in\Gamma} S_i$. On the other hand, if each $S_i \leq X$, then $X\overline{S_i} = S_i$ and so $X(\bigcup_{i\in\Gamma} \overline{S_i}) = \bigcup_{i\in\Gamma} S_i$. Hence $X\overline{\bigcup_{i\in\Gamma} S_i} = \bigcup_{i\in\Gamma} S_i$ and therefore $\vee_{i\in\Gamma} S_i = \bigcup_{i\in\Gamma} S_i \leq X$. Thus, $\vee_{i\in\Gamma} S_i = \bigcup_{i\in\Gamma} S_i$ is the join with respect to the partial order $\leq$ on the hom-set of $\mathbf{j}(\mathbf{X})(A, B)$. Clearly, for any map $M \in \mathbf{j}(\mathbf{X})(B, Y)$ and $N \in \mathbf{j}(\mathbf{X})(X, A)$ we have

$$M(\vee_{i\in\Gamma} S_i) = M(\bigcup_{i\in\Gamma} S_i) = \bigcup_{i\in\Gamma} (MS_i) = \vee_{i\in\Gamma}(MS_i),$$

$$(\vee_{i\in\Gamma} S_i)N = (\bigcup_{i\in\Gamma} S_i)N = \bigcup_{i\in\Gamma}(S_i N) = \vee_{i\in\Gamma}(S_i N),$$

and

$$\overline{\vee_{i\in\Gamma} S_i} = \overline{\bigcup_{i\in\Gamma} S_i} = \bigcup_{i\in\Gamma} \overline{S_i} = \vee_{i\in\Gamma}\overline{S_i}.$$

Hence $\mathbf{j}(\mathbf{X})$ is indeed a join restriction category. ☐

Join completion $\mathbf{j}$ is indeed a left adjoint of the inclusion functor $\mathbf{jrCat}_0 \to \mathbf{rCat}_0$.

Theorem 3.1.12 $\mathcal{I}_j : \mathbf{jrCat}_0 \to \mathbf{rCat}_0$ *has a left adjoint given by the join completion* $\mathbf{j}$.

Proof: For any restriction category $\mathbf{X}$, we have a faithful restriction functor

$$\eta_{\mathbf{X}} : \mathbf{X} \to \mathcal{I}_j\mathbf{j}(\mathbf{X})$$

given by taking $f : X \to Y$ to $\downarrow\{f\} : X \to Y$, which serves as the unit of $\mathbf{j} \dashv \mathcal{I}_j$. In fact, for any join restriction category and restriction functor $F : \mathbf{X} \to \mathcal{I}_j(\mathbf{Y})$, there is a join restriction functor $F^{\#} : \mathbf{j}(\mathbf{X}) \to \mathbf{C}$ given by sending $S : A \to B$ to $\vee \downarrow(F(S)) : F(A) \to F(B)$ such that

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\eta_{\mathbf{X}}"] \arrow[dr, "F"'] & \mathcal{I}_j(\mathbf{j}(\mathbf{X})) \arrow[d, "\mathcal{I}_j(F^{\#})"] & \mathbf{j}(\mathbf{X}) \arrow[d, dotted, "F^{\#}"] \\
 & \mathcal{I}_j(\mathbf{C}) & \mathbf{C}
\end{tikzcd}$$

commutes. Suppose that $G : \mathbf{j}(\mathbf{X}) \to \mathbf{C}$ is a join restriction functor such that $\mathcal{I}_j(G)\eta_{\mathbf{X}} = F$. For any map $S : A \to B$ in $\mathbf{j}(\mathbf{X})$, since S is down closed and $\smile$-compatible, $S = \{s\}_{s\in S} = \vee_{s\in S}(\downarrow\{s\})$, where each $s : A \to B$ is a map in $\mathbf{X}$. Clearly, for each $s \in S$, $G(\downarrow\{s\}) = F(s) = F^{\#}(\downarrow\{s\})$. Hence

$$G(S) = G(\vee_{s\in S}\downarrow\{s\}) = \vee_{s\in S} G(\downarrow\{s\}) = \vee_{s\in S} F^{\#}(\downarrow\{s\}) = F^{\#}(\vee_{s\in S}\downarrow\{s\}) = F^{\#}(S)$$

and therefore the uniqueness of $F^{\#}$ follows. Thus, $\mathbf{j} \dashv \mathcal{I}_j$. ☐

3.1.3 Join Completion for Inverse Categories

Join completion for inverse semigroups is well-known (See [31]). In this subsection, we shall describe the join completion for inverse categories and compare it with the join completion given in the last subsection. Throughout this subsection, $\mathbf{I}$ is an inverse category.

For a given $X \in \mathrm{ob}(\mathbf{I})$, we write

$$E(X) = \{f : X \to X \mid f^2 = f\}, E(\mathbf{I}) = \bigcup_{X \in \mathrm{ob}(\mathbf{I})} E(X)$$

and

$$\mathcal{O}(X) = \{f : X \to X \mid \overline{f} = f\}, \mathcal{O}(\mathbf{I}) = \bigcup_{X \in \mathrm{ob}(\mathbf{I})} \mathcal{O}(X).$$

Some basic properties of inverse categories are summarized in the following lemma.

Lemma 3.1.13 *In an inverse category* $\mathbf{I}$*,*

(1) *for each* $f \in E(\mathbf{I})$*,* $f^{(-1)} = f$*;*

(2) *for each* $X \in \mathrm{ob}(\mathbf{I})$*,* $E(X) = \mathcal{O}(X)$*. In particular, two idempotents with the same domain commute;*

(3) *for each idempotent* e *and each map* f*, both* $f^{(-1)}ef$ *and* $fef^{(-1)}$ *are idempotents whenever they are defined. In particular, for each* $f \in \mathrm{map}(\mathbf{I})$*, both* $f^{(-1)}f$ *and* $ff^{(-1)}$ *are idempotents;*

(4) *for each* $f \in \mathrm{map}(\mathbf{I})$*,* $(f^{(-1)})^{(-1)} = f$*;*

(5) *for* $f_1, \cdots, f_n \in \mathrm{map}(\mathbf{I})$ *such that* $f_1 \cdots f_n$ *is defined,* $(f_1 \cdots f_n)^{(-1)} = f_n^{(-1)} \cdots f_1^{(-1)}$*;*

(6) *for idempotents* $e : X \to X$ *and* $e' : Y \to Y$ *and a map* $f : X \to Y$*,* $fe \leq f$ *and* $e'f \leq f$*;*

(7) *the following are equivalent:*

(7.1) *$g \leq f$,*

(7.2) *$g = fe$ for some idempotent e,*

(7.3) *$g = e'f$ for some idempotent e',*

(7.4) *$g^{(-1)} \leq f^{(-1)}$,*

(7.5) *$g = gg^{(-1)}f$;*

(8) *if $g \leq f$, then $g^{(-1)} \leq f^{(-1)}$, $g^{(-1)}g \leq f^{(-1)}f$, and $gg^{(-1)} \leq ff^{(-1)}$.*

PROOF:

(1) For each $f \in E(\mathbf{I})$, $f^2 = f$. Then

$$fff = f^2 f = f^2 = f$$

and so $f^{(-1)} = f$.

(2) For each $f \in E(X)$, $\overline{f} = f^{(-1)}f = f^2 = f$ and so $f \in \mathcal{O}(X)$. Obviously, $\mathcal{O}(X) \subseteq E(X)$. Thus, $E(X) = \mathcal{O}(X)$. Since restriction idempotents are commutative, idempotents are all restriction idempotents so that they are commutative.

(3) Since idempotents are commutative,

$$(f^{(-1)}ef)^2 = f^{(-1)}eff^{(-1)}ef = f^{(-1)}eff^{(-1)}f = f^{(-1)}ef.$$

Similarly, $(fef^{(-1)})^2 = fef^{(-1)}$. So both $f^{(-1)}ef$ and $fef^{(-1)}$ are idempotents.

In particular, taking $e = 1$, one has that both $f^{(-1)}f$ and $ff^{(-1)}$ are idempotents.

(4) As $f^{(-1)}$ is the unique solution of $xfx = x$ and $fxf = f$.

(5) When $n = 2$, using idempotents with the same domain are commutative,

$$f_1f_2(f_2^{(-1)}f_1^{(-1)})f_1f_2 = f_1(f_2f_2^{(-1)})(f_1^{(-1)}f_1)f_2 = f_1(f_1^{(-1)}f_1)(f_2f_2^{(-1)})f_2 = f_1f_2$$

and

$$f_2^{(-1)}f_1^{(-1)}(f_1f_2)f_2^{(-1)}f_1^{(-1)} = f_2^{(-1)}(f_1^{(-1)}f_1)(f_2f_2^{(-1)})f_1^{(-1)}$$

$$= f_2^{(-1)}(f_2f_2^{(-1)})(f_1^{(-1)}f_1)f_1^{(-1)} = (f_2^{(-1)}f_2f_2^{(-1)})(f_1^{(-1)}f_1f_1^{(-1)}) = f_2^{(-1)}f_1^{(-1)}.$$

Then $(f_1f_2)^{(-1)} = f_2^{(-1)}f_1^{(-1)}$ and so, by mathematical induction,

$$(f_1 \cdots f_n)^{(-1)} = f_n^{(-1)} \cdots f_1^{(-1)}.$$

(6) As $f\overline{fe} = f\overline{f}e = fe$ and $\overline{fe'f} = \overline{e'}f = e'f$.

(7) (7.1) ⇒ (7.2): $g \leq f \Rightarrow g = fg^{(-1)}g$ so that there is an idempotent $e = g^{(-1)}g$ such that $g = fe$.

(7.2) ⇒ (7.3): Since $fef^{(-1)}f = ff^{(-1)}fe = fe$.

(7.3) ⇒ (7.4): Since $g = e'f \leq f$ implies $g = fg^{(1)}g$, $g^{(-1)} = g^{(-1)}gf^{(-1)} \leq f^{(-1)}$ as $g^{(-1)}g$ is an idempotent.

(7.4) ⇒ (7.5): $g^{(-1)} \leq f^{(-1)}$ implies $g^{(-1)} = f^{(-1)}gg^{(-1)}$ implies $g = gg^{(-1)}f$.

(7.5) ⇒ (7.1): As $gg^{(-1)}$ is an idempotent, $g = gg^{(-1)}f \leq f$.

(8) If $g \leq f$, then $g^{(-1)}g \leq f^{(-1)}g \leq f^{(-1)}f$ and $gg^{(-1)} \leq gf^{(-1)} \leq ff^{(-1)}$.

□

A category **C** is called a *regular-inverse category* if one of the following two conditions is satisfied:

(1) each **C**-map f has at least one map g, called a *regular-inverse* of f, such that $fgf = f$;

(2) each **C**-map f has at least one map h, called an *inverse* of f, such that $fhf = f$ and $hfh = h$.

Remark 3.1.14 The two conditions in the definition of regular-inverse categories above are equivalent. To see this, assume first that **C** is defined by (1). Then gfg satisfies $f(gfg)f = f$ and $(gfg)f(gfg) = gfg$ so that gfg is an inverse of f. Conversely, if **C** is defined by (2), then, obviously, h serves as a regular-inverse of f in (1).

As inverse semigroups can be characterized by regular semigroups in which idempotents commute, inverse categories can be characterized by regular-inverse categories in which idempotents commute. To show this, we need the following technical lemma.

Lemma 3.1.15 *In a category, if h_1 and h_2 are inverse of f, then the following are equivalent:*

(i) $(fh_1)(fh_2) = (fh_2)(fh_1)$ *and* $(h_1f)(h_2f) = (h_2f)(h_1f)$*;*

(ii) $fh_1 = fh_2$ *and* $h_1f = h_2f$*;*

(iii) $h_1fh_2 = h_2fh_1$*;*

(iv) $h_1 = h_2$.

Proof: *(i)* ⇒ *(ii)*: $fh_1 = (fh_2f)h_1 = (fh_1)(fh_2) = fh_2$. Similarly, $h_1f = h_1(fh_2f) = (h_2f)(h_1f) = h_2f$.

(ii) ⇒ *(iii)*: $h_1fh_2 = h_2fh_2 = h_2fh_1$.

(ii) ⇒ *(iv)*: $h_1 = h_1fh_1 = h_2fh_1 = h_2fh_2 = h_2$.

(iii) ⇒ *(i)* and *(iv)* ⇒ *(i)* are clear. ☐

Now we list a few characterizations of inverse categories in the following proposition.

Proposition 3.1.16 *For a category* **C**, *the following are equivalent:*

(i) **C** *is an inverse category;*

(*ii*) $\mathbf{C}$ *is a category in which each map has a unique inverse;*

(*iii*) *there is a functor* $(\ )^\circ : \mathbf{C} \to \mathbf{C}^{\mathrm{op}}$ *that is the identity on objects and satisfies* $(f^\circ)^\circ = f$*,* $ff^\circ f = f$*, and* $ff^\circ gg^\circ = gg^\circ ff^\circ$*;*

(*iv*) $\mathbf{C}$ *is a regular-inverse category in which for each pair of idempotents with the same domain commute.*

Proof: (*i*) $\Leftrightarrow$ (*ii*) $\Leftrightarrow$ (*iii*): By Theorem 2.20 in [14].

(*i*) $\Rightarrow$ (*iv*): Clearly, each inverse category is a regular-inverse category in which, by Lemma 3.1.13(2), each pair of idempotents with the same domain commute.

(*iv*) $\Rightarrow$ (*ii*): Assume that each pair of idempotents with the same domain commute and that map $f : A \to B$ has inverse h_1 and h_2. By Lemma 3.1.15, $h_1 = h_2$. Thus, the uniqueness of the inverse of f follows. Hence $\mathbf{C}$ is an inverse category. $\square$

Remark 3.1.17 The existence of a unique regular-inverse implies clearly the existence of a unique inverse but the opposite does not hold true. For example, $\mathcal{I}_X$, the set of all partial one to one transformations on a set X, forms a one-object category, called the *symmetric inverse semigroup* on X in inverse semigroup theory. The empty transformation $\emptyset : X \to X$ has a unique inverse $\emptyset$ as $\emptyset h \emptyset = \emptyset$ and $h \emptyset h = h$ implies $h = \emptyset$. But $\emptyset$ does not have a unique regular inverse as $\emptyset g \emptyset = \emptyset$ for all one to one transformation g on X. Hence the existence of a unique inverse in Proposition 3.1.16(*ii*) can not be replaced by the existence of a unique regular-inverse.

In order to give the definition of join restriction categories and construct free join restriction categories over restriction categories, at the beginning of this chapter we introduced $\smile$-compatibility in a restriction category by $f \smile g \Leftrightarrow f\overline{g} = g\overline{f}$. As each inverse category $\mathbf{I}$ is both a restriction and a co-restriction category, we can talk about $\smile$-compatibility, $\smile^{\mathrm{op}}$-compatibility, and $\stackrel{\smile}{\smile}$-compatibility, where, $f \smile^{\mathrm{op}} g$ in $\mathbf{I}$ if and only if $f \smile g$ in $\mathbf{I}^{\mathrm{op}}$, and $f \underset{\smile}{\smile} g$ if and only if $f \smile g$ and $f \smile^{\mathrm{op}} g$. Furthermore, $\smile$, $\smile^{\mathrm{op}}$, and $\underset{\smile}{\smile}$ are precisely $\sim_l$, $\sim_r$, and $\sim$, which are defined in inverse semigroup theory, respectively. To prove this, we first recall the compatible relations $\sim_l$, $\sim_r$, and $\sim$ in inverse semigroup theory.

As in [31], for **I**-maps $f, g : X \to Y$, the $\sim_l$-*compatibility* is defined by

$$f \sim_l g \Leftrightarrow fg^{(-1)} \in E(Y),$$

the $\sim_r$-*compatibility relation* is defined by

$$f \sim_r g \Leftrightarrow f^{(-1)}g \in E(X),$$

and the $\sim$-*compatible relation* is defined by

$$f \sim g \Leftrightarrow fg^{(-1)} \in E(Y) \text{ and } f^{(-1)}g \in E(X).$$

Obviously, the three relations are reflexive and symmetric but not transitive.

The relationship between $\sim$-compatibility and $\smile$-compatibility is shown in the following lemma.

Lemma 3.1.18 *In an inverse category* $\mathbf{I}$*,*

(i) $f \sim_l g \Leftrightarrow f^{(-1)} \sim_r g^{(-1)}$*;*

(ii) $f \sim_l g$ *if and only if* $fg^{(-1)}g = gf^{(-1)}f$ *if and only if* $f \smile g$*;*

(iii) $f \sim_r g$ *if and only if* $g^{(-1)}gf = f^{(-1)}fg$ *if and only if* $f \smile^{\mathrm{op}} g$*;*

(iv) $f \sim g$ *if and only if* $fg^{(-1)}g = gf^{(-1)}f$ *and* $g^{(-1)}gf = f^{(-1)}fg$ *if and only if* $f \underset{\smile}{\smile} g$*;*

(v) *if* $f \leq h$ *then* $f \sim h$ *and if* $f \leq h$ *and* $g \leq h$ *then* $f \sim g$*.*

PROOF:

(i) $f \sim_l g \Leftrightarrow (f^{(-1)})^{(-1)}g^{(-1)} = fg^{(-1)} \in E(\mathbf{I}) \Leftrightarrow f^{(-1)} \sim_r g^{(-1)}$.

(ii) If $f \sim_l g$, then $fg^{(-1)} \in E(\mathbf{I})$ and so $gf^{(-1)} = (fg^{(-1)})^{(-1)} = fg^{(-1)}$.

To prove that $fg^{(-1)}g = gf^{(-1)}f$, we first claim that $w \leq f$ and $w \leq g$ implies $w \leq gf^{(-1)}f$ and $w \leq fg^{(-1)}g$.

In fact, $w \leq f$ and $w \leq g$ implies $w^{(-1)}w \leq f^{(-1)}f$ by Lemma 3.1.13 so that

$$w = ww^{(-1)}w \leq gf^{(-1)}f.$$

Similarly, $w \leq f$ and $w \leq g$ implies $w = ww^{(-1)}w \leq fg^{(-1)}g$.

Now, since $fg^{(-1)}g \leq f$ and $fg^{(-1)}g \leq g$ as $g^{(-1)}$ and $fg^{(-1)}$ are restriction idempotents, we have

$$fg^{(-1)}g \leq gf^{(-1)}f.$$

Similarly, as $gf^{(-1)}f = fg^{(-1)}f \leq f$ and $gf^{(-1)}f \leq g$,

$$gf^{(-1)}f \leq fg^{(-1)}g.$$

Thus, $gf^{(-1)}f = fg^{(-1)}g$.

Conversely, if $gf^{(-1)}f = fg^{(-1)}g$, then

$$fg^{(-1)} = fg^{(-1)}gg^{(-1)} = gf^{(-1)}fg^{(-1)} = (fg^{(-1)})^{(-1)}fg^{(-1)} \in E(\mathbf{I})$$

and so $f \sim_l g$. Thus, $f \sim_l g$ if and only if $gf^{(-1)}f = fg^{(-1)}g$.

(iii) Similar to *(ii)*.

(iv) By *(ii)* and *(iii)*.

(v) If $f \leq h$, then $f = hf^{(-1)}f$ and so

$$f^{(-1)}h = (hf^{(-1)}f)^{(-1)}h = (f^{(-1)}f)(h^{(-1)}h)$$

and

$$fh^{(-1)} = hf^{(-1)}fh^{(-1)} = (hf^{(-1)})(hf^{(-1)})^{(-1)}.$$

Hence both $f^{(-1)}h$ and $fh^{(-1)}$ are idempotents and therefore $f \sim h$.

If $f \leq h$ and $g \leq h$, then $f = hf^{(-1)}f$ and $g = hg^{(-1)}g$ and so

$$fg^{(-1)} = hf^{(-1)}f(hg^{(-1)}g)^{(-1)} = hf^{(-1)}fg^{(-1)}gh^{(-1)}$$

is an idempotent by Lemma 3.1.13. Similarly, $f^{(-1)}g$ is an idempotent too. Thus, $f \sim g$.

$\square$

A set $S \subseteq \mathrm{map}_{\mathbf{I}}(A, B)$ is said to be $\sim_l$*-compatible* (resp. $\sim_r$*-compatible*, $\sim$*-compatible*, $\smile$*-compatible*, $\smile^{\mathrm{op}}$*-compatible*, $\underset{\smile}{\smile}$*-compatible*) if for all $f, g \in S$, $f \sim_l g$ (resp. $f \sim_r g$, $f \sim g$, $f \smile g$, $f \smile^{\mathrm{op}} g$, $f \underset{\smile}{\smile} g$).

Recall that an inverse semigroup S is *join complete* if every non-empty $\underset{\smile}{\smile}$-compatible subset has a join ([31], p.27). An inverse semigroup S is *left (right) infinitely distributive* if, whenever $A \subseteq S$ is a non-empty subset for which $\vee A$ exists, then $\vee sA$ $(\vee(As))$ exists for any $s \in S$ and $s(\vee A) = \vee(sA)$ $((\vee A)s = \vee(As))$. An inverse semigroup that is both left and right infinitely distributive is called *infinitely distributive* ([31], p.28). It is well-known that each inverse semigroup can be embedded in a complete, infinitely distributive inverse semigroup, called *join completion* ([31], p.31). One can describe the join completion $\mathbf{jj}(\mathbf{I})$ for inverse category $\mathbf{I}$ using $\underset{\smile}{\smile}$-compatible down closed sets as its maps as follows.

Let $\mathbf{jj}(\mathbf{I})$ be with

objects: $X \in \mathbf{I}$;

maps: a map $P : X \to Y$ is given by a $\underset{\smile}{\smile}$-compatible and down closed subset

$$P \subseteq \mathrm{map}_{\mathbf{I}}(X, Y);$$

identities: $1_X = \mathcal{O}(X)$;

composition: for any maps $P : X \to Y$ and $Q : Y \to Z$ in $\mathbf{jj}(\mathbf{I})$, $QP = \{gf | f \in P,\ g \in Q\}$;

restriction: $\overline{P} = P^{(-1)}P$;

join: $\vee_{i\in\Gamma} P_i = \bigcup_{i\in\Gamma} P_i$, where each $P_i : X \to Y$ is $\underset{\smile}{\smile}$-compatible and down closed in $\mathbf{I}$ and $\{P_i\}_{i\in\Gamma}$ is a $\underset{\smile}{\smile}$-compatible set in $\mathbf{jj}(\mathbf{I})$.

To show that $\mathbf{jj}(\mathbf{I})$ is an inverse category, we need the following lemma.

Lemma 3.1.19 *Given an inverse category* $\mathbf{I}$*,*

(*i*) $\mathcal{O}(X)$ *is* $\underset{\smile}{\smile}$*-compatible and down closed in* $\mathbf{I}$*;*

(*ii*) *if* $P \subseteq \mathrm{map}_{\mathbf{I}}(X, Y)$ *is* $\sim_l$*-compatible* ($\sim_r$*-compatible) and down closed, then* $P^{(-1)}$ *is* $\sim_r$*-compatible* ($\sim_l$*-compatible) and down closed;*

(*iii*) *if* $P \subseteq \mathrm{map}_{\mathbf{I}}(X, Y)$ *and* $Q \subseteq \mathrm{map}_{\mathbf{I}}(Y, Z)$ *are* $\underset{\smile}{\smile}$*-compatible (resp.* $\smile$*-compatible,* $\smile^{\mathrm{op}}$*-compatible) and down closed, then so is* $QP \subseteq \mathrm{map}_{\mathbf{I}}(X, Z)$ $\underset{\smile}{\smile}$*-compatible (resp.* $\smile$*-compatible,* $\smile^{\mathrm{op}}$*-compatible) and down closed;*

(*iv*) *if* $P \subseteq \mathrm{map}_{\mathbf{I}}(X, Y)$ *is* $\sim_r$*-compatible and down closed, then* $P^{(-1)}P = \{a^{(-1)}a | a \in P\}$ *is* $\underset{\smile}{\smile}$*-compatible and down closed and* $PP^{(-1)}P = P$*. If* $P \subseteq \mathrm{map}_{\mathbf{I}}(X, Y)$ *is* $\sim_l$*-compatible and down closed, then* $PP^{(-1)} = \{aa^{(-1)} | a \in P\}$ *is* $\underset{\smile}{\smile}$*-compatible and down closed and* $PP^{(-1)}P = P$*;*

(*v*) *if* $P \subseteq \mathrm{map}_{\mathbf{I}}(X, X)$ *is* $\underset{\smile}{\smile}$*-compatible and down closed, then* $P^2 = P$ *if and only if* $P \subseteq \mathcal{O}(X)$*.*

PROOF:

(i) If $e \leq f \in \mathcal{O}(X)$, then $e = f\overline{e} \in \mathcal{O}(X)$ and so $\mathcal{O}(X)$ is down closed. Obviously, for all $e, e' \in \mathcal{O}(X)$, $e(e')^{(-1)} = ee' \in \mathcal{O}(X)$ and $e^{(-1)}e' = ee' \in \mathcal{O}(X)$. Hence $e \overset{\smile}{\smile} e'$ and so $\mathcal{O}(X)$ is $\overset{\smile}{\smile}$-compatible. Thus, $\mathcal{O}(X)$ is $\overset{\smile}{\smile}$-compatible and down closed.

(ii) If $f \leq p^{(-1)}$ for some $p \in P$, then $f = p^{(-1)}f^{(-1)}f$ and so

$$f^{(-1)} = f^{(-1)}fp \leq p \in P$$

as $f^{(-1)}f$ is an idempotent. Hence $f^{(-1)} \in P$ as P is down closed, and therefore $f = (f^{(-1)})^{(-1)} \in P^{(-1)}$. Thus, $P^{(-1)}$ is down closed.

For each $p_1^{(-1)}, p_2^{(-1)} \in P^{(-1)}$ with $p_1, p_2 \in P$, since P is $\sim_l$-compatible, $p_1 \sim_l p_2$. Then we have $p_1^{(-1)} \sim_r p_2^{(-1)}$ and so $P^{(-1)}$ is $\sim_r$-compatible.

(iii) We only show the case of $\overset{\smile}{\smile}$ as $\smile$ and $\smile^{\mathrm{op}}$ cases are similar. If $h \leq gf \in QP$ with $f \in P$ and $g \in Q$, then $h = gfh^{(-1)}h$. Since $fh^{(-1)}h \leq f$ and P is down closed,

$$fh^{(-1)}h \in P \text{ and } h = g(fh^{(-1)}h) \in QP.$$

Hence QP is down closed.

To show that QP is $\overset{\smile}{\smile}$-compatible, let $g_1f_1, g_2f_2 \in QP$ with $f_1, f_2 \in P$ and $g_1, g_2 \in Q$. Since P and Q are $\overset{\smile}{\smile}$-compatible, $f_1f_2^{(-1)}$, $g_1g_2^{(-1)}$, $f_1^{(-1)}f_2$, and $g_1^{(-1)}g_2$ are idempotents. Then

$$(g_1f_1)^{(-1)}g_2f_2 = f_1^{(-1)}g_1^{(-1)}g_2f_2 \leq f_1^{(-1)}f_2 \in \mathcal{O}(\mathbf{I})$$

and

$$g_1f_1(g_2f_2)^{(-1)} = g_1f_1f_2^{(-1)}g_2^{(-1)} \leq g_1g_2^{(-1)} \in \mathcal{O}(\mathbf{I})$$

and so both $g_1f_1(g_2f_2)^{(-1)}$ and $(g_1f_1)^{(-1)}g_2f_2$ are idempotents. Thus, $g_1f_1 \overset{\smile}{\smile} g_2f_2$.

(iv) Obviously, $\{a^{(-1)}a|a \in P\} \subseteq P^{(-1)}P$. For each $f^{(-1)}g \in P^{(-1)}P$ with $f, g \in P$, if P is $\sim_r$-compatible, then $f^{(-1)}g \in \mathcal{O}(\mathbf{I})$. Since $ff^{(-1)}g \leq g \in P$, $ff^{(-1)}g \in P$. Hence

$$\begin{aligned} f^{(-1)}g &= (f^{(-1)}g)^{(-1)}f^{(-1)}g \\ &= g^{(-1)}ff^{(-1)}g \\ &= g^{(-1)}ff^{(-1)}ff^{(-1)}g \\ &= (ff^{(-1)}g)^{(-1)}(ff^{(-1)}g) \\ &\in \{a^{(-1)}a|a \in P\}. \end{aligned}$$

Thus, $P^{(-1)}P = \{a^{(-1)}a|a \in P\}$.

Let $e \leq f^{(-1)}f \in P^{(-1)}P$ with $f \in P$. Then

$$fe \leq ff^{(-1)}f = f \in P.$$

But

$$e = f^{(-1)}fe = f^{(-1)}fe^2 = ef^{(-1)}fe = (fe)^{(-1)} \cdot fe.$$

Hence $e \in P^{(-1)}P$ and therefore $P^{(-1)}P$ is down closed.

For all $f^{(-1)}f, g^{(-1)}g \in P^{(-1)}P$ with $f, g \in P$, clearly both

$$(f^{(-1)}f)^{(-1)}g^{(-1)}g = f^{(-1)}fg^{(-1)}g$$

and

$$f^{(-1)}f(g^{(-1)}g)^{(-1)} = f^{(-1)}fg^{(-1)}g$$

are idempotents. Hence $f^{(-1)}f \stackrel{\smile}{\smile} g^{(-1)}g$ and therefore $P^{(-1)}P$ is $\stackrel{\smile}{\smile}$-compatible and down closed.

Obviously $P \subseteq PP^{(-1)}P$ as $p = pp^{(-1)}p \in PP^{(-1)}P$ for each $p \in P$. To prove the inverse direction, let $fg^{(-1)}h \in PP^{(-1)}P$ with $f, g, h \in P$. Since $P^{(-1)}P = \{p^{(-1)}p|p \in P\}$, there exists $u \in P$ such that $g^{(-1)}h = u^{(-1)}u$. Then

$$fg^{(-1)}h = fu^{(-1)}u \leq f \in P$$

and so $fg^{(-1)}h \in P$ as P is down closed. Hence $PP^{(-1)}P \subseteq P$ and therefore $PP^{(-1)}P = P$.

Similarly, if P is $\sim_l$-compatible and down closed, then $PP^{(-1)} = \{aa^{(-1)} | a \in P\}$ is $\overset{\smile}{\smile}$-compatible and down closed and $PP^{(-1)}P = P$.

(v) For each $\overset{\smile}{\smile}$-compatible and down closed subset $P \subseteq \mathrm{map}_{\mathbf{I}}(X, Y)$ such that $P^2 = P$ and each $p \in P$, we have $p = p_1p_2$ for some $p_1, p_2 \in P$. Then

$$p = pp^{(-1)}p = p(p_1p_2)^{(-1)}p = (pp_2^{(-1)})(p_1^{(-1)}p)$$

is an idempotent as $p \overset{\smile}{\smile} p_1$ and $p \overset{\smile}{\smile} p_2$ and so $P \subseteq \mathcal{O}(X)$.

On the other hand, if a $\overset{\smile}{\smile}$-compatible and down closed subset $P \subseteq \mathcal{O}(X)$, then, clearly,

$$P^2 = PP = P^{(-1)}P = \{p^{(-1)}p | p \in P\} = \{pp | p \in P\} = \{p | p \in P\} = P.$$

So, $P : X \to X$ is an idempotent in $\mathbf{jj}(\mathbf{I})$.

□

We can now show that $\mathbf{jj}(\mathbf{I})$ is an inverse category.

Lemma 3.1.20 $\mathbf{jj}(\mathbf{I})$ *is an inverse category.*

Proof: By Lemma 3.1.19 (i) and (iii), the identity and composition in $\mathbf{jj}(\mathbf{I})$ are well-defined. To show that $\mathbf{jj}(\mathbf{I})$ is a category, we need to verify both identity and associative laws.

For each $\mathbf{jj}(\mathbf{I})$-map $P : X \to Y$,

$$1_Y P = \mathcal{O}(Y)P \supseteq P$$

as $1_Y \in \mathcal{O}(Y)$. On the other hand, for each $ep \in \mathcal{O}(Y)P$ with $e \in \mathcal{O}(Y)$ and $p \in P$,

$$ep \leq p \in P.$$

Since P is down closed, $ep \in P$. Hence $\mathcal{O}(Y)P \subseteq P$ and therefore $1_Y P = \mathcal{O}(Y)P = P$. Similarly, we have $P1_X = P\mathcal{O}(X) = P$. So identity law holds true.

For all $\mathbf{jj}(\mathbf{I})$-maps $P : X \to Y$, $Q : Y \to Z$, and $R : Z \to A$,

$$R(QP) = \{r(qp)|p \in P, q \in Q, r \in R\} = \{(rq)p|p \in P, q \in Q, r \in R\} = (RQ)P.$$

Hence associative law holds true and therefore $\mathbf{jj}(\mathbf{I})$ is a category.

For each $\mathbf{jj}(\mathbf{I})$-map $P : X \to Y$, by Lemma 3.1.19 (iv), $PP^{(-1)}P = P$. So $\mathbf{jj}(\mathbf{I})$ is a regular-inverse category.

Finally, by Lemma 3.1.19 (v), each map P is an idempotent in $\mathbf{jj}(\mathbf{I})$ if and only if $P \subseteq \mathcal{O}(\mathbf{I})$ and so they commute. Thus, $\mathbf{jj}(\mathbf{I})$ is an inverse category by Proposition 3.1.16. $\square$

The following lemma characterizes $\leq$ and $\overset{\smile}{\smile}$ and verifies that $\vee$ is well-defined in $\mathbf{jj}(\mathbf{I})$.

Lemma 3.1.21 (i) *For* $\mathbf{jj}(\mathbf{I})$*-maps* $P_1, P_2 : X \to Y$*,* $P_1 \leq P_2$ *if and only if* $P_1 \subseteq P_2$*;*

(ii) *for* $\mathbf{jj}(\mathbf{I})$*-maps* $P_i : X \to Y, i \in \Gamma$*,* $\{P_i | i \in \Gamma\}$ *is* $\overset{\smile}{\smile}$*-compatible in* $\mathbf{jj}(\mathbf{I})$ *if and only if* $\bigcup_{i\in\Gamma} P_i$ *is* $\overset{\smile}{\smile}$*-compatible and down closed in* $\mathbf{I}$*.*

PROOF:

(i) If $P_1 \leq P_2$, then the fact that $P_1^{(-1)}P_1 = \{p^{(-1)}p | p \in P_1\}$ consists of idempotents implies that $P_1 = P_2 P_1^{(-1)} P_1 \subseteq P_2$ as P_2 is down closed.

Conversely, if $P_1 \subseteq P_2$, then, clearly,

$$P_1 = P_1 P_1^{(-1)} P_1 \subseteq P_2 P_1^{(-1)} P_1.$$

On the other hand, as P_1 is $\overset{\smile}{\smile}$-compatible and down closed, $P_!^{(-1)}P_1 = \{p_1^{(-1)}p_1 | p_1 \in P_1\}$. For each $p_2 p_1^{(-1)} p_1 \in P_2 P_1^{(-1)} P_1$ with $p_1 \in P_1$ and $p_2 \in P_2$, $p_1 \in P_2$ and

$p_2 \in P_2$ implies that $p_2 p_1^{(-1)}$ is an idempotent as $p_1 \smile p_2$. Hence $p_2 p_1^{(-1)} p_1 \in P_1$ as $p_2 p_1^{(-1)} p_1 \leq p_1 \in P_1$, and therefore $P_2 P_1^{(-1)} P_1 \subseteq P_1$. Thus, $P_1 = P_2 P_1^{(-1)} P_1$.

(ii) If $\bigcup_{i\in\Gamma} P_i$ is $\smile$-compatible and down closed in $\mathbf{I}$, then, $\bigcup_{i\in\Gamma} P_i$ is a $\mathbf{jj}(\mathbf{I})$-map and for all $i, j \in \Gamma$, $P_i \leq \bigcup_{i\in\Gamma} P_i$ and $P_j \leq \bigcup_{i\in\Gamma} P_i$ and so $P_i \smile P_j$. Hence $\{P_i | i \in \Gamma\}$ is $\smile$-compatible in $\mathbf{jj}(\mathbf{I})$.

Conversely, suppose that $\{P_i\}_{i\in\Gamma}$ is $\smile$-compatible in $\mathbf{jj}(\mathbf{I})$. For all $f, g \in \bigcup_{i\in\Gamma} P_i$, assume that $f \in P_i$ and $g \in P_j$ for some $i, j \in \Gamma$. Since $\{P_i | i \in \Gamma\}$ is $\smile$-compatible in $\mathbf{jj}(\mathbf{I})$, both $P_i^{(-1)} P_j \subseteq \mathcal{O}(\mathbf{jj}(\mathbf{I}))$ and $P_i P_j^{(-1)} \subseteq \mathcal{O}(\mathbf{jj}(\mathbf{I}))$ and so both $f^{(-1)} g$ and $f g^{(-1)}$ are idempotents. Hence $f \smile g$ and therefore $\bigcup_{i\in\Gamma} P_i$ is $\smile$-compatible.

If $f \leq p \in \bigcup_{i\in\Gamma} P_i$, then $f \leq p \in P_i$ for some $i \in \Gamma$ and so $f \in P_i \subseteq \bigcup_{i\in\Gamma} P_i$ as P_i is down closed. Hence $\bigcup_{i\in\Gamma} P_i$ is $\smile$-compatible and down closed.

$\square$

Definition 3.1.22 *An inverse category* $\mathbf{I}$ *is called a* join inverse category *if for each pair of* $\mathbf{I}$*-objects* X *and* Y *and each* $\smile$*-compatible subset* $P \subseteq \mathrm{map}_{\mathbf{I}}(X, Y)$*, there is* $\vee_{p\in P} p \in \mathrm{map}_{\mathbf{I}}(X, Y)$ *such that the conditions* [**J**.**1**], [**J**.**2**], [**J**.**3**], *and* [**J**.**4**] *in the definition of join restriction categories (Definition 3.1.7) are satisfied.*

The inverse categories and join functors (functors that preserve joins) form a subcategory $\mathbf{jinvCat}_0$ of $\mathbf{invCat}_0$ so that there is an obvious inclusion $\mathcal{I}_{jj} : \mathbf{jinvCat}_0 \to \mathbf{invCat}_0$.

$\mathbf{jj}(\mathbf{I})$ is actually a join inverse category as shown in the following lemma.

Lemma 3.1.23 *The inverse category* $\mathbf{jj}(\mathbf{I})$ *is a join inverse category.*

Proof: it is clear that the inverse category $\mathbf{jj}(\mathbf{I})$ is a restriction category with the restriction $\overline{P} = P^{(-1)} P = \{p^{(-1)} p | p \in P\}$. The four join axioms are verified as follows.

[J.1] For each $\smile$-compatible set of $\mathbf{jj}(\mathbf{I})$-maps $P_i : X \to Y, i \in \Gamma$, the join $\vee_{i\in\Gamma} P_i$ is given by $\bigcup_{i\in\Gamma} P_i$. The join is well-defined as $\bigcup_{i\in\Gamma} P_i$ is $\smile$-compatible and down closed by Lemma 3.1.21. Clearly, each $P_i \leq \bigcup_{i\in\Gamma} P_i = \vee_{i\in\Gamma} P_i$. If $P_i \leq Q$ for each $i \in \Gamma$, then $P_i \subseteq Q$ and so $\vee_{i\in\Gamma} P_i = \bigcup_{i\in\Gamma} P_i \subseteq Q$.

[J.2]

$$\begin{aligned}\overline{\vee_{i\in\Gamma} P_i} &= \overline{\bigcup_{i\in\Gamma} P_i} = (\bigcup_{i\in\Gamma} P_i)^{(-1)} \bigcup_{i\in\Gamma} P_i = \{p^{(-1)} p | p \in \bigcup_{i\in\Gamma} P_i\} \\ &= \bigcup_{i\in\Gamma} \{p_i^{(-1)} p_i | p_i \in P_i\} = \vee_{i\in\Gamma} P_i^{(-1)} P_i = \vee_{i\in\Gamma} \overline{P_i}.\end{aligned}$$

[J.3] $S\big(\vee_{i\in\Gamma} P_i\big) = S\big(\bigcup_{i\in\Gamma} P_i\big) = \bigcup_{i\in\Gamma} SP_i = \vee_{i\in\Gamma} SP_i$.

[J.4] $(\vee_{i\in\Gamma} P_i)T = \bigcup_{i\in\Gamma} P_i T = \vee_{i\in\Gamma} P_i T$.

$\square$

We are now ready to show that $\mathbf{jj}(\mathbf{I})$ is free.

Theorem 3.1.24 *The inclusion $\mathcal{I}_{jj} : \mathbf{jinvCat}_0 \to \mathbf{invCat}_0$ has a left adjoint given by the join completion* $\mathbf{jj}$.

Proof: For any inverse category $\mathbf{I}$, we have a faithful functor

$$\eta_{\mathbf{I}} : \mathbf{I} \to \mathcal{I}_{jj}(\mathbf{jj}(\mathbf{I}))$$

given by taking $f : X \to Y$ to $\downarrow f : X \to Y$, which serves as the unit of $\mathbf{jj} \dashv \mathcal{I}_{jj}$. In fact, for any join inverse category $\mathbf{X}$ and any functor $F : \mathbf{I} \to \mathcal{I}_{jj}(\mathbf{X})$, there is a join functor $F^{\#} : \mathbf{jj}(\mathbf{I}) \to \mathbf{X}$ given by sending $P : A \to B$ to $\vee \downarrow(F(P)) : F(A) \to F(B)$ such that

$$\begin{tikzcd}
\mathbf{I} \arrow[r, "\eta_{\mathbf{I}}"] \arrow[dr, "F"'] & \mathcal{I}_{jj}(\mathbf{jj}(\mathbf{I})) \arrow[d, "\mathcal{I}_{jj}(F^{\#})"] & \mathbf{jj}(\mathbf{I}) \arrow[d, dotted, "F^{\#}"] \\
& \mathcal{I}_{jj}(\mathbf{X}) & \mathbf{X}
\end{tikzcd}$$

commutes. Suppose that $G : \mathbf{jj}(\mathbf{I}) \to \mathbf{X}$ is a join functor such that $\mathcal{I}_{jj}(G)\eta_{\mathbf{I}} = F$. For any map $P : A \to B$ in $\mathbf{jj}(\mathbf{I})$, since P is $\underset{\smile}{\smile}$-compatible and down closed, $P = \vee_{p \in P}(\downarrow p)$, where each $p : A \to B$ is a map in $\mathbf{I}$. Clearly, for each $p \in P$, $G(\downarrow p) = F(p) = F^{\#}(\downarrow p)$. Hence

$$G(P) = G\big(\vee_{p\in P} \downarrow p\big) = \vee_{p\in P} G(\downarrow p) = \vee_{p\in P} F^{\#}(\downarrow p) = F^{\#}\big(\vee_{p\in P} \downarrow p\big) = F^{\#}(P)$$

and therefore the uniqueness of $F^{\#}$ follows. Thus, $\mathbf{jj} \dashv \mathcal{I}_{jj}$. $\square$

Given an inverse category $\mathbf{I}$, we have a join restriction category $\mathbf{j}(\mathbf{I})$. But $\mathbf{j}(\mathbf{I})$ is not an inverse category usually as, for a $\mathbf{j}(\mathbf{I})$-map $S : X \to Y$ that is $\smile$-compatible and down closed, $S^{(-1)}$ is not necessarily $\smile$-compatible so that $S^{(-1)}$ is not a $\mathbf{j}(\mathbf{I})$-map and cannot provide a regular inverse to S, even thought $SS^{(-1)}S = S$. The difference between $\mathbf{j}(\mathbf{I})$ and $\mathbf{jj}(\mathbf{I})$ is their maps: a map $S : X \to Y$ in $\mathbf{j}(\mathbf{I})$ is a down closed and $\smile$-compatible subset $S \subseteq \mathrm{map}_{\mathbf{I}}(X, Y)$ while a map $P : X \to Y$ in $\mathbf{jj}(\mathbf{I})$ is a down closed and $\underset{\smile}{\smile}$-compatible subset $P \subseteq \mathrm{map}_{\mathbf{I}}(X, Y)$.

Given an inverse category $\mathbf{I}$, even though $\mathbf{j}(\mathbf{I})$ is not a join inverse inverse category generally, we can form the inverse subcategory $\mathbf{inv}(\mathbf{j}(\mathbf{I}))$ of $\mathbf{j}(\mathbf{I})$. Now a natural question is whether or not the inverse category $\mathbf{inv}(\mathbf{j}(\mathbf{I}))$ is a join inverse category. The answer is "Yes!" as proved in the following proposition.

Proposition 3.1.25 *For each inverse category* $\mathbf{I}$, $\mathbf{inv}(\mathbf{j}(\mathbf{I})) = \mathbf{jj}(\mathbf{I})$.

PROOF: $\mathbf{inv}(\mathbf{j}(\mathbf{I}))$ is a subcategory of $\mathbf{j}(\mathbf{I})$, consisting of all restricted isomorphisms of $\mathbf{j}(\mathbf{I})$. Since each $\underset{\smile}{\smile}$-compatible down closed subset is $\smile$-compatible and down closed, there is an embedding $\mathbf{jj}(\mathbf{I}) \hookrightarrow \mathbf{inv}(\mathbf{j}(\mathbf{I}))$.

For each $\mathbf{inv}(\mathbf{j}(\mathbf{I}))$-map $S : X \to Y$, S is down closed and $\smile$-compatible and there is a $\mathbf{j}(\mathbf{I})$-map $T : Y \to X$ such that

$$TS = \overline{S} = \{s^{(-1)}s | s \in S\} \text{ and } ST = \overline{T} = \{t^{(-1)}t | t \in T\}.$$

Clearly,

$$STS = S\{s^{(-1)}s | s \in S\} = S$$

and

$$TST = T\{t^{(-1)}t | t \in T\} = T$$

as both S and T are down closed.

On the other hand, $S^{(-1)} = \{s^{(-1)} | s \in S\}$, as a subset, is also down closed and $SS^{(-1)} = \{ss^{(-1)} | s \in S\}$, by Lemma 3.1.19, as S down closed and $\sim_l$-compatible. Hence

$$SS^{(-1)}S = S \text{ and } S^{(-1)}SS^{(-1)} = S^{(-1)}.$$

Obviously, $(ST)(SS^{(-1)}) = (SS^{(-1)})(ST)$ as both ST and $SS^{(-1)}$ are subsets of $\mathcal{O}(Y)$. We claim now that $(TS)(S^{(-1)}S) = (S^{(-1)}S)(TS)$. In fact, for each $s^{(-1)}s \in TS$, $u^{(-1)}v \in S^{(-1)}S$ with $s, u, v \in S$, we have

$$\begin{aligned} (s^{(-1)}s)(u^{(-1)}v) &= u^{(-1)}v(v^{(-1)}u(s^{(-1)}s)u^{(-1)}v) \\ &= u^{(-1)}v((su^{(-1)}v)^{(-1)}(su^{(-1)}v)) \\ &\in (S^{(-1)}S)(TS), \end{aligned}$$

as $su^{(-1)}v \in SS^{(-1)}S = S$. Hence $(TS)(S^{(-1)}S) \subseteq (S^{(-1)}S)(TS)$. Similarly, we have

$$(S^{(-1)}S)(TS) \subseteq (TS)(S^{(-1)}S).$$

Thus, $(TS)(S^{(-1)}S) = (S^{(-1)}S)(TS)$.

By Lemma 3.1.15, $T = S^{(-1)}$. Then $S^{(-1)}$ is $\sim_l$-compatible and so S is $\sim_r$-compatible. Hence S is $\overset{\smile}{\smile}$-compatible and down closed and therefore S is a map in $\mathbf{jj}(\mathbf{I})$. Thus, $\mathbf{inv}(\mathbf{j}(\mathbf{I})) = \mathbf{jj}(\mathbf{I})$. ☐

Given a join restriction category $\mathbf{X}$, we can form the inverse subcategory $\mathbf{inv}(\mathbf{X})$ of $\mathbf{X}$. Furthermore, the inverse category $\mathbf{inv}(\mathbf{X})$ has also the join inherited from $\mathbf{X}$. To see this, we need the following lemma.

Lemma 3.1.26 *In a restriction category* $\mathbf{X}$*, let* $P = \{$*restriction isomorphisms:* $p_i : X \to Y, i \in I\}$ *be such that both* $\vee_{i\in I} p_i$ *and* $\vee_{i\in I} p_i^{(-1)}$ *exist. Then*

$$(\vee_{i\in I} p_i)^{(-1)} = \vee_{i\in I} p_i^{(-1)}.$$

Proof: It suffices to show

$$\vee_{i\in I} p_i^{(-1)} p_i = (\vee_{i\in I} p_i^{(-1)})(\vee_{i\in I} p_i)$$

and

$$\vee_{i\in I} p_i p_i^{(-1)} = (\vee_{i\in I} p_i)(\vee_{i\in I} p_i^{(-1)}).$$

We only prove the first equality as the proof of the second one is similar.

Clearly, each $p_i \leq \vee_{i\in I} p_i$ and so

$$p_i^{(-1)} p_i \leq p_i^{(-1)} \vee_{i\in I} p_i \leq (\vee_{i\in I} p_i^{(-1)})(\vee_{i\in I} p_i)$$

for all $i \in I$.

If $p_i^{(-1)} p_i \leq q$ for all $i \in I$, then

$$p_i = p_i p_i^{(-1)} p_i \leq p_i q$$

and so $\vee_{i\in I} p_i \leq (\vee_{i\in I} p_i) q$. Hence

$$(\vee_{i\in I} p_i^{(-1)})(\vee_{i\in I} p_i) \leq (\vee_{i\in I} p_i^{(-1)})(\vee_{i\in I} p_i) q = \vee_{i,j\in I} p_i^{(-1)} p_j q \leq q$$

as each $p_i^{(-1)} p_j$ is a restriction idempotent. Thus, $\vee_{i\in I} p_i^{(-1)} p_i = (\vee_{i\in I} p_i)^{(-1)} (\vee_{i\in I} p_i)$. $\square$

Now we can verify that $\mathbf{inv} : \mathbf{jrCat}_0 \to \mathbf{jinvCat}_0$ is a functor:

Lemma 3.1.27 *For a given join restriction category* $\mathbf{X}$*,* $\mathbf{inv}(\mathbf{X})$ *is a join inverse category so that there is a functor* $\mathbf{inv} : \mathbf{jrCat}_0 \to \mathbf{jinvCat}_0$*.*

Proof: For each join restriction category $\mathbf{X}$, $\mathbf{inv}(\mathbf{X})$ is an inverse category. For each $\smile$-compatible set $P = \{p_i | i \in I\} \subseteq \mathrm{map}_{\mathbf{inv}(\mathbf{X})}(X, Y)$, both $\vee_{i\in I} p_i$ and $\vee_{i\in I} p_i^{(-1)}$ exist in $\mathbf{X}$. It suffices to verify that $\vee_{i\in I} p_i$ is a map in $\mathbf{inv}(\mathbf{X})$. But this is clear as, by Lemma 3.1.26,

$$(\vee_{i\in I} p_i^{(-1)})(\vee_{i\in I} p_i) = \vee_{i\in I} p_i^{(-1)} p_i = \vee_{i\in I} \overline{p_i} = \overline{\vee_{i\in I} p_i}$$

and

$$(\vee_{i\in I} p_i)(\vee_{i\in I} p_i^{(-1)}) = \vee_{i\in I} p_i p_i^{(-1)} = \vee_{i\in I} \overline{p_i^{-1}} = \overline{\vee_{i\in I} p_i^{(-1)}}.$$

Now it is routine to verify that $\mathbf{inv} : \mathbf{jrCat}_0 \to \mathbf{jinvCat}_0$ is a functor. ☐

We now construct the left adjoint of $\mathbf{inv} : \mathbf{jrCat}_0 \to \mathbf{jinvCat}_0$ using the notion of a density relation introduced by Cockett in [7]. So let us recall the definitions and the basic properties of density relations.

A *density relation* $\leq_{\mathbf{j}}$, on a restriction category $\mathbf{X}$, is a relation $\leq_{\mathbf{j}}$ on parallel maps such that the following seven density relation axioms are satisfied:

[D.1] $f \leq_{\mathbf{j}} g$ implies $f \leq g$;

[D.2] $f \leq_{\mathbf{j}} f$ (reflexivity);

[D.3] $f \leq_{\mathbf{j}} g$ and $g \leq_{\mathbf{j}} h$ imply $f \leq_{\mathbf{j}} h$ (transitivity);

[D.4] $f \leq_{\mathbf{j}} g$ implies $\overline{f} \leq_{\mathbf{j}} \overline{g}$ (restriction);

[D.5] $f \leq h \leq g$ and $f \leq_{\mathbf{j}} g$ imply $h \leq_{\mathbf{j}} g$ (gap closed);

[D.6] $f \leq_{\mathbf{j}} g$ implies $fx \leq_{\mathbf{j}} gx$ (stability);

[D.7] $f \leq_{\mathbf{j}} g$ implies $yf \leq_{\mathbf{j}} yg$ (universality).

The following lemma provides some examples of density relations.

Lemma 3.1.28 1. *(Lemma 3.1 [7]) Each restriction functor* $F : \mathbf{X} \to \mathbf{Y}$ *induces a density relation* $\leq_{\mathbf{j}_F}$ *on* $\mathbf{X}$ *given by*

$$f \leq_{\mathbf{j}_F} g \Leftrightarrow f \leq g \textit{ and } F(f) = F(g).$$

2. *For a given join inverse category* $\mathbf{I}$*, the relation* $\leq_{\mathbf{j}(\mathbf{I})}$ *on the join completion* $\mathbf{j}(\mathbf{I})$ *given by*

$$\begin{aligned} & P \leq_{\mathbf{j}(\mathbf{I})} Q \\ \Leftrightarrow \quad & P \subseteq Q \textit{ and for each } q \in Q, \textit{there exists a } \smile\textit{-compatible subset } \{p_i, i \in \Gamma_q\} \subseteq P \\ & \textit{ such that } q = \vee_{i \in \Gamma_q} p_i, \end{aligned}$$

is a density relation on $\mathbf{j}(\mathbf{I})$.

Proof:

1. One needs to check the seven density relation axioms.

   [D.1] Obviously, by the definition, $f \leq_{\mathbf{j}_F} g$ implies $f \leq g$.

   [D.2] Reflexivity is clear as $f \leq f$ and $F(f) = F(f)$.

   [D.3] Transitivity is clear too as both $\leq$ and $=$ are transitive.

   [D.4] If $f \leq_{\mathbf{j}_F} g$, then $f \leq g$ and $F(f) = F(g)$ and so $\overline{f} \leq \overline{g}$ and $F(\overline{f}) = F(\overline{g})$. Therefore, $\overline{f} \leq_{\mathbf{j}_F} \overline{g}$.

   [D.5] If $f \leq h \leq g$ and $f \leq_{\mathbf{j}_F} g$, then $F(f) = F(g)$ and so $F(f) \leq F(h) \leq F(g)$ implies $F(h) = F(g)$. Hence, $h \leq_{\mathbf{j}_F} g$.

   [D.6] If $f \leq_{\mathbf{j}_F} g$, then $f \leq g$ and $F(f) = F(g)$ and so $fh \leq gh$ and $F(fh) = F(f)F(h) = F(g)F(h) = F(gh)$. Hence $fh \leq_{\mathbf{j}_F} gh$.

   [D.7] Similar to the argument of of **[D.6]** above.

2. The seven density relation axioms are verified as follows.

[**D.1**] As $P \leq_{\mathbf{j(I)}} Q$ implies $P \subseteq Q$ so that $P \leq Q$ in $\mathbf{j}(\mathbf{I})$.

[**D.2**] Clearly.

[**D.3**] If $P \leq_{\mathbf{j(I)}} Q$ and $Q \leq_{\mathbf{j(I)}} R$, then $P \subseteq Q \subseteq R$ and for each $r \in R$, there are $\smile$-compatible subsets $\{q_i | i \in \Gamma_r\} \subseteq Q$ and $\{p_{i_k} | k \in \Lambda_{q_i}\} \subseteq P$ such that

$$r = \vee_{i \in \Gamma_r} q_i \text{ and } q_i = \vee_{k \in \Lambda_{q_i}} p_{i_k}.$$

Hence

$$r = \vee_{i,k} p_{i_k}.$$

For each pair of p_{i_k} and p_{j_l}, as $p_{i_k} \leq \vee p_{i_k} \smile \vee\, p_{j_l} \geq p_{j_l}$, we have $p_{i_k} \smile p_{j_l}$. Thus, $P \leq_{\mathbf{j(I)}} R$.

[**D.4**] If $P \leq_{\mathbf{j(I)}} Q$, then $P \subseteq Q$ and for each $q \in Q$ there exists a $\smile$-compatible $\{p_i | i \in \Gamma\} \subseteq P$ such that $q = \vee_{i \in \Gamma} p_i$ and so $\overline{P} \subseteq \overline{Q}$ and $\overline{q} = \overline{\vee_{i \in \Gamma} p_i} = \vee_{i \in \Gamma} \overline{p_i}$. Hence $\overline{P} \leq_{\mathbf{j(I)}} \overline{Q}$.

[**D.5**] If $P \leq H \leq Q$ and $P \leq_{\mathbf{j(I)}} Q$, then, clearly, $H \leq_{\mathbf{j(I)}} Q$ as $P \subseteq H$.

[**D.6**] If $P \leq_{\mathbf{j(I)}} Q$, then $P \subseteq Q$ and for each $q \in Q$, there is a $\smile$-compatible subset $\{p_i | i \in \Gamma\} \subseteq P$ such that $q = \vee_{i \in \Gamma} p_i$. For any $\mathbf{j}(\mathbf{I})$-map H such that HP and HQ are composable, $HP \subseteq HQ$ and $hq = h(\vee_{p_i \in P, i \in \Gamma} p_i) = \vee_{p_i \in P, i \in \Gamma} h p_i$. Obviously, for each $h \in H$, $\{hp_i | i \in \Gamma\}$ is $\smile$-compatible. Thus, $HP \leq_{\mathbf{j(I)}} HQ$.

[**D.7**] Similar to the proof of [**D.6**] above.

$\square$

Given a restriction category $\mathbf{X}$, each density relation $\leq_{\mathbf{j}}$ on $\mathbf{X}$ gives rise to a restriction congruence $\sim_{\mathbf{j}}$ on parallel maps by

$$f \sim_{\mathbf{j}} g \Leftrightarrow \text{ there is a map } k \text{ such that } k \leq_{\mathbf{j}} f \text{ and } k \leq_{\mathbf{j}} g.$$

One has:

Proposition 3.1.29 (Proposition 3.2 [7]) *Given a density relation* $\leq_{\mathbf{j}}$ *on a restriction category* $\mathbf{X}$, $\sim_{\mathbf{j}}$ *is a restriction congruence which has an associated quotient functor*

$$Q_{\mathbf{j}} : \mathbf{X} \to \mathbf{X}/\sim_{\mathbf{j}}$$

that induces precisely the density relation $\leq_{\mathbf{j}}$. *Furthermore, for any restriction functor* $F : \mathbf{X} \to \mathbf{Y}$ *such that* $f \leq_{\mathbf{j}} g$ *implies* $F(f) = F(g)$ *there is a unique restriction functor* $F^{\sharp} : \mathbf{X}/\sim_{\mathbf{j}} \to \mathbf{Y}$ *such that*

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "Q_{\mathbf{j}}"] \arrow[dr, "F"'] & \mathbf{X}/\sim_{\mathbf{j}} \arrow[d, "F^{\sharp}"] \\
 & \mathbf{Y}
\end{tikzcd}$$

commutes.

By Lemma 3.1.28, each restriction functor $F : \mathbf{X} \to \mathbf{Y}$ induces a density relation $\leq_{\mathbf{j}_F}$ and so a restriction congruence $\sim_{\mathbf{j}_F}$. As $f \leq_{\mathbf{j}_F} g$ implies $F(f) = F(g)$ obviously, the restriction functor F can be factored as

$$\begin{tikzcd}
\mathbf{X} \arrow[rr, "F"] \arrow[dr, "Q_{\mathbf{j}_F}"'] & & \mathbf{F} \arrow[dl, "F^{\sharp}"] \\
 & \mathbf{X}/\sim_{\mathbf{j}_F} &
\end{tikzcd}$$

A restriction functor $F : \mathbf{X} \to \mathbf{Y}$ is called *open-separated* if $e, e' \in \mathcal{O}(X)$ implies $F(e) = F(e')$. A functor G between restriction categories is called *dense* if G is bijective on the objects, full on maps, and $G(f) = G(g)$ implies there is an idempotent e with $G(\overline{f}) = G(e) = G(\overline{g})$. It is easy to see that $Q_{\mathbf{j}_F}$ and $F^{\sharp}$ above are dense and open-separated, respectively.

Proposition 3.1.30 ([7], Proposition 3.3) *The dense functors and the open-separated functors provide a factorization system on restriction functors.*

If $\mathbf{X}$ is a restriction category and $\leq_{\mathbf{j}}$ is a density relation on $\mathbf{X}$, an object $X \in \mathbf{X}$ is $\mathbf{j}$-*unitary* if for all $\mathbf{X}$-maps $f, g : Z \to X$, $f \sim_{\mathbf{j}} g$ implies $f \smile g$. The restriction category $\mathbf{X}$ is $\mathbf{j}$-*unitary* if every $\mathbf{X}$-object is $\mathbf{j}$-unitary.

A *join density relation* on a join restriction category $\mathbf{X}$ is a density relation $\leq_{\mathbf{j}}$ on $\mathbf{X}$ with the additional requirement:

[jLoc] If $\{f_i | i \in I\}$ and $\{g_i | i \in I\}$ are $\smile$-compatible families such that $f_i \leq_{\mathbf{j}} g_i, i \in I$, then $\vee_{i\in I} f_i \leq_{\mathbf{j}} \vee_{i\in I} g_i$.

If $\leq_{\mathbf{j}}$ is a join density relation on a join restriction category $\mathbf{X}$, then $\mathbf{X}/\sim_{\mathbf{j}}$ is a join restriction category shown by Cockett in [7].

Lemma 3.1.31 (Lemma 5.1 [7]) *If $\leq_{\mathbf{j}}$ is a join density relation on a $\mathbf{j}$-unitary join restriction category $\mathbf{X}$, then $\mathbf{X}/\sim_{\mathbf{j}}$ is a join restriction category and $Q_{\mathbf{j}} : \mathbf{X} \to \mathbf{X}/\sim_{\mathbf{j}}$ is a join restriction functor.*

Given a join inverse category $\mathbf{I}$, we can first form the join restriction category $\mathbf{j}(\mathbf{I})$ and then the quotient category $\mathbf{j}(\mathbf{I})/\sim_{\mathbf{j}(\mathbf{I})}$, where $\sim_{\mathbf{j}(\mathbf{I})}$ is the restriction congruence induced by the density relation $\leq_{\mathbf{j}(\mathbf{I})}$ given in Lemma 3.1.28. $\mathbf{j}(\mathbf{I})/\sim_{\mathbf{j}(\mathbf{I})}$ is indeed a join restriction category:

Lemma 3.1.32 *$\mathbf{j}(\mathbf{I})/\sim_{\mathbf{j}(\mathbf{I})}$ is a join restriction category.*

PROOF: By Lemma 3.1.28 and Proposition 3.1.29, $\mathbf{j}(\mathbf{I})/\sim_{\mathbf{j}(\mathbf{I})}$ is a restriction category. For each object $X \in \mathbf{j}(\mathbf{I})$, if $P, Q : Z \to X$ are $\mathbf{j}(\mathbf{I})$-maps such that $P \sim_{\mathbf{j}(\mathbf{I})} Q$, then there is a $\mathbf{j}(\mathbf{I})$-map K such that $K \leq_{\mathbf{j}(\mathbf{I})} P$ and $K \leq_{\mathbf{j}(\mathbf{I})} Q$ and so for each $p \in P$ and $q \in Q$ there are $\smile$-compatible $\{k_i | i \in \Gamma\} \subseteq K$ and $\{k'_j | j \in \Lambda\} \subseteq K$ such that

$$p = \vee_{i\in\Gamma} k_i \text{ and } q = \vee_{j\in\Lambda} k'_j.$$

Hence

$$\overline{p}q = \overline{\vee_{i\in\Gamma} k_i} \vee_{j\in\Lambda} k'_j = \vee_{i\in\Gamma, j\in\Lambda} \overline{k_i} k'_j = \vee_{i\in\Gamma, j\in\Lambda} \overline{k'_j} k_i = \overline{\vee_{j\in\Lambda} k'_j} \vee_{i\in\Gamma} k_i = \overline{q}p$$

as K is $\smile$-compatible and therefore $\overline{P}Q = \overline{Q}P$, namely, $P \smile Q$. Thus, $\mathbf{j}(\mathbf{I})$ is $\mathbf{j}$-unitary.

If $\{P_i | i \in I\}$ and $\{Q_i | i \in I\}$ are $\smile$-compatible families such that $P_i \leq_{\mathbf{j(I)}} Q_i$, then $P_i \subseteq Q_i$ and for each $q \in Q_i$ there is a $\smile$-compatible $\{p_k | k \in \Gamma\} \subseteq P_i$ such that $q = \vee_{k \in \Gamma} p_k$. Since

$$\vee_{i \in I} P_i = \bigcup_{i \in I} P_i \leq \bigcup_{i \in I} Q_i = \vee_{i \in I} Q_i,$$

we have $\vee_{i \in I} P_i \leq_{\mathbf{j(I)}} \vee_{i \in I} Q_i$. Then **[jLoc]** is satisfied and so, by Lemma 3.1.31, $\mathbf{j}(\mathbf{I}) / \sim_{\mathbf{j(I)}}$ is a join restriction category. $\square$

For each given join inverse category $\mathbf{I}$, let $\beta(\mathbf{I}) = \mathbf{j}(\mathbf{I}) / \sim_{\mathbf{j(I)}}$. We are now ready to show that β actually gives a left adjoint to $\mathbf{inv} : \mathbf{jrCat}_0 \to \mathbf{jinvCat}_0$.

Theorem 3.1.33 $\mathbf{inv} : \mathbf{jrCat}_0 \to \mathbf{jinvCat}_0$ *has a left adjoint given by* β*.*

Proof: For each join inverse category $\mathbf{I}$ and each $\mathbf{I}$-map $f : X \to Y$, $[\downarrow f] : X \to Y$, as a $\beta(\mathbf{I})$-map, has its restriction inverse $[\downarrow f^{(-1)}] : Y \to X$ as

$$[\downarrow f][\downarrow f^{(-1)}][\downarrow f] = [\downarrow (ff^(- 1)f)] = [\downarrow f]$$

and

$$[\downarrow f^{(-1)}][\downarrow f][\downarrow f^{(-1)}] = [\downarrow (f^{(-1)} f f^{(-1)}] = [\downarrow f^{(-1)}].$$

Hence $[\downarrow f] : X \to Y$ is a map in $\mathbf{inv}(\beta(\mathbf{I}))$. Define the join functor

$$\eta_{\mathbf{I}} : \mathbf{I} \to \mathbf{inv}(\beta(\mathbf{I}))$$

by taking $f : X \to Y$ to $[\downarrow f] : X \to Y$, which serves as the unit of $\beta \dashv \mathbf{inv}$. In fact, for any join inverse category $\mathbf{X}$ and any join functor $F : \mathbf{I} \to \mathbf{inv}(\mathbf{X})$, by Theorem 3.1.12 $\mathbf{j}$ provides a left adjoint to the inclusion $\mathbf{jrCat}_0 \hookrightarrow \mathbf{rCat}_0$. Then there is a unique join restriction functor $F^* : \mathbf{j}(\mathbf{I}) \to \mathbf{inv}(\mathbf{X})$ such that

$$\begin{tikzcd}
\mathbf{I} \arrow[r, "\eta'_{\mathbf{I}}"] \arrow[dr, "F"'] & \mathbf{j}(\mathbf{I}) \arrow[d, "F^*"] \\
& \mathbf{inv}(\mathbf{X})
\end{tikzcd}$$

commutes, where $\eta'_{\mathbf{I}}$ is given by sending f to $\downarrow f$ F^* sending P to $\vee \downarrow(F(P))$. As each $\mathbf{I}$-map $f : A \to B$ has a inverse $f^{(-1)}$, $\downarrow f$ has an inverse $\downarrow f^{(-1)} : B \to A$ in $\mathbf{j}(\mathbf{I})$. So we have the following commutative diagram:

$$\begin{tikzcd}
\mathbf{I} \arrow[r, "\eta'_{\mathbf{I}}"] \arrow[dr, "F"'] & \mathbf{inv}(\mathbf{j}(\mathbf{I})) \arrow[d, "\mathbf{inv}(F^*)"] \\
 & \mathbf{inv}(\mathbf{X})
\end{tikzcd}$$

For $\mathbf{j}(\mathbf{I})$-maps $P \leq_{\mathbf{j}(\mathbf{I})} Q$ and each $q \in Q$, there is a $\smile$-compatible $\{p_i | i \in \Gamma\} \subseteq P$ such that $q = \vee_{i \in \Gamma} p_i$. Then

$$F(q) = F(\vee_{i \in \Gamma} p_i) = \vee_{i \in \Gamma} F(p_i)$$

and so

$$F^*(Q) = \vee \downarrow F(Q) = \vee \downarrow F(P) = F^*(P).$$

Hence, by Proposition 3.1.29, there is a join restriction functor $F^\sharp : \mathbf{j}(\mathbf{I})/ \sim_{\mathbf{j}(\mathbf{I})}$ such that

$$\begin{tikzcd}
\mathbf{j}(\mathbf{I}) \arrow[r, "Q_{\mathbf{j}}"] \arrow[dr, "F^*"'] & \beta(\mathbf{I}) \arrow[d, "F^\sharp"] \\
 & \mathbf{Y}
\end{tikzcd}$$

commutes. Thus, we have the following commutative diagram:

$$\begin{tikzcd}
\mathbf{I} \arrow[r, bend left, "\eta_{\mathbf{I}}"] \arrow[r, "\mathbf{inv}(Q_{\mathbf{j}})\eta'_{\mathbf{I}}"] \arrow[dr, "F"'] & \mathbf{inv}(\beta(\mathbf{I})) \arrow[d, "\mathbf{inv}(F^{\#})"] \\
 & \mathbf{inv}(\mathbf{X})
\end{tikzcd}$$

Suppose that $G : \beta(\mathbf{I}) \to \mathbf{X}$ is a join restriction functor such that $\mathbf{inv}(G)\eta_{\mathbf{I}} = F$. For any map $[P] : A \to B$ in $\beta(\mathbf{I})$, since P is $\smile$-compatible and down closed, $P = \vee_{p \in P}(\downarrow p)$, where each $p : A \to B$ is a map in $\mathbf{I}$. Clearly, for each $p \in P$, $G([\downarrow p]) = F(p) = F^{\#}([\downarrow p])$. Hence

$$G([P]) = G\big(\vee_{p \in P} [\downarrow p]\big) = \vee_{p \in P} G([\downarrow p]) = \vee_{p \in P} F^{\#}([\downarrow p]) = F^{\#}\big(\vee_{p \in P} [\downarrow p]\big) = F^{\#}([P])$$

and therefore the uniqueness of $F^{\#}$ follows. Thus, $\beta \dashv \mathbf{inv}$. □

Summarizing the adjoints between restriction categories, join restriction categories, inverse categories, and join inverse categories, we have:

Theorem 3.1.34 *There is an adjunction situation:*

$$\begin{tikzcd}
\mathbf{jrCat}_0 \arrow[r, hook, shift right, "\mathcal{I}_j"'] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathbf{inv}"] \arrow[d, phantom, "\dashv"] & \mathbf{rCat}_0 \arrow[l, shift right, "\mathbf{j}"'] \arrow[d, shift left, "\mathbf{inv}"] \arrow[d, phantom, "\dashv"] \\
\mathbf{jinvCat}_0 \arrow[u, shift left, "\beta"] \arrow[r, hook, shift right, "\mathcal{I}_{jj}"'] \arrow[r, phantom, "\perp"] & \mathbf{invCat}_0 \arrow[l, shift right, "\mathbf{jj}"'] \arrow[u, hook, shift left, "\mathcal{I}_i"]
\end{tikzcd}$$

in which

1. $\mathcal{I}_{jj} \circ \mathbf{inv} = \mathbf{inv} \circ \mathcal{I}_j$;

2. $\beta \circ \mathbf{jj} = \mathbf{j} \circ \mathcal{I}_i$;

3. $\mathbf{inv} \circ \mathbf{j} \circ \mathcal{I}_i = \mathbf{jj}$;

Proof: Theorems 3.1.12, 3.1.24, and 3.1.33 give $\mathbf{j} \dashv \mathcal{I}_j$, $\mathbf{jj} \dashv \mathcal{I}_{jj}$, and $\beta \dashv \mathbf{inv}$, respectively while Proposition 2.24 [14] implies $\mathcal{I}_i \dashv \mathbf{inv}$. So we have the adjunction situation.

1. It is clear.

2. As $\beta \circ \mathbf{jj}$ and $\mathbf{j} \circ \mathcal{I}_i$ are the left adjoints of $\mathcal{I}_{jj} \circ \mathbf{inv}$ and $\mathbf{inv} \circ \mathcal{I}_j$, respectively, $\beta \circ \mathbf{jj} = \mathbf{j} \circ \mathcal{I}_i$.

3. By Proposition 3.1.25.

□

3.2 $\mathcal{M}$-adhesive Categories and $\mathcal{M}$-gaps

Adhesive categories, introduced in [28], are a class of categories where pushouts along monics exist and are well-behaved with respect to pullbacks. They are instances of Van Kampen squares. In [22], characterized being Van Kampen as a universal property: Van

Kampen cocones are precisely those diagrams in a category that induce bicolimit diagrams in its associated bicategory of spans, provided that the category has pullbacks and enough colimits. In [20], Garner and Lack gave a general framework for describing categorical structures consisting of the existence of finite limits as well as certain types of colimits, along with exactness conditions stating that the limits and colimits interact in the same way as they do in a topos. In [21], Garner and Lack provided four characterization theorems dealing with adhesive categories and their variants. In this section, we develop some elementary results on van Kampen colimits. *van Kampen squares* are a special case of these colimits and we start by describing these. The results on van Kampen colimits facilitate the definition of $\mathcal{M}$-adhesive categories and lead us into a discussion of the properties of $\mathcal{M}$-gaps.

3.2.1 van Kampen Squares

Recall, as in [28, 29] by Lack and Sobociński, a pushout square $ABCD$ is a *van Kampen (VK) square* if for any commutative cube

$$\begin{tikzcd}
& A' \arrow[rr] \arrow[dl] \arrow[dd] & & C' \arrow[dl] \arrow[dd] \\
B' \arrow[rr] \arrow[dd] & & D' \arrow[dd] & \\
& A \arrow[rr] \arrow[dl] & & C \arrow[dl] \\
B \arrow[rr] & & D &
\end{tikzcd}$$

such that $ABA'B'$ and $ACA'C'$ are pullback squares, $BDB'D'$ and $CDC'D'$ are pullback squares if and only if $A'B'C'D'$ is a pushout square.

Equivalently, a van Kampen (VK) square in a category with pullbacks is a pushout diagram:

$$\begin{tikzcd}
A \arrow[r, "m_2"] \arrow[d, "m_1"'] & C \arrow[d, "m_3"] \\
B \arrow[r, "m_4"'] & D
\end{tikzcd}$$

such that given a commutative diagram $(I)+(II)$, (I) and (II) are pullback diagrams if and only if there is an object E and some maps from E such that (III) and (IV) are pullback diagrams and (V) is a pushout diagram:

$$\begin{tikzcd}
B' \arrow[r, "m_4'"] \arrow[d, "b"'] \arrow[dr, phantom, "(I)"] & D' \arrow[d, "d"] \arrow[dr, phantom, "(II)"] & C' \arrow[l, "m_3'"'] \arrow[d, "c"] \\
B \arrow[r, "m_4"'] & D & C \arrow[l, "m_3"]
\end{tikzcd}
\qquad
\begin{tikzcd}
B' \arrow[d, "b"'] \arrow[dr, phantom, "(III)"] & E \arrow[l, "m_1'"'] \arrow[r, "m_2'"] \arrow[d, "e"] \arrow[dr, phantom, "(IV)"] & C' \arrow[d, "c"] \\
B & A \arrow[l, "m_1"] \arrow[r, "m_2"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "m_2'"] \arrow[d, "m_1'"'] \arrow[dr, phantom, "(V)"] & C' \arrow[d, "m_3'"] \\
B' \arrow[r, "m_4'"'] & D'
\end{tikzcd}$$

Lemma 3.2.1 *In a VK square*

$$\begin{tikzcd}
A \arrow[r, "m_2"] \arrow[d, "m_1"'] & C \arrow[d, "m_3"] \\
B \arrow[r, "m_4"'] & D
\end{tikzcd}$$

if m_1 *(or* m_2*) is monic, so is* m_3 *(or* m_4*) and such a square is a pullback diagram.*

Proof: See [29], Lemma 2.3. ☐

VK squares are stable.

Lemma 3.2.2 *VK squares are stable.*

Proof: Suppose that the square $A'B'C'D'$ is the pullback of a VK square $ABCD$ along a map f:

$$\begin{tikzcd}
 & B' \arrow[rr] \arrow[dd] & & D' \arrow[dd, "f"] \\
A' \arrow[ur] \arrow[rr] \arrow[dd] & & C' \arrow[ur] \arrow[dd] & \\
 & B \arrow[rr] & & D \\
A \arrow[ur] \arrow[rr] & & C \arrow[ur] &
\end{tikzcd}$$

which means that front faces and back faces are pullback diagrams. Then $A'B'C'D'$ is a pushout diagram. Now it is easy to test $A'B'C'D'$ is indeed a VK square by using pullback cancellation and gluing laws. ☐

3.2.2 van Kampen Colimits

We generalize van Kampen squares to van Kampen colimits.

Definition 3.2.3 *Let* $\mathbf{C}$ *be a category and let* $D : \mathbf{S} \to \mathbf{C}$ *be a diagram on* $\mathbf{S}$. *A colimit* $\alpha : D \Rightarrow C$ *in* $\mathbf{C}$ *is called a* van Kampen colimit *on* $\mathbf{S}$ *if for any diagram* $D' : \mathbf{S} \to \mathbf{C}$ *on* $\mathbf{S}$, *any cocone* $\alpha' : D' \Rightarrow X$ *under* D', *and any commutative diagram*

$$\begin{tikzcd} D' \arrow[r, Rightarrow, "\alpha'"] \arrow[d, Rightarrow, "\beta"'] & X \arrow[d, "r"] \\ D \arrow[r, Rightarrow, "\alpha"] & C \end{tikzcd}$$

in which β *is cartesian,* $\alpha' : D' \Rightarrow X$ *is a colimit if and only if for each* $s \in \mathbf{S}$

$$\begin{tikzcd} D'(s) \arrow[r, "\alpha'(s)"] \arrow[d, "\beta(s)"'] & X \arrow[d, "r"] \\ D(s) \arrow[r, "\alpha(s)"] & C \end{tikzcd}$$

is a pullback diagram, where a natural transformation $\begin{tikzcd} \mathbf{S} \arrow[r, bend left, "D'"{name=U}] \arrow[r, bend right, "D"'{name=L}] & \mathbf{C} \arrow[from=U, to=L, phantom, "\Downarrow\beta"] \end{tikzcd}$ *is* cartesian *if for each map* $f : s_1 \to s_2$ *in* $\mathbf{S}$,

$$\begin{tikzcd} D'(s_1) \arrow[r, "D'(f)"] \arrow[d, "\beta(s_1)"'] & D'(s_2) \arrow[d, "\beta(s_2)"] \\ D(s_1) \arrow[r, "D(f)"] & D(s_2) \end{tikzcd}$$

is a pullback diagram.

van Kampen colimits are stable too.

Lemma 3.2.4 *van Kampen colimits are stable.*

PROOF: Let $D : \mathbf{S} \to \mathbf{C}$ be a diagram, $\alpha : D \Rightarrow C$ a van Kampen colimit, and

$$\begin{tikzcd} D' \arrow[r, Rightarrow, "\alpha'"] \arrow[d, Rightarrow, "\beta"'] & X \arrow[d, "r"] \\ D \arrow[r, Rightarrow, "\alpha"] & C \end{tikzcd}$$

a pullback diagram of diagrams, which means that for each $s \in \mathbf{S}$ each

$$\begin{tikzcd}
D'(s) \arrow[r, "\alpha'(s)"] \arrow[d, "\beta(s)"'] & X \arrow[d, "r"] \\
D(s) \arrow[r, "\alpha(s)"] & C
\end{tikzcd}$$

is a pullback diagram, where $D' : \mathbf{S} \to \mathbf{C}$ is a diagram from $\mathbf{S}$. Then $\alpha' : D' \Rightarrow X$ is a colimit since $\alpha : D \Rightarrow C$ is a van Kampen colimit. To prove that $\alpha' : D' \Rightarrow Z$ is indeed a van Kampen colimit, suppose that

$$\begin{tikzcd}
D'' \arrow[r, Rightarrow, "\alpha''"] \arrow[d, Rightarrow, "\gamma"'] & X' \arrow[d, "p"] \\
D' \arrow[r, Rightarrow, "\alpha'"] & X
\end{tikzcd}$$

is a commutative diagram, where $D'' : \mathbf{S} \to \mathbf{C}$ is a diagram and $\alpha'' : D'' \Rightarrow X'$ is a cocone. Suppose that for each $f : s_1 \to s_2$ in $\mathbf{S}$

$$\begin{tikzcd}
D''(s_1) \arrow[r, "D''(f)"] \arrow[d, "\gamma(s_1)"'] & D''(s_2) \arrow[d, "\gamma(s_2)"] \\
D'(s_1) \arrow[r, "D'(f)"] & D'(s_2)
\end{tikzcd}$$

is a pullback diagram. Then we have a commutative diagram:

$$\begin{tikzcd}
D'' \arrow[r, Rightarrow, "\alpha''"] \arrow[d, Rightarrow, "\beta\gamma"'] & X' \arrow[d, "rp"] \\
D \arrow[r, Rightarrow, "\alpha"] & C
\end{tikzcd}$$

If $\alpha'' : D'' \Rightarrow Z'$ is a colimit, then for each $s \in \mathbf{S}$

$$\begin{tikzcd}
D''(s) \arrow[r, "\alpha''(s)"] \arrow[d, "\beta(s)\gamma(s)"'] & X' \arrow[d, "rp"] \\
D(s) \arrow[r, "\alpha(s)"] & C
\end{tikzcd}$$

is a pullback diagram and so each

$$\begin{tikzcd}
D''(s) \arrow[r, "\alpha''(s)"] \arrow[d, "\gamma(s)"'] & X' \arrow[d, "p"] \\
D'(s) \arrow[r, "\alpha'(s)"] & X
\end{tikzcd}$$

is a pullback diagram.

Conversely, if each

$$\begin{tikzcd}
D''(s) \arrow[r, "\alpha''(s)"] \arrow[d, "\gamma(s)"'] & X' \arrow[d, "p"] \\
D'(s) \arrow[r, "\alpha'(s)"] & X
\end{tikzcd}$$

is a pullback diagram, then

$$\begin{tikzcd}
D''(s) \arrow[r, "\alpha''(s)"] \arrow[d, "\beta(s)\gamma(s)"'] & X' \arrow[d, "rp"] \\
D(s) \arrow[r, "\alpha(s)"] & C
\end{tikzcd}$$

is a pullback diagram and so $\alpha'' : D'' \Rightarrow X'$ is a colimit. Thus, $\alpha' : D' \Rightarrow X$ is a van Kampen colimit, which means that the van Kampen colimit is stable. ☐

3.2.3 Pullbacks of van Kampen Colimits

Obviously, a pullback of a van Kampen colimit $\alpha : D \Rightarrow C$ along a given map $f : X \to C$:

$$\begin{tikzcd}
D' \arrow[r, Rightarrow, "\alpha'"] \arrow[d, Rightarrow, "\beta"'] & X \arrow[d, "f"] \\
D \arrow[r, Rightarrow, "\alpha"] & C
\end{tikzcd}$$

can be given by the following pullback diagram:

$$\begin{tikzcd}
D'(s) \arrow[r, "\alpha'(s)"] \arrow[d, "\beta(s)"'] & X \arrow[d, "f"] \\
D(s) \arrow[r, "\alpha(s)"] & C
\end{tikzcd}$$

for each $s \in \mathbf{S}$.

Given two van Kampen colimits $\alpha_i : D_i \Rightarrow X, i = 1, 2$, we can form a new van Kampen colimit $\alpha_1 \times_X \alpha_2 : D_1 \times_X D_2 \Rightarrow X$. To do this, we need the following technical lemma on colimits.

Lemma 3.2.5 *Let $\partial : \delta(\mathbf{S}) \to \mathbf{S}$ be an opfibration and let F, G be functors such that for any $s \in \mathbf{S}$ the colimit $\alpha^s : F|_{\partial^{(-1)}(s)} \Rightarrow G(s)$ exists and for each map $f : s \to s'$ in $\mathbf{S}$ and each $t \in \partial^{(-1)}(s)$ the following diagram*

$$\begin{tikzcd}
F(t) \arrow[r, "\alpha^s(t)"] \arrow[d, "F(\vartheta_f(t))"'] & G(s) \arrow[d, "G(f)"] \\
F(f_*(t)) \arrow[r, "\alpha^{s'}(f_*(t))"] & G(s')
\end{tikzcd}$$

commutes, where $\vartheta_f(t) : t \to f_*(t)$ is an opcartesian lifting of f at t.

$$\begin{tikzcd}
\delta(\mathbf{S}) \arrow[r, "F"] \arrow[d, "\partial"'] & \mathbf{C} \\
\mathbf{S} \arrow[ur, "G"'] &
\end{tikzcd}$$

Then the colimit $\beta : F \Rightarrow V$ exists if and only if the colimit $\gamma : G \Rightarrow V$ exists.

Proof: "only if" Suppose that $\beta : F \Rightarrow V$ is a colimit. For a given $s \in \mathbf{S}$ and each $t \in \partial^{(-1)}(s)$, since $\alpha^s : F|_{\partial^{(-1)}(s)} \Rightarrow G(s)$ is a colimit and $\beta : F|_{\partial^{(-1)}(s)} \Rightarrow V$ is a cocone, there is a unique map $\gamma(s) : G(s) \to V$ such that for any $t \in \partial^{(-1)}(s)$

$$\begin{tikzcd}
F(t) \arrow[rr, "\beta(t)"] \arrow[dr, "\alpha^s(t)"'] & & V \\
& G(s) \arrow[ur, dotted, "\gamma(s)"'] &
\end{tikzcd}$$

commutes. Since $\partial : \delta(\mathbf{S}) \to \mathbf{S}$ is an opfibration and

$$\begin{tikzcd}
F(t) \arrow[r, "\alpha^s(t)"] \arrow[d, "F(\vartheta_f(t))"'] & G(s) \arrow[d, "G(f)"] \\
F(f_*(t)) \arrow[r, "\alpha^{s'}(f_*(t))"] & G(s')
\end{tikzcd}$$

commutes, $\gamma(s)\alpha^s(t) = \beta(t) = \gamma(s')\alpha^{s'}(f_*(t))F(\vartheta_f(t)) = \gamma(s')G(f)\alpha^s(t)$. Since $\{\alpha^s(t)|t \in \partial^{(-1)}(s)\}$ is jointly epic, $\gamma(s) = \gamma(s')G(f)$ for each map $f : s \to s'$ in $\mathbf{S}$. Hence $\gamma : G \Rightarrow V$ is a cocone.

For any cocone $\chi : G \Rightarrow X$, we have a cocone $\chi\alpha^{\partial(-)} : F \Rightarrow X$ given by $(\chi\alpha^{\partial(-)})(t) = \chi(\partial(t))\alpha^{\partial(t)}(t)$. So there is a unique $k : V \to X$ such that $\chi(\partial(t))\alpha^{\partial(t)}(t) = k \cdot \beta(t)$ for each $t \in \delta(\mathbf{S})$. In particular, for each $s \in \mathbf{S}$ and each $t \in \partial^{(-1)}(s)$,

$$\chi(s)\alpha^s(t) = \chi(\partial(t))\alpha^{\partial(t)}(t) = k \cdot \beta(t) = k \cdot \gamma(s)\alpha^s(t).$$

Then $\chi(s) = k \cdot \gamma(s)$ for each $s \in \mathbf{S}$ since $\{\alpha^{\partial(t)}(t_1) \mid t_1 \in \partial^{(-1)}(\partial(t))\}$ is jointly epic. Clearly, k is the unique map such that $\chi(s) = k \cdot \gamma(s)$ for each $s \in \mathbf{S}$. So $\gamma : G \Rightarrow V$ is a colimit.

“if” Suppose now that $\gamma : G \Rightarrow V$ is a colimit. For each $t \in \delta(\mathbf{S})$, we have $t \in \partial^{(-1)}(\partial(t))$. Define $\beta(t) = \gamma(\partial(t))\alpha^{\partial(t)}(t)$. Then $\beta : F \Rightarrow V$ is a cocone since ∂ is an opfibration and the last diagram is commutative. For any cocone $\chi : F \Rightarrow X$, there is a $\kappa(\partial(t)) : G(\partial(t)) \to X$ such that

$$\kappa(\partial(t)) \cdot \alpha^{\partial(t)}(t) = \chi(t)$$

since $\alpha^{\partial(t)} : F|_{\partial^{(-1)}(t)} \Rightarrow X$ is a colimit. Now it is easy to see that $\kappa : G \Rightarrow X$ is a cocone. Since $\gamma : G \Rightarrow V$ is a colimit, there is a unique map $x : V \to X$ such that $x\gamma(\partial(t)) = k(\partial(t))$. Then $x\beta(t) = \chi(t)$ and x is the unique such a map and so $\beta : F \Rightarrow V$ is a colimit.

$$\begin{tikzcd}
& & X \\
F(t) \arrow[rru, "\chi(t)"] \arrow[rr, "\beta(t)"] \arrow[rd, "\alpha^{\partial(t)}(t)"'] & & V \arrow[u, dotted, "x"] \\
& G(\partial(t)) \arrow[ru, "\gamma(\partial(t))"'] \arrow[ruu, dotted, bend right, "\kappa(\partial(t))"'] &
\end{tikzcd}$$

□

The pullbacks of van Kampen colimits can be formed as follows.

Lemma 3.2.6 *Let D_i be diagrams from $\mathbf{S}_i$, $i = 1, 2$. If both $\alpha_1 : D_1 \Rightarrow X$ and $\alpha_2 : D_2 \Rightarrow X$ are van Kampen colimits, then so is $\alpha_1 \times_X \alpha_2 : D_1 \times_X D_2 \Rightarrow X$, where*

$D_1 \times_X D_2 : \mathbf{S}_1 \times \mathbf{S}_2 \to \mathbf{C}$ *is given by the following pullback diagram:*

$$\begin{tikzcd}
(D_1 \times_X D_2)(s_1, s_2) \arrow[r, "\beta(s_1,s_2)"] \arrow[d, "\gamma(s_1,s_2)"'] & D_2(s_2) \arrow[d, "\alpha_2(s_2)"] \\
D_1(s_1) \arrow[r, "\alpha_1(s_1)"] & X
\end{tikzcd}$$

and $(\alpha_1 \times_X \alpha_2)(s_1, s_2) = \alpha_1(s_1)\gamma(s_1, s_2) = \alpha_2(s_2)\beta(s_1, s_2)$, *for each* $(s_1, s_2) \in \mathbf{S}_1 \times \mathbf{S}_2$.

Proof: Clearly, $D_1 \times_X D_2 : \mathbf{S}_1 \times \mathbf{S}_2 \to \mathbf{C}$ is a diagram from $\mathbf{S}_1 \times \mathbf{S}_2$. For each $s_2^* \in \mathbf{S}_2$, we have a diagram $(D_1 \times_X D_2)(-, s_2^*) : \mathbf{S}_1 \to \mathbf{C}$ and a commutative diagram:

$$\begin{tikzcd}
(D_1 \times_X D_2)(-, s_2^*) \arrow[r, Rightarrow, "\beta(-,s_2^*)"] \arrow[d, Rightarrow, "\gamma(-,s_2^*)"'] & D_2(s_2^*) \arrow[d, "\alpha_2(s_2^*)"] \\
D_1 \arrow[r, Rightarrow, "\alpha_1"] & X
\end{tikzcd}$$

For each $f : s_1 \to s_1'$ in $\mathbf{S}_1$,

$$\begin{tikzcd}
(D_1 \times_X D_2)(s_1, s_2^*) \arrow[r, "(D_1 \times_X D_2)(f,s_2^*)"] \arrow[d, "\gamma(s_1,s_2^*)"'] & (D_1 \times_X D_2)(s_1', s_2^*) \arrow[d, "\gamma(s_1',s_2^*)"] \\
D_1(s_1) \arrow[r, "D_1(f)"] & D(s_1')
\end{tikzcd}$$

is a pullback diagram since $(D_1 \times_X D_2)(s_1, s_2^*)$ and $(D_1 \times_X D_2)(s_1', s_2^*)$ are the pullbacks of $\alpha_1(s_1), \alpha_2(s_2^*)$ and $\alpha_1(s_1'), \alpha_2(s_2^*)$, respectively, as in the following commutative diagram:

$$\begin{tikzcd}
(D_1 \times_X D_2)(s_1, s_2^*) \arrow[rr, "\beta(s_1,s_2^*)"] \arrow[dr, "D_1 \times_X D_2(f)"'] \arrow[dd, "\gamma(s_1,s_2^*)"'] & & D_2(s_2^*) \arrow[dd, "\alpha_2(s_2^*)"] \\
& (D_1 \times_X D_2)(s_1', s_2^*) \arrow[ur, "\beta(s_1',s_2^*)"'] \arrow[dd, "\gamma(s_1',s_2^*)"] & \\
D_1(s_1) \arrow[rr, "\alpha_1(s_1)"] \arrow[dr, "D_1(f)"'] & & X \\
& D_1(s_1') \arrow[ur, "\alpha_1(s_1')"'] &
\end{tikzcd}$$

Then both β and γ are cartesian on each variable. Since $\alpha_1 : D_1 \Rightarrow X$ is a van Kampen colimit and each

$$\begin{tikzcd}
(D_1 \times_X D_2)(s_1, s_2^*) \arrow[r, "\beta(s_1,s_2^*)"] \arrow[d, "\gamma(s_1,s_2^*)"'] & D_2(s_2^*) \arrow[d, "\alpha_2(s_2^*)"] \\
D_1(s_1) \arrow[r, "\alpha_1(s_1)"] & X
\end{tikzcd}$$

is a pullback diagram, for each $s_2^* \in \mathbf{S}_2$

$$\beta(-, s_2^*) : (D_1 \times_X D_2)(-, s_2^*) \Rightarrow D_2(s_2^*)$$

is a van Kampen colimit. So we have the following situation

$$\begin{tikzcd}
\mathbf{S}_1 \times \mathbf{S}_2 \arrow[r, "D_1 \times_X D_2"] \arrow[d, "Pr"'] & \mathbf{C} \\
\mathbf{S}_2 \arrow[ur, "D_2"'] &
\end{tikzcd}$$

where the projection Pr is a bifibration that is surjective on objects and for each $s_2 \in \mathbf{S}_2$, $\beta(-, s_2) : (D_1 \times_X D_2)|_{Pr^{(-1)}(s_2)} \Rightarrow D_2(s_2)$ is a van Kampen colimit. Hence, by Lemma 3.2.5, $\alpha_1 \times_X \alpha_2 : D_1 \times_X D_2 \Rightarrow X$ is a colimit.

For any cocone $\alpha : D \Rightarrow Y$ from $\mathbf{S}_1 \times \mathbf{S}_2$ and any commutative diagram

$$\begin{tikzcd}
D \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\nu"'] & Y \arrow[d, "x"] \\
D_1 \times_X D_2 \arrow[r, Rightarrow, "\alpha_1 \times_X \alpha_2"] & X
\end{tikzcd}$$

in which ν is cartesian, if $\alpha : D \Rightarrow Y$ is a colimit, then each

$$\begin{tikzcd}
D(s_1, s_2) \arrow[r, "\alpha(s_1,s_2)"] \arrow[d, "\nu(s_1,s_2)"'] & Y \arrow[dd, "x"] \\
(D_1 \times_X D_2)(s_1, s_2) \arrow[d, "\beta(s_1,s_2)"'] & \\
D_2(s_2) \arrow[r, "\alpha_2(s_2)"] & X
\end{tikzcd}$$

is a pullback diagram since $\alpha_2 : D_2 \Rightarrow X$ is a van Kampen colimit. Since $\beta(-, s_2) : (D_1 \times_X D_2)(-, s_2) \Rightarrow D_2(s_2)$ is a van Kampen colimit, we know that

$$\begin{tikzcd}
D(s_1, s_2) \arrow[rr, "\alpha(s_1,s_2)"] \arrow[d, "\nu(s_1,s_2)"'] & & Y \arrow[d, "x"] \\
(D_1 \times_X D_2)(s_1, s_2) \arrow[r, "\beta(s_1,s_2)"] & D_2(s_2) \arrow[r, "\alpha_2(s_2)"] & X
\end{tikzcd}$$

is a pullback diagram.

Conversely, suppose that each last square is a pullback diagram. Let $\mu : D_2' \Rightarrow Y$ be a pullback of the van Kampen colimit $\alpha_2 : D_2 \Rightarrow X$ along x. That is $(D_2'(s_2), \mu(s_2), \rho(s_2))$ is a pullback of $\alpha_2(s_2)$ and x. Then there is a unique map $\lambda(s_1, s_2) : D(s_1, s_2) \Rightarrow D_2(s_2)$ such that

$$\begin{tikzcd}
D(s_1, s_2) \arrow[r, dotted, "\lambda(s_1,s_2)"] \arrow[d, "\nu(s_1,s_2)"'] \arrow[rr, bend left, "\alpha(s_1,s_2)"] & D_2'(s_2) \arrow[r, "\mu(s_2)"] \arrow[d, "\rho(s_2)"] & Y \arrow[d, "x"] \\
(D_1 \times_X D_2)(s_1, s_2) \arrow[r, "\beta(s_1,s_2)"] & D_2(s_2) \arrow[r, "\alpha_2(s_2)"] & X
\end{tikzcd}$$

It is routine to check that $\lambda(-, s_2) : D(-, s_2) \Rightarrow D_2'(s_2)$ is a cocone. Since both the outside and right hand side squares are pullback diagrams, the left hand side square is a pullback diagram. Hence $\lambda(-, s_2) : D(-, s_2) \Rightarrow D_2'(s_2)$, as a pullback of $\beta(-, s_2) : (D_1 \times_X D_2)(-, s_2) \Rightarrow D_2(s_2)$, is a van Kampen colimit. Now, by Lemma 3.2.5, $\alpha : D \Rightarrow Y$ is a colimit since $\alpha(s_1, s_2) = \mu(s_2)\lambda(s_1, s_2)$. Thus, $\alpha_1 \times_X \alpha_2 : D_1 \times_X D_2 \Rightarrow X$ is a van Kampen colimit. ☐

3.2.4 $\mathcal{M}$-adhesive Categories

Recall that a category is said to be *adhesive* if it has pushouts along monics and pullbacks and if pushouts along monics are VK-squares [28, 29]. Toposes are adhesive [30]. In [25], Johnstone, Lack, and Sobociński introduced several examples which fail to be adhesive. In [21], Garner and Lack reformulated the van Kampen condition in the definition of an adhesive category. By Theorem 3.2 [21], for each category $\mathbf{C}$ with pullbacks, $\mathbf{C}$ is adhesive if and only if $\mathbf{C}$ has pushouts along monics and these pushouts are stable and are pullbacks.

From now on, we shall consider diagrams of stable meet semilattices. Let $\mathbf{S}$ be a *stable meet semilattice*, namely, a poset with a binary meet (no top is assumed), and let $D : \mathbf{S} \to \mathbf{C}$ be a diagram. A cocone $\alpha : D \Rightarrow X$, is called an *amalgam cocone* if for all

$s_1, s_2 \in \mathbf{S}$ each

$$\begin{tikzcd}
D(s_1 \wedge s_2) \arrow[r, "D(\leq)"] \arrow[d, "D(\leq)"'] & D(s_1) \arrow[d, "\alpha(s_1)"] \\
D(s_2) \arrow[r, "\alpha(s_2)"] & X
\end{tikzcd}$$

is a pullback diagram. Observe that if $\alpha : D \Rightarrow X$ is an amalgam cocone then each component $\alpha(s)$ is necessarily monic since

$$\begin{tikzcd}
D(s \wedge s) \arrow[r, "="] \arrow[d, "="'] & D(s) \arrow[d, "\alpha(s)"] \\
D(s) \arrow[r, "\alpha(s)"] & X
\end{tikzcd}$$

is a pullback diagram.

When we have an $\mathcal{M}$-category we can consider diagrams and cocones which are restricted to the subcategory determined by $\mathcal{M}$; these give $\mathcal{M}$-diagrams and $\mathcal{M}$-cocones. An $\mathcal{M}$-diagram $D : \mathbf{S} \to \mathcal{M}$ is $\mathcal{M}$-*amalgamable* if there is an amalgam $\mathcal{M}$-cocone for D.

Definition 3.2.7 *An $\mathcal{M}$-category, $(\mathbf{C}, \mathcal{M})$, is* $\mathcal{M}$-adhesive *if each amalgamable $\mathcal{M}$-diagram has a van Kampen colimit. In this case, we call* $\mathbf{C}$ *an* $\mathcal{M}$-adhesive category.

Associated with an $\mathcal{M}$-adhesive category is a rather important class of maps: $\mathcal{M}$-gaps. But $\mathcal{M}$-gaps can be defined in $\mathcal{M}$-categories.

Definition 3.2.8 *A map $g : X \to Y$ in an $\mathcal{M}$-category is called an* $\mathcal{M}$-gap *if there is an $\mathcal{M}$-amalgamable colimit $\nu : D \Rightarrow X$ such that each $g\nu(s) \in \mathcal{M}$ for each $s \in \mathbf{S}$:*

$$\begin{tikzcd}
D \arrow[r, Rightarrow, "\nu"] \arrow[dr, Rightarrow, "\alpha"'] & X \arrow[d, "g"] \\
& Y
\end{tikzcd}$$

Our first observation on $\mathcal{M}$-gaps is:

Lemma 3.2.9 *In an $\mathcal{M}$-adhesive category, each $\mathcal{M}$-gap is a monic.*

Proof: Let $g : X \to Y$ be an $\mathcal{M}$-gap such that $g\nu(s) \in \mathcal{M}$ for all $s \in \mathbf{S}$ for some van Kampen colimit $\nu : D \Rightarrow X$ from $\mathbf{S}$. Let $x, y : W \to X$ be maps such that $gx = gy$ and let $\nu_x : D_x \Rightarrow W$ and $\nu_y : D_y \Rightarrow W$ be the pullbacks of $\nu : D \Rightarrow X$ along x, y, respectively. Then we have the following commutative diagram:

$$\begin{tikzcd}
& D_x \times_W D_y \arrow[dl, Rightarrow, "\alpha_x"'] \arrow[d, Rightarrow, "\nu_x \times_W \nu_y"] \arrow[dr, Rightarrow, "\alpha_y"] & \\
D_x \arrow[r, Rightarrow, "\nu_x"] \arrow[d, Rightarrow, "\gamma_x"'] & W \arrow[d, "x"', "y"] & D_y \arrow[l, Rightarrow, "\nu_y"'] \arrow[d, Rightarrow, "\gamma_y"] \\
D \arrow[r, Rightarrow, "\nu"] \arrow[dr, Rightarrow, "\alpha"'] & X \arrow[d, "g"'] & D \arrow[l, Rightarrow, "\nu"'] \arrow[dl, Rightarrow, "\alpha"] \\
& Y &
\end{tikzcd}$$

Note that $gx = gy$ implies $gx(\nu_x \times_W \nu_y) = gy(\nu_x \times_W \nu_y)$ so that $g\nu\gamma_x\alpha_x = g\nu\gamma_y\alpha_y$. Since $g\nu(s) \in \mathcal{M}$, $\gamma_x\alpha_x = \gamma_y\alpha_y$. Hence $x(\nu_x \times_W \nu_y) = y(\nu_x \times_W \nu_y)$ and therefore $x = y$ since $(\nu_x \times_W \nu_y) : D_x \times_W D_y \Rightarrow W$ is a van Kampen colimit. Thus, g is a monic. ☐

Let $g : X \to Y$ be a $\mathcal{M}$-gap and $\nu : D \Rightarrow X$ a van Kampen colimit such that $g\nu(s) \in \mathcal{M}$. Since g is a monic, by the left-cancellable property of $\mathcal{M}$ (Lemma 1.6.15) each $\nu(s)$ is in $\mathcal{M}$.

In an $\mathcal{M}$-adhesive category $\mathbf{C}$, an $\mathcal{M}$-gap $g : X \to Y$ can be determined by a canonical van Kampen colimit $\alpha_g : D_g \Rightarrow X$, where $\mathbf{S}_g$ is the stable meet semilattice with

objects: (A, a), $a : A \to X$ is an $\mathcal{M}$-map with $ga \in \mathcal{M}$;

maps: a map $m : (A, a) \to (B, b)$ is a map $m : A \to B$ in $\mathbf{C}$ such that $bm = a$. Clearly, such a map m must be in $\mathcal{M}$;

meet: $(A, a) \wedge (B, b) = (C, c)$ is given by the following pullback diagram:

$$\begin{tikzcd}
C \arrow[r, "b'"] \arrow[d, "a'"'] & A \arrow[d, "a"] \\
B \arrow[r, "b"] & X
\end{tikzcd}$$

where $c = ab' = ba'$.

$\alpha_g : D_g \Rightarrow X$ is defined by

$$D_g(m : (A, a) \to (B, b)) = (m : A \to B) \text{ and } \alpha_g(A, a) = a.$$

Obviously, α_g is an amalgam $\mathcal{M}$-cocone.

Lemma 3.2.10 *In an $\mathcal{M}$-adhesive category, a map $g : X \to Y$ is an $\mathcal{M}$-gap if and only if $\alpha_g : D_g \Rightarrow X$ is a van Kampen colimit such that $g\alpha_g(s) \in \mathcal{M}, \forall s \in \mathbf{S}_g$.*

Proof: "if" is clear. We prove the "only if" part as follows. Since α_g is an amalgam $\mathcal{M}$-cocone, D_g has a van Kampen colimit $\alpha : D_g \Rightarrow Z$. Hence there is a unique $\mathcal{M}$-gap $c : Z \to X$ such that $c\alpha(s) = \alpha_g(s)$ for all $s \in \mathbf{S}_g$. It suffices to show that the monic $\mathcal{M}$-gap c is an isomorphism.

Since $g : X \to Y$ is an $\mathcal{M}$-gap, there is an $\mathcal{M}$-van Kampen colimit $\nu : D \Rightarrow X$ from a stable meet semilattice $\mathbf{T}$ such that $g\nu(t) \in \mathbf{T}$ for all $t \in \mathbf{T}$.

$$\begin{tikzcd}
 & Z \arrow[d, dotted, bend right, "c"'] & \\
D_g \arrow[ur, Rightarrow, "\alpha"] \arrow[r, Rightarrow, "\alpha_g"] \arrow[dr, Rightarrow] & X \arrow[u, dotted, bend right, "k"'] \arrow[d, "g"] & D \arrow[l, Rightarrow, "\nu"'] \arrow[ul, Rightarrow, "\alpha'"'] \arrow[dl, Rightarrow] \\
 & Y &
\end{tikzcd}$$

Note that there is a stable meet semilattice map $F : \mathbf{T} \to \mathbf{S}_g$ given by $F(t) = (D(t), \nu(t) : D(t) \to X)$. So the cocone $\alpha : D_g \Rightarrow Z$ gives rise to a cocone $\alpha' : D \Rightarrow Z$ from $\mathbf{T}$ by $\alpha'(t) = \alpha(F(t))$. Hence there is a unique map $k : X \to Z$ such that $k\nu(t) = \alpha'(t)$ for all

$t \in \mathbf{T}$. Since

$$\begin{aligned} ck\nu(t) &= c\alpha'(t) \\ &= c\alpha(F(t)) \\ &= \alpha_g(F(t)) \\ &= \alpha_g(D(t), \nu(t) : D(t) \to X) \\ &= \nu(t), \end{aligned}$$

$ck = 1_X$. Hence the monic c is an isomorphism, as desired. □

All $\mathcal{M}$-gaps form a stable system of monics.

Proposition 3.2.11 *The class $\mathcal{M}_{\text{gap}}$ of all $\mathcal{M}$-gaps in an $\mathcal{M}$-adhesive category $\mathbf{C}$ is a stable system of monics in $\mathbf{C}$ with $\mathcal{M} \subseteq \mathcal{M}_{\text{gap}}$.*

Proof: For each $m \in \mathcal{M}$, since

$$\begin{tikzcd} \bullet \arrow[r, "1"] \arrow[d, "1"'] & \bullet \arrow[d, "1"] \\ \bullet \arrow[r, "1"] & \bullet \end{tikzcd}$$

is a van Kampen colimit (a pushout diagram) and since $m1 \in \mathcal{M}$, we have $m \in \mathcal{M}_{\text{gap}}$. Hence $\mathcal{M} \subseteq \mathcal{M}_{\text{gap}}$.

Let $g : A \to B$ be an $\mathcal{M}$-gap and let $\alpha : D \Rightarrow A$ be a van Kampen colimit such that $g\alpha(s) \in \mathcal{M}$. Then, for each map $x : X \to B$, we form the pullback of g along x and the pullback of α along x':

$$\begin{tikzcd} D' \arrow[r, Rightarrow, "\alpha'"] \arrow[d, Rightarrow, "\beta"'] & Y \arrow[r, "g'"] \arrow[d, "x'"] & X \arrow[d, "x"] \\ D \arrow[r, Rightarrow, "\alpha"] & A \arrow[r, "g"] & B \end{tikzcd}$$

Then, by Lemma 3.2.4, $\alpha' : D' \Rightarrow Z$ is a van Kampen diagram and each $g'\alpha'(s) \in \mathcal{M}$. Hence $g' \in \mathcal{M}_{\text{gap}}$.

To prove that $\mathcal{M}_{\text{gap}}$ is a stable system of monics in $\mathbf{C}$, it suffices to prove that $\mathcal{M}_{\text{gap}}$ is closed under composition. Let $g_1 : X_1 \to X_2$ and $g_2 : X_2 \to A$ be $\mathcal{M}$-gaps and let

$\alpha_i : D_i \Rightarrow X_i$ be van Kampen colimits such that $g_i\alpha_i(s_i) \in \mathcal{M}$ for all $s_i \in \mathbf{S}_i$, where $\mathbf{S}_i$ are stable meet semilattices and $D_i : \mathbf{S}_i \to \mathcal{M}$ are stable meet semilattice $\mathcal{M}$-diagrams, $i = 1, 2$. Let $\alpha'_2 : D'_2 \Rightarrow X_1$ be the pullback of α_2 along g_1. Then $\alpha'_2 : D'_2 \Rightarrow X_1$ is van Kampen colimit. Now, we form the van Kampen colimit $\alpha_1 \times_{X_1} \alpha'_2 : D_1 \times_{X_1} D'_2 \Rightarrow X_1$ by the following pullback diagrams:

$$\begin{tikzcd}
(D_1 \times_{X_1} D'_2)(s_1, s_2) \arrow[r, "\alpha'_1(s_2)"] \arrow[d, "\alpha''_2(s_1)"'] & D'_2(s_2) \arrow[r, "g'_1"] \arrow[d, "\alpha'_2(s_2)"'] & D_2(s_2) \arrow[d, "\alpha_2(s_2)"] & \\
D_1(s_1) \arrow[r, "\alpha_1(s_1)"] & X_1 \arrow[r, "g_1"] & X_2 \arrow[r, "g_2"] & A
\end{tikzcd}$$

Then $g_2g_1(\alpha_1 \times_{X_1} \alpha'_2)(s_1, s_2) = g_2g_1(\alpha'_2(s_2)\alpha'_1(s_1)) = g_2\alpha_2(s_2)g'_1\alpha'_1(s_2) \in \mathcal{M}$ since $g'_1\alpha'_1(s_2)$ is a pullback of $g_1\alpha_1(s_1)$ and so $g_2g_1 \in \mathcal{M}_{\text{gap}}$, as desired. ☐

Since $\mathcal{M}_{\text{gap}}$ is a stable system of monics in an $\mathcal{M}$-adhesive category $\mathbf{C}$, it is natural to ask if $\mathbf{C}$ is an $\mathcal{M}_{\text{gap}}$-adhesive category and if $(\mathcal{M}_{\text{gap}})_{\text{gap}}$ is larger than $\mathcal{M}_{\text{gap}}$. In order to answer these questions, we first observe that any amalgam $\mathcal{M}_{\text{gap}}$-cocone gives rise to an amalgam $\mathcal{M}$-cocone.

Given an amalgam $\mathcal{M}_{\text{gap}}$-cocone $\beta : D \Rightarrow X$ from $\mathbf{S}$, we have a stable meet semilattice $\delta(\mathbf{S})$ with

objects: $(s, m : A \to D(s))$ with $s \in \mathbf{S}$, $m \in \mathcal{M}$, and $\beta(s)m \in \mathcal{M}$;

maps: a map $(\leq, f) : (s, m : A \to D(s)) \to (s', m' : A' \to D(s'))$ is a pair of a map $s \leq s'$ in $\mathbf{S}$ and a map $f : A \to A'$ in $\mathbf{C}$ such that

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, "f"'] & D(s) \arrow[d, "D(s\leq s')"] \\
A' \arrow[r, "m'"] & D(s')
\end{tikzcd}$$

commutes, when $s \leq s'$. Since $\beta(s')m'f = \beta(s)m \in \mathcal{M}$, f must be an $\mathcal{M}$-map;

meet $(s, m : A \to D(s)) \wedge (m', m' : A' \to D(s')) = (s \wedge s', m \wedge m' : A \wedge A' \to D(s \wedge s'))$

is given by the pullback of $\beta(s)m$ and $\beta(s')m'$:

$$\begin{tikzcd}
 & A \arrow[rr, "m"] & & D(s) \arrow[dr, "\beta(s)"] & \\
A \wedge A' \arrow[ur] \arrow[dr] \arrow[rr, dotted, "m \wedge m'"] & & D(s \wedge s') \arrow[ur] \arrow[dr] & & X \\
 & A' \arrow[rr, "m'"'] & & D(s') \arrow[ur, "\beta(s')"'] &
\end{tikzcd}$$

Clearly, $m \wedge m' : A \wedge A' \to D(s \wedge s')$ is in $\mathcal{M}$.

Note that there is a stable meet map $\partial : \delta(\mathbf{S}) \to \mathbf{S}$ taking

$$(\leq, f) : (s, m : A \to D(s)) \to (s', m' : A' \to D(s'))$$

to $s \leq s'$. For each $s \in \mathbf{S}$, since $\beta(s) : D(s) \to X$ is an $\mathcal{M}$-gap, there is van Kampen $\mathcal{M}$-colimit $\alpha_{\beta(s)} : D_{\beta(s)} \Rightarrow D(s)$ from $\mathbf{T}$ such that each $\beta(s)\alpha_{\beta(s)}(t) \in \mathcal{M}$ for $t \in \mathbf{T}$. Hence $(s, \alpha_{\beta(s)}(t) : D_{\beta(s)}(t) \to D(s))$ are in $\delta(\mathbf{S})$ such that $\partial(s, \alpha_{\beta(s)}(t) : D_{\beta(s)}(t) \to D(s)) = s$ and therefore ∂ is surjective on objects.

Given $s \leq s'$ in $\mathbf{S}$, for each $(s', m' : A' \to D(s'))(s', m' : A' \to D(s')) \in \partial^{(-1)}(s')$, the cartesian lifting $(\leq)^*(s', m' : A' \to D(s')) = (s, n : A^* \to D(s))$ is given by the following pullback diagram:

$$\begin{tikzcd}
A^* \arrow[r, "a"] \arrow[d, "n"'] & A' \arrow[d, "m'"] \\
D(s) \arrow[r, "D(\leq)"] & D(s')
\end{tikzcd}$$

On the other hand, for each $(s, m : A \to D(s)) \in \partial^{(-1)}(s)$, the opcartesian lefting $(\leq)_*(s, m : A \to D(s)) = (s', D(\leq)m : A \to D(s'))$. Hence $\partial : \delta(\mathbf{S}) \to \mathbf{S}$ is a bifibration.

The amalgam $\mathcal{M}_{\text{gap}}$-cocone $\beta : D \Rightarrow X$ from $\mathbf{S}$ gives rise to an amalgam $\mathcal{M}$-cocone $\delta(\beta) : \delta(D) \Rightarrow X$ from $\delta(\mathbf{S})$, where $\delta(D)$ takes $(\leq, f) : (s, m : A \to D(s)) \to (s', m' : A' \to D(s'))$ to $f : A \to A'$ and $\delta(\beta)(s, m : A \to D(s)) = \beta(s)m$. It is easy to see that

$\delta(\beta) : \delta(D) \Rightarrow X$ is an amalgam $\mathcal{M}$-cocone. So we have the following situation:

$$\begin{tikzcd}
\delta(\mathbf{S}) \arrow[r, "\delta(D)"] \arrow[d, "\partial"'] & \mathbf{C} \\
\mathbf{S} \arrow[ur, "D"'] &
\end{tikzcd}$$

where ∂ is a bifibration that is surjective on objects and for each $s \in \mathbf{S}$ and the van Kampen colimit $\alpha_{\beta(s)} : \delta(D)|_{\partial^{(-1)}(s)} \Rightarrow D(s)$ of $\delta(D)|_{\partial^{(-1)}(s)}$ exists by Lemma 3.2.10.

For any map $f : s \to s'$ in $\mathbf{S}$ and any $(s, m : A \to D(s)) \in \partial^{(-1)}(s)$, the opcartesian lifting ϑ_f of f at $(s, m : A \to D(s))$ is $(1_A, f) : (s, m : A \to D(s)) \to (s', D(f)m : A \to D(s'))$. Hence

$$\begin{tikzcd}
\delta(D)(s,m) \arrow[r, "\alpha_{\beta(s)}"] \arrow[d, "\delta(\vartheta_f)"'] & D(s) \arrow[d, "D(f)"] \\
\delta(D)(f_*(s,m)) \arrow[r, "\alpha_{\beta(s')}"] & D(s')
\end{tikzcd}$$

commutes since it is actually the following commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "m"] \arrow[d, "1_A"'] & & D(s) \arrow[d, "D(f)"] \\
A \arrow[r, "m"] & D(s) \arrow[r, "D(f)"] & D(s')
\end{tikzcd}$$

Lemma 3.2.12 *The van Kampen colimit $\kappa : D \Rightarrow C$ of D exists, where C is the colimit cocone vertex of $\delta(D)$.*

Proof: Given any amalgam $\mathcal{M}_{\text{gap}}$-cocone $\beta : D \Rightarrow X$, we have an amalgam $\mathcal{M}$-cocone $\delta(\beta) : \delta(D) \Rightarrow X$. Since $\mathbf{X}$ is an $\mathcal{M}$-adhesive category, $\delta(\beta)$ has colimit $\nu : \delta(D) \Rightarrow C$. By Lemma 3.2.5, $\kappa : D \Rightarrow C$, given by the following commutative diagram

$$\begin{tikzcd}
\delta(D)(t) \arrow[rr, "\nu(t)"] \arrow[dr, "\alpha^{\delta(t)}(t)"'] & & C \\
& D(\delta(t)) \arrow[ur, "\kappa(\delta(t))"'] &
\end{tikzcd}$$

is a colimit. We need to check that $\kappa : D \Rightarrow C$ is indeed a van Kampen colimit. For any amalgamable diagram E and any commutative diagram

$$\begin{tikzcd}
E \arrow[r, Rightarrow, "\chi"] \arrow[d, Rightarrow, "\mu"'] & C' \arrow[d, "r"] \\
D \arrow[r, Rightarrow, "\kappa"] & C
\end{tikzcd}$$

in which μ is cartesian, form the pullback $(\psi(t), \phi(t)$ of $\alpha^{\delta(t)}(t)$ along $\mu(\delta(t))$:

$$\begin{tikzcd}
\delta(D)'(t) \arrow[d, "\phi(t)"'] \arrow[dr, "\psi(t)"'] \arrow[drr] & & \\
\delta(D)(t) \arrow[dr, "\alpha^{\delta(t)}(t)"'] \arrow[drr, "\nu(t)"] & E(\delta(t)) \arrow[r, "\chi(\delta(t))"'] \arrow[d, "\mu(\delta(t))"'] & C' \arrow[d, "r"] \\
& D(\delta(t)) \arrow[r, "\kappa(\delta(t))"'] & C
\end{tikzcd}$$

If the front square is a pullback diagram, then the back square is a pullback diagram and so $\chi\psi = i : \delta(D)' \Rightarrow C'$, as a pullback of a van Kampen colimit $\nu : \delta(D) \Rightarrow C$, is a van Kampen colimit. Thus, by Lemma 3.2.5, $\chi : E \Rightarrow C'$ is a colimit.

Conversely, suppose that $\chi : E \Rightarrow C'$ is a colimit, then $\chi\psi : \delta(D)' \Rightarrow C'$ is a colimit. Hence we have the following situation:

$$\begin{tikzcd}
\delta(D)'(t) \arrow[r, "\psi(t)"] \arrow[d, "\phi(t)"'] & E(\delta(t)) \arrow[r, "\chi(\delta(t))"] \arrow[d, "\mu(\delta(t))"'] & C' \arrow[d, "r"] \\
\delta(D)(t) \arrow[r, "\alpha^{\delta(t)}(t)"] & D(\delta(t)) \arrow[r, "\kappa(\delta(t))"] & C
\end{tikzcd}$$

in which both the left two rows are van Kampen colimits, both the right two rows are colimits, and both the left and the outer squares are pullback diagrams. We want to show the right square is a pullback diagram. To do this, we form pullback (u, v) of $\kappa(\delta(t))$ along r. Then there is a unique map $x : E(\delta(t)) \to P$ such that $ux = \chi(\delta(t))$ and

$vx = \mu(\delta(t))$:

$$\begin{tikzcd}
\delta(D)'(t) \arrow[r, "\psi(t)"] \arrow[dd, "\phi(t)"'] \arrow[drr, dotted] & E(\delta(t)) \arrow[rr, "\chi(\delta(t))"] \arrow[dd, "\mu(\delta(t))"'] \arrow[dr, dotted, "x"] & & C' \arrow[dd, "r"] \\
& & P \arrow[ur, "u"'] \arrow[dl, "v"] \arrow[ul, dotted, bend right, "y"'] & \\
\delta(D)(t) \arrow[r, "\alpha^{\delta(t)}(t)"'] & D(\delta(t)) \arrow[rr, "\kappa(\delta(t))"'] & & C
\end{tikzcd}$$

Since $x\psi(t)$ is a pullback of $\alpha^{\delta(t)}(t)$ along v and since the left-lower row is a van Kampen colimit, $x\psi : \delta(D)' \Rightarrow P$ is a colimit so that there is a unique map $y : Y \to E(\delta(t))$ such that

$$yx\psi(t) = \psi(t).$$

Then $yx = 1$. Similarly, since the left-upper row is a colimit, $xy = 1$. Hence the right square is a pullback diagram, as desired. Thus, $\kappa : D \Rightarrow C$ is a van Kampen colimit. $\square$

For $\mathcal{M}$-adhesivity and $\mathcal{M}$-gaps, we have:

Proposition 3.2.13 *If* $\mathbf{C}$ *is an* $\mathcal{M}$*-adhesive category, then*

(i) $\mathbf{C}$ *is an* $\mathcal{M}_{\text{gap}}$*-adhesive category;*

(ii) $(\mathcal{M}_{\text{gap}})_{\text{gap}} = \mathcal{M}_{\text{gap}}$.

PROOF:

(i) For any $\mathcal{M}_{\text{gap}}$-diagram D from a stable meet semilattice $\mathbf{S}$ such that there is an amalgam $\mathcal{M}_{\text{gap}}$-cocone $\alpha : D \Rightarrow X$, we have an amalgam $\mathcal{M}$-cocone $\delta(\alpha) : \delta(D) \Rightarrow X$ so that $\delta(D)$ has van Kampen colimit $\nu : \delta(D) \Rightarrow C$. By Lemma 3.2.12, D has van Kampen colimit $\kappa : D \Rightarrow C$. Thus, $\mathbf{C}$ is an $\mathcal{M}_{\text{gap}}$-adhesive category.

(ii) It suffices to prove that $(\mathcal{M}_{\text{gap}})_{\text{gap}} \subseteq \mathcal{M}_{\text{gap}}$. For any $g : X \to Y$ in $(\mathcal{M}_{\text{gap}})_{\text{gap}}$, there is an $\mathcal{M}_{\text{gap}}$-van Kampen colimit $\alpha_g : D_g \Rightarrow X$ from $\mathbf{S}_g$ such that $g{\cdot}\alpha_g(s) \in \mathcal{M}_{\text{gap}}$ for each $s \in \mathbf{S}$. Since $\alpha_g : D_g \Rightarrow X$ is a van Kampen $\mathcal{M}_{\text{gap}}$-cocone, $\delta(\alpha_g) : \delta(D_g) \Rightarrow X$

is an amalgam $\mathcal{M}$-cocone from $\delta(\mathbf{S}_g)$ that is a van Kampen colimit cocone such that $g\delta(\alpha_g)(t) \in \mathcal{M}$ for all $t \in \delta(\mathbf{S}_g)$. Hence g is an $\mathcal{M}$-gap and therefore $(\mathcal{M}_{\text{gap}})_{\text{gap}} \subseteq \mathcal{M}_{\text{gap}}$.

□

3.2.5 Binary Gaps

Let's consider a partial map category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ with $\mathcal{M} = \{\text{all monics}\}$. If $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ has van Kampen colimits for amalgamable pushouts of monic, then all binary joins of $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ exist, if there is a strict initial object, it follows that all non-empty finite joins given by van Kampen colimits exist in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$. Hence all diagrams D on a non-empty finite stable meet semilattice have the van Kampen colimits. Further as gaps are monic we immediately have that the partial map category is a join restriction category.

Our objective is to generalize these observations to any $\mathcal{M}$-adhesive category so that we can reexpress the condition in terms of Van Kampen squares.

Definition 3.2.14 *In an $\mathcal{M}$-adhesive category, a map $b : X \to Y$ is called a* binary $\mathcal{M}$-gap *if there is a van Kampen square*

$$\begin{tikzcd}
 & \bullet \arrow[dr, "\alpha_1"] \arrow[drr, bend left] & & \\
\bullet \arrow[ur, "m_1"] \arrow[dr, "m_2"'] & & X \arrow[r, dotted, "b"] & \bullet \\
 & \bullet \arrow[ur, "\alpha_2"'] \arrow[urr, bend right] & &
\end{tikzcd}$$

such that $b\alpha_1, b\alpha_2 \in \mathcal{M}$. Similarly, a map $f : X \to Y$ is called a finite $\mathcal{M}$-gap *if there is a van Kampen colimit $\nu : D \Rightarrow X$ on a finite stable semilattice $\mathbf{S}$ such that $f\nu(s) \in \mathcal{M}$ for all $s \in \mathbf{S}$.*

Let

$$\mathcal{M}_{\text{bgap}} = \{\text{all binary gaps in } \mathbf{C}\}$$

and

$$\mathcal{M}_{\text{fgap}} = \{\text{all finite gaps in } \mathbf{C}\}.$$

Let $f_1 : X \to Y$ and $f_2 : Y \to Z$ be in $\mathcal{M}_{\text{fgap}}$. Suppose the finite gaps f_1 and f_2 are given by van Kampen colimits $\nu_1 : D_1 \Rightarrow X$ on $\mathbf{S}_1$ and $\nu_2 : D_2 \Rightarrow Y$ on $\mathbf{S}_2$, respectively. Then, by the proof of Lemma 3.2.11, $f_2 f_1$ is a gap given by $D_1 \times_X D_2$ on $\mathbf{S}_1 \times \mathbf{S}_2$. Since $|\mathbf{S}_1 \times \mathbf{S}_2| < \infty$, $f_2 f_1 \in \mathcal{M}_{\text{fgap}}$. It follows that $\mathcal{M}_{\text{fgap}}$ is a stable system of monics. Let $C(\mathcal{M}_{\text{bgap}})$ be the composition closure of $\mathcal{M}_{\text{bgap}}$. Then, clearly,

$$\mathcal{M} \subseteq \mathcal{M}_{\text{bgap}} \subseteq C(\mathcal{M}_{\text{bgap}}) \subseteq \mathcal{M}_{\text{fgap}} \subseteq \mathcal{M}_{\text{gap}}.$$

Proposition 3.2.15 *Suppose that* $(\mathbf{C}, \mathcal{M})$ *has van Kampen colimits (squares) for* $\mathcal{M}$*-amalgamable diagram of the form* $\mathbf{V} = \begin{tikzcd}[row sep=small, column sep=small] & \bullet \\ \bullet \arrow[ur] \arrow[dr] & \\ & \bullet \end{tikzcd}$ *and* $\mathcal{M}_{\text{bgap}} = \mathcal{M}$*, then* $\mathbf{C}$ *has van kampen colimits for all diagram* $D : \mathbf{S} \to \mathcal{M}$ *which are* $\mathcal{M}$*-amalgamable, when* $2 \leq |\mathbf{S}| \leq \infty$*. Moreover,* $\mathcal{M}_{\text{fgap}} = \mathcal{M}_{\text{bgap}} \cup \{$*empty gaps, unary gaps*$\}$.

Proof: Since $|\mathbf{S}| \leq \infty$, $\mathbf{S}$ has at least one maximal element. We distinguish 2 cases.

If $\mathbf{S}$ has a unique maximal element, namely, its maximum $\top$, then it is easy to see that D has van Kampen colimit $\leq : D \Rightarrow D(\top)$.

Assume now that $\mathbf{S}$ has more than one but finitely many maximal elements $\top_1, \ldots, \top_n$. Since $D : \mathbf{S} \to \mathcal{M}$ is an amalgam $\mathcal{M}$-cocone, so is $D|_{\{\top_1 \wedge \top_2, \top_1, \top_2\}}$. Hence $D|_{\{\top_1 \wedge \top_2, \top_1, \top_2\}}$ has van Kampen colimit $D(\top_1) \vee D(\top_2)$:

$$\begin{tikzcd}
 & D(\top_1) \arrow[dr] & \\
D(\top_1 \wedge \top_2) \arrow[ur] \arrow[dr] & & D(\top_1) \vee D(\top_2) \\
 & D(\top_2) \arrow[ur] &
\end{tikzcd}$$

Let $\mathbf{S}_1 = \mathbf{S} \cup \{\top_1 \vee \top_2\}$. Then $\mathbf{S}_1$ becomes a stable meet semilattice by adding

$$s \wedge (\top_1 \vee \top_2) = (s \wedge \top_1) \vee (s \wedge \top_2)$$

for each $s \in \mathbf{S}$ that is not below $\top_1$ or $\top_2$. Now, we can extend $D : \mathbf{S} \to \mathcal{M}$ to $D_1 : \mathbf{S}_1 \to \mathcal{M}$ by adding the gaps induced by the van Kampen colimits $\top_1 \vee \top_2$ and $(s \wedge \top_1) \vee (s \wedge \top_2)$. Clearly, D_1 is $\mathcal{M}$-amalgamable. So, we have an $\mathcal{M}$-amalgamable diagram $D_1 : \mathbf{S}_1 \to \mathcal{M}$ such that both D and D_1 have the same colimit if they exist and $\mathbf{S}_1$ has $n-1$ maximal elements $\top_1 \vee \top_2, \top_3, \ldots, \top_n$. Continuing in this way, we have a stable meet semilattice $\mathbf{S}'$ in which D can be extended from $\mathbf{S}$ to $\mathbf{S}'$ such that $D' : \mathbf{S}' \to \mathcal{M}$ is also an $\mathcal{M}$-amalgamable diagram and both D and D' have the same colimit. So D has van Kampen colimit since D' does.

To prove that $\mathcal{M}_{\text{fgap}} = \mathcal{M}_{\text{bgap}}$, suppose that $f \in \mathcal{M}_{\text{fgap}}$ is given by a van Kampen colimit $\nu : D \Rightarrow X$ on a finite stable meet semilattice $\mathbf{S}$ with $|\mathbf{S}| \geq 2$, namely, $f\nu(s) \in \mathcal{M}$ for all $s \in \mathbf{S}$. Now, we look at the number $\max(\mathbf{S})$ of maximal elements of $\mathbf{S}$

If $\max(\mathbf{S}) = 1$, then the van Kampen colimit of D must be $\leq : D \Rightarrow D(\top)$ that is, actually, a van Kampen square. So $f \in \mathcal{M}_{\text{bgap}}$.

Otherwise, if $\max(\mathbf{S}) > 1$, by the first part of the proof of the proposition, there is a van Kampen colimit $\nu : D' \Rightarrow X$ on $\mathbf{S}'$ such that $\nu'(s') \in \mathcal{M}$ for all $s' \in \mathbf{S}'$ and $\max(\mathbf{S}') = 1$. Hence $f \in \mathcal{M}_{\text{bgap}}$, too. $\square$

3.3 Completeness for Join Restriction Categories

The main goal of this section is to prove the completeness theorem for join restriction categories.

3.3.1 Joins in Partial Map Categories and $\mathcal{M}$-adhesive Categories

In the following lemma, we demonstrate that if $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ has the bottom element then $\mathbf{C}$ has a strict initial object that is a van Kampen colimit from the stable meet semilattice $\emptyset$.

Lemma 3.3.1 *If each hom-set in* $\mathsf{Par}(\mathbf{C},\mathcal{M})$ *has the bottom element, then* $\mathbf{C}$ *has the strict initial object* 0 *and each map* $0 \to X$ *is in* $\mathcal{M}$*.*

PROOF: Suppose that each hom-set in $\mathsf{Par}(\mathbf{C},\mathcal{M})$ has the bottom element. For each pair of objects A, B in $\mathbf{C}$, let $\perp_{AB} = (m_A, 0_B)$, with $0_B : 0_{AB} \to B$, be the bottom element of $\hom(A, B)$ in $\mathsf{Par}(\mathbf{C},\mathcal{M})$. First, we claim that the map $0_B : 0_{AB} \to B$ is unique in $\mathbf{C}$. In fact, if there were two maps $f, g : 0_{AB} \to B$, then there were a map $\leq : 0_{AB} \to 0_{AB}$ such that

$$\begin{tikzcd}
& 0_{AB} \arrow[ld, "m_A"'] \arrow[rd, shift left, "f"] \arrow[rd, shift right, "g"'] & \\
A & & B \\
& 0_{AB} \arrow[lu, "m_A"] \arrow[ru, "0_B"'] \arrow[uu, dotted, "\leq"] &
\end{tikzcd}$$

commutes since $\perp_{AB}= (m_A, 0_B)$ is the bottom element in $\hom_{\mathsf{Par}(\mathbf{C},\mathcal{M})}(A, B)$. Hence $m_A = m_A \leq$ and therefore $\leq= 1$. Thus, $f = 0_B = g$.

For any object X in $\mathbf{C}$, we can compose the two bottom elements by forming the pullback diagram $(*)$:

$$\begin{tikzcd}
& & 0_{AX} \arrow[ld, "k"'] \arrow[rd, "0'_B"] & & \\
& 0_{AB} \arrow[ld, "m_A"'] \arrow[rd, "0_B"] & (*) & 0_{BX} \arrow[ld, "m_B"'] \arrow[rd, "0_X"] & \\
A & & B & & X
\end{tikzcd}$$

Since $(m_A, 0_B) \leq (m_A k, m_B 0'_B)$, there is a map $k' : 0_{AB} \to 0_{AX}$. Hence $kk' = 1_{0_{AB}}$ and $k'k = 1_{0_{AX}}$ and therefore k is an isomorphism. Hence there is a unique map from 0_{AB} to X. Thus, 0_{AB} is an initial object in $\mathbf{C}$.

The initial object 0 is also strict: if there is a map $x : X \to 0$, then one can form the join $(1_X, x) \vee (1_0, 0_X)$ by taking pullback diagram $(**)$:

$$\begin{tikzcd}
& & \bullet \arrow[dl, "l"'] \arrow[dr] & & \\
& X \arrow[dl, "1_X"'] \arrow[dr, "x"] & (**) & 0 \arrow[dl, "1_0"'] \arrow[dr, "0_X"] & \\
X & & 0 & & X
\end{tikzcd}$$

But $(1_X, x) \vee (1_0, 0_X)$ must be $(1_X, 1_X)$. Note that

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "x"'] & X \arrow[d, "x"] \\
0 \arrow[r, equal] & 0
\end{tikzcd}$$

is a pullback diagram. Hence l is an isomorphism and $X \cong 0$ and therefore x is an isomorphism. $\square$

Clearly, each strict initial object is stable. However, the existence of a stable initial object 0 and $0 \to X \in \mathcal{M}$ imply each hom-set in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ has the bottom element.

Lemma 3.3.2 *If* $\mathbf{C}$ *has a stable initial object* 0 *and each map* $0 \to X$ *is in* $\mathcal{M}$*, then each hom-set in* $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ *has the bottom element.*

Proof: If $\mathbf{C}$ has a stable initial object 0, then, for any objects A, B in $\mathbf{C}$, it is easy to check that $\perp_{\mathrm{map}_{\mathbf{C}}(A,B)} = (0_A, 0_B)$, where 0_A is the unique map $0 \to A$. In fact, for any partial map $(m, f) : A \to B$,

$$\begin{array}{rcl} (m,f)\overline{(0_A, 0_B)} & = & (m,f)(0_A, 0_A) \\ & = & (0_A, 0_B) \end{array}$$

as the initial object 0 is stable so that $(***)$ is a pullback diagram:

$$\begin{tikzcd}
& & 0 \arrow[dl, equal, "1"'] \arrow[dr, "0_{A'}"] & & \\
& 0 \arrow[dl, "0_A"'] \arrow[dr, "0_A"] & (***) & A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\
A & & A & & B
\end{tikzcd}$$

Hence $(0_A, 0_A) \leq (m, f)$. $\square$

For example, since $\mathbf{Set}_{\text{ffib}}$, defined in Example 1.6.16, does not have any strict inital objects, each hom-set in $\mathsf{Par}(\mathbf{Set}_{\text{ffib}}, \mathcal{M})$ does not have the bottom element but does have binary joins, where $\mathcal{M} = \{\text{injections } i : A \hookrightarrow B \mid |B \setminus i(A)| < +\infty\}$.

Now we consider arbitrary joins in a partial map category. In one direction, we have:

Theorem 3.3.3 *Let $\mathbf{C}$ be a category with a stable system of monics $\mathcal{M}$. If each amalgamable $\mathcal{M}$-diagram has a stable colimit and each $\mathcal{M}$-gap is in $\mathcal{M}$, then $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a join restriction category.*

PROOF: Since each amalgamable $\mathcal{M}$-diagram has a stable colimit, it has a stable colimit from $\emptyset$ so that it has a stable initial object. Hence each hom set in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ has a bottom element by Lemma 3.3.2. For any $\smile$-compatible set $\{(m_i, f_i) | i \in I\} \subseteq \mathrm{Map}_{\mathsf{Par}(\mathbf{C},\mathcal{M})}(A, B)$, with $I \neq \emptyset$, where $m_i : A_i \to A$ and $f_i : A_i \to B$ are maps with $m_i \in \mathcal{M}$, let the diagram $D : I \to \mathbf{C}$ be such that $D(i) = A_i$. Clearly, $\{A_i | i \in I\}$ forms a stable meet semilattice diagram with the binary meet given by pullbacks and $\nu : D \Rightarrow A$, given by $\nu(i) = m_i$, is an amalgam $\mathcal{M}$-cocone. Since each amalgamable $\mathcal{M}$-diagram has a stable colimit, D has a stable colimit $(\vee_{j \in I} A_j, \alpha)$. Since $\{m_i | i \in I\}$ provides a D-cocone, there is a unique map $m : \vee_{j \in I} A_j \to A$ such that $m\alpha(A_i) = m_i$. Clearly, m is an $\mathcal{M}$-gap and so it is an $\mathcal{M}$-map, Clearly, $\{f_i | i \in I\}$ forms a D-cocone and so there is a map $f : \vee_{j \in I} A_j \to B$ such that $f\alpha(A_i) = f_i$.

$$\begin{tikzcd}
& A_i \arrow[ddl, "m_i"'] \arrow[ddr, "f_i"] \arrow[d, "\alpha(A_i)"] & \\
& \vee_{j \in I} A_j \arrow[dl, dotted, "m"] \arrow[dr, dotted, "f"'] & \\
A & & B
\end{tikzcd}$$

Now it is routine to check that (m, f) is the join of the $\smile$-compatible set $\{(m_i, f_i) | i \in I\}$ and that $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is indeed a join restriction category since $\vee_{j \in I} A_j$ is a stable colimit. The four join axioms are verified as follows.

[J.1] Since for each $i \in I$

$$\begin{tikzcd}
A_i \arrow[r, "\alpha(A_i)"] \arrow[d, "1_{A_i}"'] & \vee_{j \in I} A_j \arrow[d, "m"] \\
A_i \arrow[r, "m_i"] & A
\end{tikzcd}$$

is a pullback square, $1_{A_i} : A_i \to A_i$ is an isomorphism, and $f = f_i\alpha(A_i)1_{A_i}^{-1}$, by Lemma 1.6.20 $(m_i, f_i) \leq (m, f)$ for each $i \in I$.

Suppose that each $(m_i, f_i) \leq (n, g)$ for all $i \in I$. Then, by Lemma 1.6.20, π_{ni} is an isomorphism and $g\pi_{m_i}\pi_{ni}^{-1} = f_i$, where π_{m_i} and π_{ni} are given by the following pullback square:

$$\begin{tikzcd}
C_i \arrow[r, "\pi_{m_i}"] \arrow[d, "\pi_{ni}"'] & A' \arrow[d, "n"] \\
A_i \arrow[r, "m_i"] & A
\end{tikzcd}$$

Assume that

$$\begin{tikzcd}
C \arrow[r, "\rho_m"] \arrow[d, "\rho_n"'] & A' \arrow[d, "n"] \\
\vee_{i \in I} A_i \arrow[r, "m"] & A
\end{tikzcd}$$

is a pullback diagram. Similarly, $D' : I \to \mathbf{C}$, given by $D'(i) = C_i$, forms an amalgamable $\mathcal{M}$-diagram. Then the colimit $\vee_{i \in I} C_i$ of D' exists and is stable and so $\vee_{i\in I} C_i = C$:

$$\begin{tikzcd}
 & C_i \arrow[dl, "\alpha(C_i)"'] \arrow[dr, "\pi_{m_i}"] \arrow[dd, "\pi_{ni}"] & \\
C \arrow[rr, "\rho_m"] \arrow[dd, "\rho_n"'] & & A' \arrow[dd, "n"] \\
 & A_i \arrow[dl, "\alpha(A_i)"'] \arrow[dr, "m_i"] & \\
\vee_{i\in I} A_i \arrow[rr, "m"] & & A
\end{tikzcd}$$

As monics and retracts are stable and each π_{ni} is an isomorphism, ρ_n is an isomorphism. Since

$$f\rho_n\alpha(C_i) = f\alpha(A_i)\pi_{ni} = f_i\pi_{ni} = g\pi_{m_i} = g\pi_m\alpha(C_i),$$

$f\rho_n = g\rho_m$. Hence $(m, f) \leq (n, g)$. Thus, $(m, f) = \vee_{i\in I}(m_i, f_i)$.

[J.2] Clearly, as m is the $\mathcal{M}$-gap given by the stable colimit $(\vee_{j\in I} A_j, \alpha)$ of $\{m_i : A_i \to A\}_{i\in I}$, we have

$$\overline{\vee_{i\in I}(m_i, f_i)} = \overline{(m, f)} = (m, m) = \vee_{i\in I}(m_i, m_i) = \vee_{i\in I}\overline{(m_i, f_i)}.$$

[J.3] For any partial map $(m, f) : A \to B$ and $\smile$-compatible partial maps $\{(n_i, g_i) : B \to C,\ i \in J\}$, $\vee_{i\in J}\big((n_i, g_i)(m, f)\big) = (k, h)$ is given by the stable colimit $(\vee_{i\in J} D_i, \alpha(D_i))$:

$$\begin{tikzcd}
 & & D_i \arrow[dl, "n'_i"'] \arrow[d, "\alpha(D_i)"] \arrow[dr, "f'_i"] & & \\
 & A_1 \arrow[dl, "m"'] & \vee_{i\in J} D_i \arrow[dll, dotted, "k"] \arrow[drr, dotted, "h"'] & B_i \arrow[dr, "g_i"] & \\
B & & & & C
\end{tikzcd}$$

where each

$$\begin{tikzcd}
D_i \arrow[r, "f'_i"] \arrow[d, "n'_i"'] \arrow[dr, phantom, "(*_i)"] & B_i \arrow[d, "n_i"] \\
A_1 \arrow[r, "f"] & B
\end{tikzcd}$$

is a pullback diagram. On the other hand, $\vee_{i\in J}(n_i, g_i) = (n, g)$ is given by the stable colimit $(\vee_{i\in J}B_i, \alpha(B_i))$:

$$\begin{tikzcd}
& B_i \arrow[ddl, "n_i"'] \arrow[d, "\alpha(B_i)"] \arrow[ddr, "g_i"] & \\
& \vee_{i\in J}B_i \arrow[dl, dotted, "n"] \arrow[dr, dotted, "g"'] & \\
B & & C
\end{tikzcd}$$

and $\big(\vee_{i\in J}\,(n_i, g_i)\big)(m, f) = (mn', gf'')$ is given by

$$\begin{tikzcd}
& & D \arrow[dl, "n'"'] \arrow[dr, "f''"] & & \\
& A_1 \arrow[dl, "m"'] \arrow[dr, "f"] & (**) & \vee_{i\in J}B_i \arrow[dl, "n"'] \arrow[dr, "g"] & \\
A & & B & & C
\end{tikzcd}$$

where $(**)$ is a pullback diagram. As $\vee_{i\in J}B_i$ is stable, $D = \vee_{i\in J}D_i$:

$$\begin{tikzcd}
& & D_i \arrow[dddl, "n'_i"'] \arrow[dd, "\alpha(D_i)"] \arrow[dr, "f'_i"] & & \\
& & & B_i \arrow[dddl, "n_i"'] \arrow[dd, "\alpha(B_i)"] \arrow[dddr, "g_i"] & \\
& & D \arrow[dl, "n'"'] \arrow[dr, "f''"] & & \\
& A_1 \arrow[dl, "m"'] \arrow[dr, "f"] & (**) & \vee_{i\in J}B_i \arrow[dl, "n"'] \arrow[dr, "g"] & \\
A & & B & & C
\end{tikzcd}$$

Hence

$$\vee_{i\in J}\big((n_i, g_i)(m, f)\big) = \vee_{i\in J}(mn'_i, g_i f'_i) = (mn', gf'') = \big(\vee_{i\in J}\,(n_i, g_i)\big)(m, f).$$

[J.4] Similar to the proof of [**J.3**] above or by Lemma 3.1.8.

□

By Theorem 3.3.3 and Proposition 3.2.13, immediately we have:

Corollary 3.3.4 *$\mathsf{Par}(\mathbf{C}, \mathcal{M}_{\text{gap}})$ is a join restriction category.*

Conversely, we have:

Theorem 3.3.5 *Let $\mathbf{C}$ be a category with a stable system of monics $\mathcal{M}$. If $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a join restriction category, then $\mathbf{C}$ is an $\mathcal{M}$-adhesive category and $\mathcal{M}_{\text{gap}} \subseteq \mathcal{M}$.*

Proof: If $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a join restriction category, then its hom-sets have the bottom elements and so, by Lemma 3.3.1, $\mathbf{C}$ has van Kampen colimit from $\emptyset$, given by its strict initial object. For every $\mathcal{M}$-diagram $D : \mathbf{S} \to \mathcal{M}$ with a stable meet semilattice $\mathbf{S} \neq \emptyset$ such that there is an amalgam $\mathcal{M}$-cocone $\alpha : D \Rightarrow X$, for each $s_1, s_2 \in \mathbf{S}$

$$\begin{tikzcd}
D(s_1 \wedge s_2) \arrow[r, "D(\leq)"] \arrow[d, "D(\leq)"'] & D(s_2) \arrow[d, "\alpha(s_2)"] \\
D(s_1) \arrow[r, "\alpha(s_1)"] & X
\end{tikzcd}$$

is a pullback diagram. Then, by Lemma 3.1.4, $\{(\alpha(s), \alpha(s)) | s \in \mathbf{S}\}$ is $\smile$-compatible and so the join $\vee_{s\in\mathbf{S}}(\alpha(s), \alpha(s))$ exists, say (m, m), in $\mathrm{Map}_{\mathsf{Par}(\mathbf{C},\mathcal{M})}(X, X)$, where $m : C \to X$ is in $\mathcal{M}$. Since $(\alpha(s), \alpha(s)) \leq (m, m)$, there is an $\mathcal{M}$-map $\iota(s) : D(s) \to C$ for each $s \in \mathbf{S}$. ($\iota(s)$ is the pullback of $\alpha(s)$ along m by Lemma 1.6.20.) So we have an amalgam $\mathcal{M}$-cocone $\iota : D \Rightarrow C$. We claim that $\iota : D \Rightarrow C$ is a colimit. For any cocone $\beta : D \Rightarrow Y$ in $\mathbf{C}$, since $\{(\iota(s), \beta(s))\}$ is compatible in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$, $\vee_{s\in\mathbf{S}}(\iota(s), \beta(s))$ must exist and must be $(1, g)$ for some map $g : C \to Y$ for it is total by the fact that

$$(m, m) = \vee_{s\in\mathbf{S}}(\alpha(s), \alpha(s)) = \vee_{s\in\mathbf{S}}(m\iota(s), m\iota(s))$$

$$= \vee_{s\in\mathbf{S}}(1, m)(\iota(s), \iota(s))(m, 1) = (1, m)\big(\vee_{s\in\mathbf{S}} (\iota(s), \iota(s))\big)(m, 1).$$

It follows that $\iota(s)g = \beta(s)$ for each $s \in \mathbf{S}$ from the fact that $(\iota(s), \beta(s)) \leq (1, g)$. If $g' : C \to Y$ is a map such that $g'\iota(s) = \beta(s)$ for all $s \in \mathbf{S}$, then $(1, g) = \vee_{s\in\mathbf{S}}(\iota(s), \beta(s)) = \vee_{s\in\mathbf{S}}(\iota(s), g'\iota(s)) \leq (1, g')$ and so $g = g'$ by Lemma 1.6.20, as desired.

$$\begin{tikzcd}
D(s) \arrow[r, "\iota(s)"] \arrow[dr, "\beta(s)"'] & C \arrow[d, dotted, "g"] \\
 & Y
\end{tikzcd}$$

So, we proved:

In a join restriction category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$*, for a stable meet semilattice* $\mathcal{M}$*-diagram* D*, there is an* $\mathcal{M}$*-cocone* $\alpha : D \Rightarrow X$ *if and only if the colimit* $\iota : D \Rightarrow C$ *exists and it is an* $\mathcal{M}$*-cocone, where* C *is given by* $\vee_{s\in\mathbf{S}}(\alpha(s), \alpha(s))$*.*

Letting $\mathbf{X} = \mathsf{Par}(\mathbf{C}, \mathcal{M})$ and applying the functor $\mathcal{M}\mathsf{Total}$, we actually have:

In a split restriction category $\mathbf{X}$*, for a stable* $\mathcal{M}_\mathbf{X}$ *diagram* $D : \mathbf{S} \to \mathcal{M}_\mathbf{X}$*, there is an* $\mathcal{M}_\mathbf{X}$*-cocone* $\alpha : D \Rightarrow X$ *if and only if the colimit* $\iota : D \Rightarrow C$ *exists in* $\mathsf{Total}(\mathbf{X})$ *and it is an* $\mathcal{M}_\mathbf{X}$*-cocone, where* C *is given by* $\vee_{s\in\mathbf{S}} e_{\alpha(s)}$*.*

Throughout the rest of the proof, we work either with a join restriction category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ or with a split join restriction category $\mathbf{X} = \mathsf{Par}(\mathsf{Total}(\mathbf{X}), \mathcal{M}_\mathbf{X})$.

$\iota : D \Rightarrow C$ is indeed an $\mathcal{M}_\mathbf{X}$-van Kampen colimit in $\mathsf{Total}(\mathbf{X})$. In fact, for any $\mathcal{M}_\mathbf{X}$-diagram $D' : \mathbf{S} \to \mathcal{M}_\mathbf{X}$ such that there is an amalgam $\mathcal{M}_\mathbf{X}$-cocone under D' and any commutative diagram of stable meet semilattice diagram:

$$\begin{tikzcd}
D' \arrow[r, Rightarrow, "\alpha'"] \arrow[d, Rightarrow, "\beta"'] & Z \arrow[d, "r"] \\
D \arrow[r, Rightarrow, "\iota"] & C
\end{tikzcd}$$

in which β is cartesian. Suppose that $\alpha' : D' \Rightarrow Z$ is a colimit. Then we want to show that for each $s \in \mathbf{S}$

$$\begin{tikzcd}
D'(s) \arrow[r, "\alpha'(s)"] \arrow[d, "\beta(s)"'] & Z \arrow[d, "r"] \\
D(s) \arrow[r, "\iota(s)"] & C
\end{tikzcd}$$

is a pullback diagram. To do this, it suffices to prove $\overline{e_{\iota(s)}r} = e_{\alpha'(s)}$ by Lemma 1.6.21.

For any $s, s' \in \mathbf{S}$, since β is cartesian, we have the following commutative diagram

$$\begin{tikzcd}
 & D'(s) \arrow[rr, "\alpha'(s)"] \arrow[dd, "\beta(s)"] & & Z \arrow[dd, "r"] \\
D'(s \wedge s') \arrow[ur] \arrow[rr, "u'"] \arrow[dd, "\beta(s \wedge s')"'] & & D'(s') \arrow[ur, "\alpha'(s')"] \arrow[dd, "\beta(s')"] & \\
 & D(s) \arrow[rr, "\alpha(s)"] & & C \\
D(s \wedge s') \arrow[ur] \arrow[rr, "u"] & & D(s') \arrow[ur, "\alpha(s')"'] &
\end{tikzcd}$$

in which the left, front, top, and bottom faces are pullback diagrams. Hence, by Lemma 1.6.21,

$$\overline{e_{\alpha(s)} r \alpha'(s')} = e_{u'} = \overline{e_{\alpha'(s)} \alpha'(s')}$$

and therefore

$$\overline{e_{\alpha(s)} r \alpha'(s')} \cdot \alpha'(s')^{(-1)} = \overline{e_{\alpha'(s)} \alpha'(s')} \cdot \alpha'(s')^{(-1)}.$$

Since $\vee_{s' \in \mathbf{S}} \big(\alpha'(s')\alpha'(s')^{(-1)}\big) = 1$,

$$\overline{e_{\alpha(s)} r} = \vee_{s' \in \mathbf{S}} \overline{e_{\alpha(s)} r \alpha'(s') \alpha'(s')^{(-1)}} = \vee_{s' \in \mathbf{S}} \overline{e_{\alpha'(s)} \alpha'(s') \alpha'(s')^{(-1)}} = e_{\alpha'(s)},$$

as desired. Thus, if $\alpha' : D' \Rightarrow Z$ is a colimit then

$$\begin{tikzcd}
D'(s) \arrow[r, "\alpha'(s)"] \arrow[d, "\beta(s)"'] & Z \arrow[d, "r"] \\
D(s) \arrow[r, "\iota(s)"] & C
\end{tikzcd}$$

is a pullback diagram.

Suppose now that for each $s \in \mathbf{S}$

$$\begin{tikzcd}
D'(s) \arrow[r, "\alpha'(s)"] \arrow[d, "\beta(s)"'] & Z \arrow[d, "r"] \\
D(s) \arrow[r, "\iota(s)"] & C
\end{tikzcd}$$

is a pullback diagram. Since $\alpha' : D' \Rightarrow Z$ is an $\mathcal{M}_{\mathbf{X}}$-cocone, the colimit $\iota' : D' \Rightarrow C'$ of D' exists and it is an $\mathcal{M}_{\mathbf{X}}$-cocone. So there is a total map $r' : C' \to C$ such that $r'\iota'(s) = \iota(s)\beta(s)$. Obviously, $rg' = r'$. That is the front-right square is commutative.

$$\begin{tikzcd}
D'(s) \arrow[rr, "\alpha'(s)"] \arrow[dr, "\iota'(s)"'] \arrow[dd, "\beta(s)"'] & & Z \arrow[dd, "r"] \\
& C' \arrow[ur, "g'"'] \arrow[dd, "r'"] & \\
D(s) \arrow[rr, "\iota(s)"] \arrow[dr, "\iota(s)"'] & & C \arrow[dl, equal] \\
& C &
\end{tikzcd}$$

Since each back square is a pullback diagram, we have

$$\overline{e_{\iota(s)}r} = e_{\alpha'(s)}, \forall s \in \mathbf{S}.$$

Hence

$$\overline{r} = \overline{(\vee_{s\in\mathbf{S}} e_{\iota(s)})r} = \overline{\vee_{s\in\mathbf{S}}(e_{\iota(s)}r)} = \vee_{s\in\mathbf{S}}\overline{e_{\iota(s)}r} = \vee_{s\in\mathbf{S}} e_{\alpha'(s)} = e_{g'}$$

an therefore, by Lemma 1.6.21, the front-right square is a pullback diagram. Then g' is an isomorphism and so $\alpha' : D' \Rightarrow Z$ is a colimit. Thus, $\iota : D \Rightarrow C$ is a van Kampen colimit.

For any $g : X \to Y$ in $\mathcal{M}_{\text{gap}}$, there is an $\mathcal{M}$-van Kampen colimit $\nu : D \Rightarrow X$ from $\mathbf{S}$ such that $g \cdot \nu(s) \in \mathcal{M}$ for all $s \in \mathbf{S}$. So we have an $\mathcal{M}$-cocone $\alpha : D \Rightarrow Y$ from $\mathbf{S}$, given by $\alpha(s) = g \cdot \nu(s)$.

$$\begin{tikzcd}
D(s) \arrow[r, "\nu(s)"] \arrow[dr, "\alpha(s)"'] & X \arrow[d, "g"] \\
& Y
\end{tikzcd}$$

Since both $\nu : D \Rightarrow X$ and g are determined by $\vee_{s\in\mathbf{S}}(\alpha(s), \alpha(s))$ as seen in constructing colimit of D, $g \in \mathcal{M}$. Hence $\mathcal{M}_{\text{gap}} \subseteq \mathcal{M}$. $\square$

Since van Kampen colimits are stable, by Theorems 3.3.3 and 3.3.5, clearly we have:

Theorem 3.3.6 (Characterization of Partial Map Categories with Joins) *For each $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, the following are equivalent:*

(i) *Each amalgamable $\mathcal{M}$-diagram has a stable colimit and each $\mathcal{M}$-gap is in $\mathcal{M}$;*

(ii) *$\mathbf{C}$ is an $\mathcal{M}$-adhesive category and $\mathcal{M}_{\text{gap}} \subseteq \mathcal{M}$;*

(iii) *$\mathsf{Par}(\mathbf{C}, \mathcal{M})$ is a join restriction category.*

Proof: *(i)* $\Rightarrow$ *(iii)*: By Theorem 3.3.3.

(iii) $\Rightarrow$ *(ii)*: By Theorem 3.3.5.

(ii) $\Rightarrow$ *(i)*: By definitions of a $\mathcal{M}$-adhesive category and $\mathcal{M}_{\text{gap}}$ and the fact that van Kampen colimits are stable. $\square$

By Theorem 3.3.6, immediately we have the following completeness theorem of join restriction categories.

Theorem 3.3.7 (Completeness of Join Restriction Categories) *Each join restriction category $\mathbf{X}$ can be fully and faithfully embedded, in a join and restriction preserving manner, into a partial map category $\mathsf{Par}(\mathbf{Y}, \mathcal{M})$, where $\mathbf{Y}$ is an $\mathcal{M}$-adhesive category and $\mathcal{M}_{\text{gap}} \subseteq \mathcal{M}$.*

Proof: For a given join restriction category $\mathbf{X}$, by Proposition 1.6.11, there is a full and faithful embedding

$$\mathbf{X} \overset{\eta_{\mathbf{X}}}{\to} \mathsf{Split}(\mathbf{X}) \approx \mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{X})), \mathcal{M}_{\mathsf{Split}(\mathbf{X})}).$$

For each $\smile$-compatible subset $S \subseteq \hom(e_1, e_2)$ in $\mathsf{Split}(\mathbf{X})$, where $e_1 : X \to X$ and $e_2 : Y \to Y$ are restriction idempotents in $\mathbf{X}$, clearly S is a $\smile$-compatible subset of $\hom(X, Y)$. Hence the join $\vee_{s \in S} s$ exists in $\hom(X, Y)$ as $\mathbf{X}$ is a join restriction category. Now it is easy to check that $\vee_{s \in S} s$ is a join in the hom-set $\hom(e_1, e_2)$ of $\mathsf{Split}(\mathbf{X})$ so that $\mathsf{Split}(\mathbf{X})$ is a join restriction category. By Theorem 3.3.6, $\mathbf{Y} = \mathsf{Total}(\mathsf{Split}(\mathbf{X}))$ is $\mathcal{M}_{\mathsf{Split}(\mathbf{X})}$-adhesive and $(\mathcal{M}_{\mathsf{Split}(\mathbf{X})})_{\text{gap}} \subseteq \mathcal{M}_{\mathsf{Split}(\mathbf{X})}$ as $\mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{X})), \mathcal{M}_{\mathsf{Split}(\mathbf{X})}) \approx \mathsf{Split}(\mathbf{X})$ is a join restriction category. Obviously, the restriction functor $\eta_{\mathbf{X}}$ preserves joins. $\square$

3.3.2 Join Completion Using $\mathcal{M}$-gaps

Let $(\mathbf{C}, \mathcal{M})$ be an $\mathcal{M}$-category and let $P : \mathbf{C} \to \mathbf{D}$ be a full functor which preserves pullbacks along $\mathcal{M}$-maps. Suppose that $\mathbf{D}$ has pullbacks. Then, by [15], there are a least class of monics $\mathcal{M}_P$ and a greatest class of monics $\mathcal{M}^P$, which make both $P : (\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{M}_P)$ and $P : (\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{M}^P)$ be $\mathcal{M}$-functors, where $\mathcal{M}_P$ is the composition closure of all pullbacks of $P(m)$ with $m \in \mathcal{M}$ and $\mathcal{M}^P = \mathcal{M}_0^P \cap \{\text{monics in } \mathbf{D}\}$ with $\mathcal{M}_0^P$ being the class of maps $n : D' \to D$ in $\mathbf{D}$ for which if $u : P(C) \to D$ is a $\mathbf{D}$-map then there is an $\mathcal{M}$-map $m : C' \to C$ such that $P(m)$ is a pullback of n along u:

$$\begin{tikzcd}
P(C') \arrow[r, "P(m)"] \arrow[d, "u'"'] & P(C) \arrow[d, "u"] \\
D' \arrow[r, "n"] & D
\end{tikzcd}$$

If P generates $\mathbf{D}$, then $\mathcal{M}_0^P = \mathcal{M}^P$. In particular, the Yoneda embedding $\mathcal{Y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ generates $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$. Thus, for any $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, one has two $\mathcal{M}$-functors

$$\mathcal{Y} : (\mathbf{C}, \mathcal{M}) \to (\mathbf{Set}^{\mathbf{C}^{\text{op}}}, \mathcal{M}_{\mathcal{Y}})$$

and

$$\mathcal{Y} : (\mathbf{C}, \mathcal{M}) \to (\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}, \mathcal{M}^{\mathcal{Y}}).$$

Given an $\mathcal{M}$-functor F, it is natural to ask when $\mathsf{Par}(F)$ is (full and) faithful. We have:

Lemma 3.3.8 *Let $F : (\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{N})$ be an $\mathcal{M}$-functor such that $F : \mathbf{C} \to \mathbf{D}$ is full and faithful. Then*

(1) $\mathsf{Par}(F) : \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to \mathsf{Par}(\mathbf{D}, \mathcal{N})$ *is faithful.*

(2) *The following are equivalent:*

  (*a*) $\mathsf{Par}(F) : \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to \mathsf{Par}(\mathbf{D}, \mathcal{N})$ *is full and faithful.*

  (*b*) *For any $\mathcal{N}$-map $n : D \to F(C)$, there is an $\mathcal{M}$-map m and an isomorphism $\alpha : D \to F(\mathrm{dom}(m))$ such that $F(m)\alpha = n$.*

  (*c*) *$F(m) \in \mathcal{N}$ implies $m \in \mathcal{M}$ and for any $\mathcal{N}$-map $n : D \to F(C)$ there is a $\mathbf{C}$-object X_D such that $F(X_D) \cong D$.*

Proof: (1) Let $(m, f) : A \to B$ and $(m', f') : A \to B$ be maps in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ such that $\mathsf{Par}(F)(m, f) = \mathsf{Par}(F)(m', f')$. Then there is an isomorphism $\alpha : F(\mathrm{dom}(m')) \to F(\mathrm{dom}(m))$ such that $F(m)\alpha = F(m')$ and $F(f)\alpha = F(f')$. Since F is full and faithful, there is an isomorphism $\beta : \mathrm{dom}(m') \to \mathrm{dom}(m)$ such that $F(\beta) = \alpha$ and so $m\beta = m'$ and $f\beta = f'$. Hence $(m, f) = (m', f')$ and therefore F is faithful.

(2) It is proved as follows.

"$(a) \Rightarrow (b)$" Since $(n, n) : F(C) \to F(C)$ is a map in $\mathsf{Par}(\mathbf{D}, \mathcal{N})$ and

$$\mathsf{Par}(F) : \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to \mathsf{Par}(\mathbf{D}, \mathcal{N})$$

is full and faithful, there is a map $(m, f) : C \to C$ in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ such that

$$\mathsf{Par}(F)(m, f) = (F(m), F(f)) = (n, n)$$

and so there is an isomorphism $\alpha : D \to F(\mathrm{dom}(m))$ such that $F(m)\alpha = n$, as desired.

"$(b) \Rightarrow (c)$" By (b), there is an $\mathcal{M}$-map m and an isomorphism $\alpha : D \to F(\mathrm{dom}(m))$ such that $F(m)\alpha = n$. Clearly, let $X_D = \mathrm{dom}(m)$, then $F(X_D) \cong D$. If $m : A \to B$ is a $\mathbf{C}$-map such that $F(m) \in \mathcal{N}$, then, by (b) there are an $\mathcal{M}$-map m' and an isomorphism α such that $F(m')\alpha = F(m)$ and so there is an isomorphism β in $\mathbf{C}$ such that $F(\beta) = \alpha$. But F is full and faithful. It follows that $m = m'\beta \in \mathcal{M}$ since $m', \beta \in \mathcal{M}$.

"$(c) \Rightarrow (a)$" By (1), we only need to prove that $\mathsf{Par}(F)$ is full. For any objects A, B in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ and any map $(n, f) : F(A) \to F(B)$ in $\mathsf{Par}(\mathbf{D}, \mathcal{N})$, by hypothesis there is an object X_D in $\mathbf{C}$ and an isomorphism $\alpha : F(X_D) \to D$, where $D = \mathrm{dom}(n)$. Clearly, $(n\alpha, f\alpha) = (n, f)$. Since F is full and faithful, there are an map m and $g : X_D \to B$ in $\mathbf{C}$ such that $F(m) = n\alpha$ and $F(g) = f\alpha$. Since $F(m) = n\alpha \in \mathcal{N}$, by (c) $m \in \mathcal{M}$. Then (m, g) is a map in $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ and $\mathsf{Par}(F)(m, g) = (n, f)$ and so $\mathsf{Par}(F)$ is full, as desired.

$\square$

Proposition 3.3.9 *Let $P : (\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{N})$ be a full and faithful $\mathcal{M}$-functor then*

$$\mathsf{Par}(F) : \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to \mathsf{Par}(\mathbf{D}, \mathcal{N})$$

is full and faithful whenever $\mathcal{M}_P \subseteq \mathcal{N} \subseteq \mathcal{M}^P$.

Proof: It suffices to prove that

$$\mathsf{Par}(F) : \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to \mathsf{Par}(\mathbf{D}, \mathcal{M}^P)$$

is full and faithful.

For any $\mathcal{M}^P$-map $n : D \to P(C)$, by the definition of $\mathcal{M}^P$ there is an $\mathcal{M}$-map $m : C' \to C$ such that

$$\begin{tikzcd}
P(C') \arrow[r, "P(m)"] \arrow[d] & P(C) \arrow[d, "1_{P(C)}"] \\
D \arrow[r, "n"] & P(C)
\end{tikzcd}$$

is a pullback diagram. But

$$\begin{tikzcd}
D \arrow[r, "n"] \arrow[d, "1_D"'] & P(C) \arrow[d, "1_{P(C)}"] \\
D \arrow[r, "n"] & P(C)
\end{tikzcd}$$

is also a pullback diagram since n is a monic. Hence there is an isomorphism α such that $P(m)\alpha = n$ and therefore, by Lemma 3.3.8, $\mathsf{Par}(P)$ is full and faithful. ☐

Since the Yoneda embedding

$$\mathcal{Y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$$

is full and faithful, by Proposition 3.3.9 we have:

Corollary 3.3.10 *Both*

$$\mathsf{Par}(\mathcal{Y}) : \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to \mathsf{Par}(\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}, \mathcal{M}_{\mathcal{Y}})$$

and

$$\mathsf{Par}(\mathcal{Y}) : \mathsf{Par}(\mathbf{C}, \mathcal{M}) \to \mathsf{Par}(\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}, \mathcal{M}^{\mathcal{Y}})$$

are full and faithful.

Recall that for any given restriction category $\mathbf{C}$, by Propositions 1.6.13 and 1.6.19 and Theorem 1.6.22 there are an $\mathcal{M}$-category $(\mathsf{Total}(\mathsf{Split}(\mathbf{C})), \mathcal{M}_{\mathsf{Split}(\mathbf{C})})$ and a full and faithful functor $J_{\mathbf{C}} : \mathbf{C} \to \mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{C})), \mathcal{M}_{\mathbf{C}})$. Clearly, $J_{\mathbf{C}}$ is the composition of the full and faithful unit $\eta_{\mathbf{C}} : \mathbf{C} \to \mathsf{Split}(\mathbf{C})$ and the equivalence

$$\mathsf{Split}(\mathbf{C}) \approx \mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{C})), \mathcal{M}_{\mathsf{Split}(\mathbf{C})}).$$

Explicitly, $J_{\mathbf{C}}$ is given by

$$\begin{tikzcd} A \arrow[d, "f"'] & & & \overline{f} \arrow[dl, "\overline{f}"'] \arrow[dr, "f"] & \\ B & \mapsto & 1_A & & 1_B \end{tikzcd}$$

Since any elementary topos is adhesive by Proposition 3.7 in [29] and since a colimit of an amalgam $\mathcal{M}_{\mathbf{C}}$-cocone is in $\mathcal{M}_{\mathbf{C}}$, $\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))^{\mathrm{op}}}$ is an $\mathcal{M}_{\mathbf{C}}$-adhesive category. Since $\mathcal{M}_{\mathbf{C}} \subseteq (\mathcal{M}_{\mathbf{C}})_{\mathrm{gap}}$, there is a faithful embedding

$$\mathcal{E} : \mathsf{Par}((\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))})^{\mathrm{op}}, \mathcal{M}_{\mathbf{C}}) \to \mathsf{Par}((\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))})^{\mathrm{op}}, (\mathcal{M}_{\mathbf{C}})_{\mathrm{gap}}).$$

Hence there is a unique restriction functor $\mathcal{F} : \mathbf{j}(\mathbf{C}) \to \mathsf{Par}((\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))})^{\mathrm{op}}, (\mathcal{M}_{\mathbf{C}})_{\mathrm{gap}})$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\mathsf{Par}(\mathcal{Y})J_{\mathbf{C}}"] \arrow[d, "\eta_{\mathbf{C}}"'] & \mathsf{Par}((\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))})^{\mathrm{op}}, \mathcal{M}_{\mathbf{C}}) \arrow[d, "\mathcal{E}"] \\ \mathbf{j}(\mathbf{C}) \arrow[r, dotted, "\mathcal{F}"] & \mathsf{Par}((\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))})^{\mathrm{op}}, (\mathcal{M}_{\mathbf{C}})_{\mathrm{gap}}) \end{tikzcd}$$

The restriction functor $\mathcal{F}$ is indeed full and faithful.

Proposition 3.3.11 *The functor $\mathcal{F}$ in the last commutative diagram is full and faithful.*

PROOF: Since $\mathsf{Par}(\mathcal{Y})J_{\mathbf{C}}$ is full and faithful and since

$$\mathbf{j}(\mathbf{C}) \text{ and } \mathsf{Par}((\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))})^{\mathrm{op}}, (\mathcal{M}_{\mathbf{C}})_{\mathrm{gap}})$$

are generated from $\mathbf{C}$ and $\mathsf{Par}((\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))})^{\mathrm{op}}, \mathcal{M}_{\mathbf{C}})$ by adding joins respectively, $\mathcal{F}$ is full.

For faithfulness of $\mathcal{F}$, note that the embedding

$$\mathcal{D} : \mathsf{Par}((\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))})^{\mathrm{op}}, (\mathcal{M}_{\mathbf{C}})_{\mathrm{gap}}) \to \mathsf{Par}((\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))})^{\mathrm{op}}, \{\text{monics}\})$$

is faithful so

$$\mathcal{F} : \mathbf{j}(\mathbf{C}) \to \mathsf{Par}((\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))})^{\mathrm{op}}, (\mathcal{M}_{\mathbf{C}})_{\mathrm{gap}})$$

is faithful if and only if

$$\mathcal{DF} : \mathbf{j}(\mathbf{C}) \to \mathsf{Par}((\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))})^{\mathrm{op}}, \{\text{monics}\})$$

is faithful.

To prove this, suppose that $S_1, S_2 \in \mathbf{j}(\mathbf{C})(A, B)$ with $S_1 \neq S_2$. Without loss of generality, we may assume that $S_1 \cap S_2 \neq S_2$ so that it suffices to show that if $S_1 \subsetneqq S_2$ then $\mathcal{D}F(S_1) \subsetneqq \mathcal{D}F(S_2)$.

Suppose on the contrary that $S_1 \subsetneqq S_2$ but $\mathcal{D}F(S_1) = \mathcal{D}F(S_2)$. Then there is an $s \in S_2 \setminus S_1$, where $s \in \mathbf{C}(A, B)$. But $(\mathcal{D}F)(s)$ is

$$\begin{tikzcd}
 & \mathsf{TS}(\mathbf{C})^{\mathrm{op}}(-, \overline{s}) \arrow[dl, "\mathcal{D}F(\overline{s})"'] \arrow[dr, "\mathcal{D}F(s)"] & \\
\mathsf{TS}(\mathbf{C})^{\mathrm{op}}(-, A) & & \mathsf{TS}(\mathbf{C})^{\mathrm{op}}(-, B)
\end{tikzcd}$$

where $\mathsf{TS}(\mathbf{C})^{\mathrm{op}} = \mathsf{Total}(\mathsf{Split}(\mathbf{C}))^{\mathrm{op}}$.

Now, compute

$$\begin{aligned}
(\mathcal{D}F)(\overline{S_1}) &= \cup_{s_1 \in S_1} (\mathcal{D}F)(\overline{s_1}) \\
&= \cup_{s_1 \in S_1} \mathsf{TS}(\mathbf{C})^{\mathrm{op}}(-, \overline{s_1}) \\
&\subseteq \mathsf{TS}(\mathbf{C})^{\mathrm{op}}(-, A).
\end{aligned}$$

Since $\overline{s} \in \mathcal{D}F(S_2) = \mathcal{D}F(S_1)$, we have

$$\mathcal{D}F(\overline{s}) \subseteqq \cup_{s_1 \in S_1} \mathcal{D}F(\overline{s_1})$$

so that there is a natural transformation $\iota : \mathcal{D}F(\overline{s}) \to \cup_{s_1 \in S_1} \mathcal{D}F(\overline{s_1})$ such that the following diagram

$$\begin{tikzcd}
\mathcal{D}F(\overline{s}) \arrow[rr, dotted, "\iota"] \arrow[dr, "\mathcal{D}F(\overline{s})"'] & & \cup_{s_1 \in S_1} \mathcal{D}F(\overline{s_1}) \arrow[dl, "\mathcal{D}F(\cup_{s_1 \in S_1} \overline{s_1})"] \\
 & \mathcal{D}F(A) &
\end{tikzcd}$$

is commutative. By Yoneda Lemma, ι corresponds to

$$\cup_{s_1 \in S_1} \mathcal{D}F(\overline{s_1})(\overline{s}) = \cup_{s_1 \in S_1} \mathsf{TS}(\mathbf{C})^{\mathrm{op}}(\overline{s}, \overline{s_1})$$

which is a contradiction since each $\mathsf{TS}(\mathbf{C})^{\mathrm{op}}(\overline{s}, \overline{s_1}) = \emptyset$ by noting that $\overline{s} \leq \overline{s_1}$ and $s \smile s_1$ implies $s \leq s_1$ and so $s \in S_1$. Hence $\mathcal{D}F(S_1) \subsetneqq \mathcal{D}F(S_2)$ and therefore $\mathcal{D}F$ is faithful, as desired. $\square$

Chapter 4

Meet Restriction Categories

In Chapter 3, we introduced joins to restriction categories so that we had the notion of join restriction categories. In this chapter, we first consider meet structures in restriction categories to introduce the notion of meet restriction categories. Then we show when a partial maps category $\mathsf{Par}(\mathbf{X}, \mathcal{M})$ is a meet restriction category and how to add meet structures to restriction categories freely.

4.1 Meet Restriction Categories

We consider binary meets on hom sets of a restriction category.

Definition 4.1.1 *A* meet restriction category *is a restriction category equipped with a meet operation on parallel maps*

$$\frac{f, g : X \to Y}{f \wedge g : X \to Y}$$

such that the following three meet axioms are satisfied:

[M.1] *for each map f, $f \wedge f = f$;*

[M.2] *for each pair of parallel maps $f, g : X \to Y$, $f \wedge g \leq f$ and $f \wedge g \leq g$;*

[M.3] *for each map $f : X \to Y$ and each pair of parallel maps $g, h : Y \to Z$, $(g \wedge h)f = gf \wedge hf$.*

Some properties of a meet in a restriction category are summarized in the following lemma.

Lemma 4.1.2 *In a meet restriction category, for any maps f, g, f', g', h that make senses, we have* (*i*) $f \leq f'$ *and* $g \leq g' \Rightarrow f \wedge g \leq f' \wedge g'$. *In particular,* $f \leq f' \Leftrightarrow f \wedge x \leq f' \wedge x$ *for all maps* x *that have the same domain and codomain with* f*;*

(*ii*) $h \leq f, h \leq g \Leftrightarrow h \leq f \wedge g$*;*

(*iii*) $f \wedge g = g \wedge f$*;*

(*iv*) $(f \wedge g) \wedge h = f \wedge (g \wedge h)$*;*

(*v*) $f \leq g \Leftrightarrow f \wedge g = g \wedge f = f$*;*

(*vi*) $f\overline{f \wedge g} = g\overline{f \wedge g} = f \wedge g$*;*

(*vii*) $\overline{h}(f \wedge g) = \overline{h}f \wedge g = f \wedge \overline{h}g = \overline{h}f \wedge \overline{h}g$*;*

(*viii*) $(f \wedge g)\overline{h} = f\overline{h} \wedge g = f \wedge g\overline{h} = f\overline{h} \wedge g\overline{h}$*;*

(*ix*) $\overline{f} \wedge \overline{g} = \overline{f}\overline{g}$*;*

(*x*) $f \wedge \overline{g}$ *is a restriction idempotent. Moreover, if* $f^{k+1} = f^k$ *for some positive integer* k *and if the restriction idempotent* $f \wedge \overline{g}$ *splits as* $f \wedge \overline{g} = mr$ *with* $rm = 1$*, then* $\overline{g}f^k$ *splits as* $\overline{g}f^k = m \cdot (rf^k)$ *with* $rf^km = 1$*.*

(*xi*) $1 \wedge f = f \Leftrightarrow f = \overline{f}$*. Moreover,* $f \wedge \overline{g} = f\overline{g}$ *if and only if* $f\overline{g}$ *is a restriction idempotent.*

PROOF:

(*i*) Since

$$\begin{aligned} f \leq f', g \leq g' &\Rightarrow f \wedge g \leq f \leq f', f \wedge g \leq g \leq g' \\ &\Rightarrow f'\overline{f \wedge g} = f \wedge g, g'\overline{f \wedge g} = f \wedge g \\ &\Rightarrow (f' \wedge g')\overline{f \wedge g} = f'\overline{f \wedge g} \wedge g'\overline{f \wedge g} = f \wedge g \\ &\Rightarrow f \wedge g \leq f' \wedge g'. \end{aligned}$$

(ii) Since

$$h \leq f, h \leq g \Rightarrow h = h \wedge h \leq f \wedge g$$

and

$$h \leq f \wedge g \Rightarrow h \leq f \wedge g \leq f, h \leq f \wedge g \leq g.$$

(iii) Note that

$$f \wedge g \leq g, f \wedge g \leq f \Rightarrow f \wedge g \leq g \wedge f$$

and symmetrically $g \wedge f \leq f \wedge g$. Hence $f \wedge g = g \wedge f$.

(iv) Since

$$(f \wedge g) \wedge h \leq f \wedge g \leq f, (f \wedge g) \wedge h \leq g \wedge h \Rightarrow (f \wedge g) \wedge h \leq f \wedge (g \wedge h)$$

and

$$f \wedge (g \wedge h) \leq f \wedge g, f \wedge (g \wedge h) \leq g \wedge h \leq h \Rightarrow f \wedge (g \wedge h) \leq (f \wedge g) \wedge h.$$

(v) Since $f \leq g$, we have $f = f \wedge f \leq f \wedge g$ and $f = f \wedge f \leq g \wedge f$. Obviously, $f \wedge g \leq f$ and $g \wedge f \leq f$. Then $f \wedge g = g \wedge f = f$. Conversely, clearly $f = f \wedge g \Rightarrow f = f \wedge g \leq g$.

(vi) Since

$$f \wedge g \leq f, f \wedge g \leq g \Rightarrow f\overline{f \wedge g} = f \wedge g = g\overline{f \wedge g}.$$

(vii) $\overline{h}f \wedge g \leq \overline{h}f$ gives

$$\begin{aligned}
\overline{h}f \wedge g &= (\overline{h}f)\overline{\overline{h}f \wedge g} \\
&= f\overline{hf}\;\overline{f\overline{hf} \wedge g} \quad (\textbf{[R.4]}) \\
&= f\overline{f\overline{hf} \wedge g}\;\overline{hf} \quad (\textbf{[R.2]}) \\
&= f\overline{(f\overline{hf} \wedge g)\overline{hf}} \quad (\textbf{[R.3]}) \\
&= f\overline{f \wedge g}\;\overline{hf} \quad (\textbf{[M.3] and [R.3]}) \\
&= \overline{h}f\;\overline{f \wedge g} \quad (\textbf{[R.4]}) \\
&= \overline{h}(f \wedge g) \quad (f \wedge g \leq f).
\end{aligned}$$

Note also that

$$f \wedge \overline{h}g = \overline{h}g \wedge f = \overline{h}(g \wedge f) = \overline{h}(f \wedge g)$$

and

$$\overline{h}(f \wedge g) = \overline{h}(\overline{h}f \wedge g) = \overline{h}f \wedge \overline{h}g.$$

Hence

$$\overline{h}(f \wedge g) = \overline{h}f \wedge g = f \wedge \overline{h}g = \overline{h}f \wedge \overline{h}g.$$

(viii) $f\overline{h} \wedge g \leq f\overline{h}$ implies

$$\begin{aligned}
f\overline{h} \wedge g &= (f\overline{h})\overline{f\overline{h} \wedge g} \\
&= f\overline{(f\overline{h} \wedge g)\overline{h}} \quad (\textbf{[R.2] and [R.3]}) \\
&= f\overline{f \wedge g}\;\overline{h} \quad (\textbf{[M.3] and [R.3]}) \\
&= (f \wedge g)\overline{h} \quad (f \wedge g \leq f).
\end{aligned}$$

It follows that

$$f \wedge g\overline{h} = g\overline{h} \wedge f = (g \wedge f)\overline{h} = (f \wedge g)\overline{h} = f\overline{h} \wedge g\overline{h}.$$

(ix) Since $\overline{f} \wedge \overline{g} = \overline{f}\;\overline{f} \wedge \overline{g}\;\overline{g} = \overline{f}\overline{g} \wedge \overline{f}\overline{g} = \overline{f}\overline{g}$.

(x) Since

$$\overline{f \wedge \overline{g}} = \overline{(f \wedge 1)\overline{g}} = \overline{f \wedge 1}\,\overline{g} = 1 \cdot \overline{f \wedge 1} \cdot \overline{g} = (f \wedge 1)\overline{g} = f \wedge \overline{g}.$$

$f \wedge \overline{g}$ is a restriction idempotent. Note that

$$\overline{g} f^k = \overline{g} f^{k+1} \wedge \overline{g} f^k = \overline{g}(f \wedge \overline{g}) f^k = (f \wedge \overline{g}) f^k = m \cdot r f^k$$

and

$$r f^k \cdot m = r f^k \cdot (f \wedge \overline{g}) m = \cdots = r(f \wedge \overline{g}) m = 1.$$

(xi) Since $1 \wedge f = f \Leftrightarrow \overline{f} \wedge f = 1 \wedge f\overline{f} = f \Leftrightarrow \overline{f} \geq f \Leftrightarrow f = \overline{f}$. Note that $f \wedge \overline{g} = f\overline{g} \wedge 1$. Then $f \wedge \overline{g} = f\overline{g} \Leftrightarrow f\overline{g} = \overline{f\overline{g}} \Leftrightarrow f\overline{g}$ is a restriction idempotent.

□

4.2 Completeness of Meet Restriction Categories

This section is intended to characterize when a partial maps category is a meet restriction category and the completeness of meet restriction categories in partial maps categories using equalizers.

4.2.1 Meet Restriction Implies Equalizers

Recall that by Proposition 1.6.11 if $\mathbf{C}$ is a restriction category, then so is $\mathsf{Split}(\mathbf{C})$ with a split restriction structure given by the restriction in the category $\mathbf{C}$ and there is a restriction preserving inclusion $\mathbf{C} \to \mathsf{Split}(\mathbf{C})$ sending $f : X \to Y$ to $f : 1_X \to 1_Y$.

If a restriction category $\mathbf{C}$ has a meet, then it is natural to ask what the special properties of the partial maps category $\mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{C})), \mathcal{M}_{\mathsf{Split}(\mathbf{C})})$ has. We shall see that in some sense, $\mathbf{C}$'s having a meet is equivalent to $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$'s having equalizers.

Since $\mathsf{Split}(\mathbf{C})$ is a split restriction category, let $\mathbf{D}$ be a restriction category with split restriction. Recall the following class of maps:

$$\mathcal{M}_{\mathbf{D}} = \{m : X \to Y \,\text{in}\, \mathsf{Total}(\mathbf{D}) \,|\, \exists\, r : Y \to X \text{ in } \mathbf{D},\ rm = 1_X \text{ and } \overline{r} = mr\}.$$

If $\mathbf{C}$ is a meet restriction category, then not only is $\mathsf{Split}(\mathbf{C})$ a split restriction category but also $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$ has equalizers.

Proposition 4.2.1 *If* $\mathbf{C}$ *is a meet restriction category, then* $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$ *has equalizers and regular monics are restriction monics in* $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$*.*

PROOF: For each pair of maps $f, g : e_1 \to e_2$ in $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$, we have

$$e_2 f = f, e_2 g = g, f e_1 = f, g e_1 = g, \overline{f} = \overline{g} = 1_{e_1} = e_1.$$

Then

$$\overline{f \wedge g} e_1 = \overline{f e_1 \wedge g e_1} = \overline{f \wedge g}$$

and so all triangles in

$$\begin{tikzcd}
\bullet \arrow[r, "\overline{f\wedge g}"] \arrow[d, "\overline{f\wedge g}"'] \arrow[dr, "\overline{f\wedge g}"] & \bullet \arrow[d, "e_1"] \\
\bullet \arrow[r, "\overline{f\wedge g}"'] & \bullet
\end{tikzcd}$$

are commutative. That is, $\overline{f \wedge g} : \overline{f \wedge g} \to e_1$ is a map in $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$. We claim that

$$\begin{tikzcd}
\overline{f \wedge g} \arrow[r, "\overline{f\wedge g}"] & e_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & e_2
\end{tikzcd}$$

is an equalizer diagram in $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$. For each map $x : e \to e_1$ such that $fx = gx$ in $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$, we have

$$e_1 x = x, xe = x, \overline{x} = 1_e = e.$$

Then

$$\overline{f \wedge g} x = x\overline{(f \wedge g)x} = x\overline{fx \wedge gx} = x\overline{fx} = \overline{f}x = e_1 x = x$$

and so $x : e \to \overline{f \wedge g}$ is a map in $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$. Since $\overline{f \wedge g}\,\overline{f \wedge g} = \overline{f \wedge g} = 1_{\overline{f \wedge g}}$, $\overline{f \wedge g}$ is a restriction monic splitting $\overline{f \wedge g}$ and there is a unique map $x : e \to \overline{f \wedge g}$ such that

$$\begin{tikzcd}
\overline{f \wedge g} \arrow[r, "\overline{f\wedge g}"] & e_1 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & e_2 \\
e \arrow[u, dotted, "!x"] \arrow[ur, "x"'] & &
\end{tikzcd}$$

is commutative. Therefore, $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$ has equalizers and each regular monic $\overline{f \wedge g}$ is a restriction monic. □

4.2.2 Equalizers Imply Meet Restriction

First, let's study when a partial maps category $\mathsf{Par}(\mathbf{X}, \mathcal{M})$ is a meet restriction category.

Proposition 4.2.2 *For an $\mathcal{M}$-category $(\mathbf{X}, \mathcal{M})$, if $\mathbf{X}$ has equalizers and*

$$\{\textit{regular monics in } \mathbf{X}\} \subseteq \mathcal{M},$$

then $\mathsf{Par}(\mathbf{X}, \mathcal{M})$ is a meet restriction category.

PROOF: Recall that $\mathsf{Par}(\mathbf{X}, \mathcal{M})$ is restriction category with the split restriction $\overline{(m, f)} = (m, m)$. It suffices to show that the split restriction category $\mathsf{Par}(\mathbf{X}, \mathcal{M})$ has a meet.

For each pair of maps $(m, f), (n, g) : X \to Y$ in $\mathsf{Par}(\mathbf{X}, \mathcal{M})$, we form the pullback square of $\mathcal{M}$-maps m and n:

$$\begin{tikzcd}
P \arrow[r, "m'"] \arrow[d, "n'"'] & Z_2 \arrow[d, "n"] \\
Z_1 \arrow[r, "m"] & X
\end{tikzcd}$$

and then we equalize fn' and gm' in $\mathbf{X}$:

$$\begin{tikzcd}
E \arrow[r, "k"] & P \arrow[r, shift left, "fn'"] \arrow[r, shift right, "gm'"'] & Y
\end{tikzcd}$$

Since the regular monic $k : E \to P$ is in $\mathcal{M}$, $mn'k' \in \mathcal{M}$ and $(mn'k, fn'k)$ is a partial map in $(\mathbf{X}, \mathcal{M})$. We claim that

$$(m, f) \wedge (n, g) = (mn'k, fn'k) : X \to Y$$

gives a meet on $\mathsf{Par}(\mathbf{X}, \mathcal{M})$. The three meet axioms are showed as follows.

[M.1] For each partial map $(m, f) : X \to Y$, since

$$\begin{tikzcd}
Z \arrow[r, "1_Z"] \arrow[d, "1_Z"'] & Z \arrow[d, "m"] \\
Z \arrow[r, "m"] & X
\end{tikzcd}$$

is a pullback diagram and

$$\begin{tikzcd}
Z \arrow[r, "1_Z"] & Z \arrow[r, shift left, "f"] \arrow[r, shift right, "f"'] & Y
\end{tikzcd}$$

is an equalizer diagram, we have

$$(m, f) \wedge (m, f) = (m \cdot 1 \cdot 1, f \cdot 1 \cdot 1) = (m, f).$$

[M.2] For each pair of parallel partial maps $(m, f), (n, g) : X \to Y$, looking at the following

$$\begin{tikzcd}
 & & & & E \arrow[dl, "1_E"'] \arrow[dr, "k"] & & & & \\
 & & & E \arrow[dl, "k"'] \arrow[dr, "k"] & & P \arrow[dl, "1_P"'] \arrow[dr, "n'"] & & & \\
 & & P \arrow[dl, "n'"'] & & P \arrow[dr, "n'"] & & Z_1 \arrow[dl, "1_{Z_1}"'] \arrow[dr, "1_{Z_1}"] & & \\
 & Z_1 \arrow[dl, "m"'] & & & & Z_1 \arrow[dr, "m"'] & & Z_1 \arrow[dl, "m"'] \arrow[dr, "f"] & \\
X & & & & & & X & & Y
\end{tikzcd}$$

where each square is a pullback diagram, we have

$$\begin{aligned}(m,f)\overline{(m,f)\wedge(n,g)} &= (m,f)\overline{(mn'k,gm'k)}\\ &= (m,f)(mn'k,mn'k)\\ &= (mn'k,fn'k)\\ &= (m,f)\wedge(n,g).\end{aligned}$$

Then $(m,f)\wedge(n,g)=(mn'k,fn'k)\leq(m,f)$. Similarly, we have $(m,f)\wedge(n,g)=(mn'k,gm'k)\leq(n,g)$.

[M.3] For each partial map $(m,f):X\rightarrow Y$ and each pair of parallel partial maps $(n_1,g_1),(n_2,g_2):Y\rightarrow Z$, we want to show that

$$((n_1,g_1)\wedge(n_2,g_2))(m.f)=((n_1,g_1)(m,f))\wedge((n_2,g_2)(m.f)).$$

To do this, first, note that $(n_1,g_1)\wedge(n_2,g_2)=(n_1te,g_1te)$ is given by the pullback square

$$\begin{tikzcd} G \arrow[r, "s"] \arrow[d, "t"'] & G_2 \arrow[d, "n_2"] \\ G_1 \arrow[r, "n_1"] & Y \end{tikzcd}$$

and the equalizer diagram

$$\begin{tikzcd} E \arrow[r, "e"] & G \arrow[r, shift left, "g_1t"] \arrow[r, shift right, "g_2s"'] & Z. \end{tikzcd}$$

So

$$\begin{aligned}((n_1,g_1)\wedge(n_2,g_2))(m,f) &= (n_1te,g_1te)(m,f)\\ &= (n_2se,g_2se)(m,f)\\ &= (mn_1't'e',g_1tef_3)\end{aligned}$$

is given by

$$\begin{tikzcd}
 & & & & C \arrow[dl, "e'"'] \arrow[dr, "f_1''"] & & & & \\
 & & & X_1' \arrow[dl, "t'"'] \arrow[dr, "f_1'"] & & E \arrow[dl, "e"'] \arrow[dr, "e"] & & & \\
 & & X_1 \arrow[dl, "n_1'"'] \arrow[dr, "f_1"] & & G \arrow[dl, "t"'] & & G \arrow[dr, "t"] & & \\
 & X' \arrow[dl, "m"'] \arrow[dr, "f"] & & G_1 \arrow[dl, "n_1"'] & & & & G_1 \arrow[dr, "g_1"] & \\
X & & Y & & & & & & Z
\end{tikzcd}$$

where each square is a pullback diagram.

On the other hand, $(n_1, g_1)(m, f) = (mn_1', g_1f_1)$ is given by the pullback square:

$$\begin{tikzcd}
 & & X_1 \arrow[dl, "n_1'"'] \arrow[dr, "f_1"] & & \\
 & X' \arrow[dl, "m"'] \arrow[dr, "f"'] & & G_1 \arrow[dl, "n_1"] \arrow[dr, "g_1"] & \\
X & & Y & & Z
\end{tikzcd}$$

and $(n_2, g_2)(m, f) = (mn_2', g_2f_2)$ is given by the pullback square:

$$\begin{tikzcd}
 & & X_2 \arrow[dl, "n_2'"'] \arrow[dr, "f_2"] & & \\
 & X' \arrow[dl, "m"'] \arrow[dr, "f"'] & & G_2 \arrow[dl, "n_2"] \arrow[dr, "g_2"] & \\
X & & Y & & Z
\end{tikzcd}$$

So

$$(n_1, g_1)(m, f) \wedge (n_2, g_2)(m, f) = (mn_1', g_1f_1) \wedge (mn_2', g_2f_2) = (mn_1'n_2''d, g_1f_1n_2''d)$$

is given by the pullback squares

$$\begin{tikzcd}
X'' \arrow[r, "n''_1"] \arrow[d, swap, "n''_2"] & X'_2 \arrow[r, "1_{X'_2}"] \arrow[d, "n'_2"] & X'_2 \arrow[d, "n'_2"] \arrow[dr, bend left, "f_2"] & \\
X'_1 \arrow[r, "n'_1"] \arrow[d, swap, "1_{X'_1}"] & X' \arrow[r, "1_{X'}"] \arrow[d, "1_{X'}"] & X' \arrow[d, "m"] & G_2 \arrow[dd, "g_2"] \\
X'_1 \arrow[r, "n'_1"] \arrow[dr, bend right, swap, "f_1"] & X' \arrow[r, "m"] & X & \\
 & G_1 \arrow[rr, swap, "g_1"] & & Z
\end{tikzcd}$$

and an equalizer diagram

$$\begin{tikzcd}
D \arrow[r, "d"] & X'' \arrow[r, shift left, "g_1 f_1 n''_2"] \arrow[r, shift right, swap, "g_2 f_2 n''_1"] & Z.
\end{tikzcd}$$

Note that $n'_1 t' = n'_2 s'$ (and $X_1 = X'_2$, up to isomorphism) since they are pullbacks of $n_1 t = n_2 s$ along f. Then $m n'_1 t' = m n'_2 s'$ and so there is a unique map $w : X'_1 \to X''$ such that

$$n''_1 w = s' \text{ and } n''_2 w = t'.$$

Hence

$$g_1 f_1 n''_2 w e' = g_1 f_1 t' e' = g_1 t e f''_1 = g_2 s e f''_2 = g_2 f_2 n''_1 w e''$$

and therefore there exists a unique map $u : C \to D$ such that

$$du = we' = we''.$$

Since m is a monic, we have

$$\begin{aligned} m n'_1 n''_2 = m n'_2 n''_1 \quad &\Rightarrow \quad n'_1 n''_2 = n'_2 n''_1 \\ &\Rightarrow \quad n_1 f_1 n''_2 = f n'_1 n''_2 = f n'_2 n''_1 = n_2 f_2 n''_1. \end{aligned}$$

So there is a unique map $p : X'' \to G$ such that

$$sp = f_2 n''_1 \text{ and } tp = f_1 n''_2.$$

Since

$$g_1tpd = g_1f_1n_2''d = g_2f_2n_1''d = g_2spd,$$

there exists a unique map $q : D \to E$ such that

$$eq = pd.$$

It follows that

$$fn_1'n_2''d = n_1f_1n_2''d = n_1tpd = n_1teq.$$

Then there is a unique map $v : D \to C$ such that

$$f_1''v = q \text{ and } t'e'v = n_2''d.$$

Since $n_2''d$ is a monic,

$$n_2''duv = n_2''we'v = t'e'v = n_2''d \Rightarrow uv = 1_D$$

and since t' is a monic,

$$t'e'vu = n_2''du = n_2''we' = t'e' \Rightarrow vu = 1_C.$$

Hence

$$\begin{aligned}
((n_1, g_1) \wedge (n_2, g_2))(m, f) &= (mn_1't'e', g_1tef_3) \\
&= (mn_1'n_2''d, g_1f_1n_2''d) \\
&\qquad (\text{as } uv = 1_D, vu = 1_C) \\
&= (n_1, g_1)(m, f) \wedge (n_2, g_2)(m, f).
\end{aligned}$$

$$\begin{tikzcd}
D \arrow[ddrr, "d"] \arrow[dddd, dotted, "v"'] \arrow[ddddd, dotted, bend right, "q"'] & & & & C \arrow[llll, dotted, "u"'] \arrow[r, "f''_2"] \arrow[d, "e''"'] & E \arrow[d, "e"] \\
& & & & X'_2 \arrow[r, "f'_2"] \arrow[d, "s'"'] \arrow[dll, dotted, "w"] & G \arrow[d, "s"] \\
& & X'' \arrow[rr, "n''_1"] \arrow[dd, "n''_2"] \arrow[dddl, dotted, bend right, "p"'] & & X_2 \arrow[r, "f_2"] \arrow[d, "n'_2"'] & G_2 \arrow[d, "n_2"'] \arrow[ddd, bend left, "g_2"] \\
& & & & X' \arrow[r, "f"'] \arrow[d, "m"] & Y \\
C \arrow[r, "e'"] \arrow[d, "f''_1"'] & X'_1 \arrow[r, "t'"] \arrow[d, "f'_1"] \arrow[uur, dotted, "w"] & X_1 \arrow[r, "n'_1"] \arrow[d, "f_1"'] & X' \arrow[r, "m"] \arrow[d, "f"] & X & \\
E \arrow[r, "e"] & G \arrow[r, "t"] & G_1 \arrow[r, "n_1"] \arrow[rrr, bend right, "g_1"'] & Y & & Z
\end{tikzcd}$$

□

Now we are ready to show:

Theorem 4.2.3 *A category is a meet restriction category if and only if it is a full subcategory of* $\mathsf{Par}(\mathbf{X}, \mathcal{M})$ *for some* $\mathcal{M}$*-category* $(\mathbf{X}, \mathcal{M})$ *in which* $\mathbf{X}$ *has equalizers and every regular monics of* $\mathbf{X}$ *is in* $\mathcal{M}$*.*

PROOF: "if" part: By Proposition 4.2.2, $\mathsf{Par}(\mathbf{X}, \mathcal{M})$ is a meet restriction category and so is its full subcategory.

"only if" part: Conversely, a meet restriction category $\mathbf{C}$ is a full sub-restriction category of $\mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{C})), \mathcal{M}_{\mathsf{Split}(\mathbf{C})})$. By Proposition 4.2.1, $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$ has equalizers and every monic of $\mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{C})))$ is in $\mathcal{M}_{\mathsf{Split}(\mathbf{C})}$, as desired. □

As an example, $\mathsf{Par}(\mathbf{Set}_{\text{ffib}}, \mathcal{M})$, given in Example 1.6.16, is not a meet restriction category as not every regular monic of $\mathbf{Set}_{\text{ffib}}$ is in $\mathcal{M}$ clearly, where $\mathcal{M} = \{\text{injections } i : A \hookrightarrow B \mid |B \setminus i(A)| < +\infty\}$.

4.3 Meet Completion for Restriction Categories

This section is intended to construct the free meet restriction category over a given restriction category. Let $\mathbf{X}$ be a restriction category. Define the meet restriction category $\mathrm{Mt}(\mathbf{X})$ (the meet completion of $\mathbf{X}$) as follows:

objects: the same as the objects of $\mathbf{X}$;

maps: a map from X to Y in $\mathrm{Mt}(\mathbf{X})$ is a pair (f, Q) of a $\mathbf{X}$-map f and a finite set $Q = \{(f_i, f_i')|i = 1, \cdots, n\}$ of parallel $\mathbf{X}$-map pairs: $f : X \to Y$, $f_1, f_1' : X \to X_1, \cdots, f_n, f_n' : X \to X_n$

$$\begin{tikzcd}
Q= & X \arrow[rrr, bend left, "f"] \arrow[dl, shift right, "f_1"'] \arrow[dl, shift left, "f_1'"] \arrow[dr, shift left, "f_n'"] \arrow[dr, shift right, "f_n"'] & & & Y \\
X_1 & \cdots & X_n & &
\end{tikzcd}$$

factored out by the equivalence relation $(f, Q) \sim (g, R)$, where $(f, Q) \sim (g, R)$ if (g, R) can be obtained from (f, Q) by finite steps $\simeq$:

[MC.1] if $(f, g) \in Q$, then $(f, Q) \simeq (g, Q)$ and $(g, Q) \simeq (f, Q)$;

[MC.2] $(f, Q \cup \{(1, 1)\}) \simeq (f, Q)$ and $(f, Q) \simeq (f, Q \cup \{(1, 1)\})$;

[MC.3] for each pair of parallel maps h and k which have the same domain as that of f,

$$(f, Q \cup \{(h, k)\}) \simeq (f, Q \cup \{(k, h)\});$$

[MC.4] if $(h, k), (k, w) \in Q$, then $(f, Q) \simeq (f, Q \cup \{(h, w)\})$ and $(f, Q \cup \{(h, w)\}) \simeq (f, Q)$;

[MC.5] if $\overline{v}h = h$ and if $(h, k) \in Q$, then

$$(f, Q) \simeq (f, Q \cup \{(vh, vk)\})$$

and

$$(f, Q \cup \{(vh, vk)\}) \simeq (f, Q);$$

[MC.6] for any restriction idempotent $e = \overline{e} : X \to X$, if $(h, k) \in Q$, then

$$(fe, Q) \simeq (f, Q \cup \{(he, k)\})$$

and

$$(f, Q \cup \{(he, k)\}) \simeq (fe, Q).$$

We shall prove that $\sim$ is an equivalence relation in Lemma 4.3.1 below. The equivalence class of (f, Q) is also denoted by (f, Q). So $(f, Q) = (f_1, Q_1)$ usually means that (f, Q) and (f_1, Q_1) have the same equivalence class, that is, $(f, Q) \sim (f_1, Q_1)$.

composition: for maps $(f, Q) : X \to Y$ and $(g, R) : Y \to Z$ in $\mathrm{Mt}(\mathbf{X})$,

$$(g, R)(f, Q) = (gf, Q \cup Rf) :$$

$$\begin{tikzcd}
 & X \arrow[rrr, bend left, "f"] \arrow[dl, shift right, "f_1"', "Q="' near end] \arrow[dl, shift left, "f_1'"] \arrow[dr, shift left, "f_n'"] \arrow[dr, shift right, "f_n"'] & & & Y \arrow[rr, bend left, "g"] \arrow[dl, shift right, "g_1"', "R="' near end] \arrow[dl, shift left, "g_1'"] \arrow[dr, shift left, "g_n'"] \arrow[dr, shift right, "g_n"'] & & Z \\
X_1 & \cdots & X_n & Y_1 & \cdots & Y_m &
\end{tikzcd}$$

identities: for each object X, $1_X = (1_X, \emptyset)$;

restriction: $\overline{(f, Q)} = (\overline{f}, Q)$;

meet: for a pair of parallel maps; $(f, Q), (g, R) : X \to Y$ in $\mathrm{Mt}(\mathbf{X})$,

$$(f, Q) \wedge (g, R) = (f, Q \cup R \cup \{(f, g)\}).$$

We first observe:

Lemma 4.3.1 $\sim$ *in the definition of* $\mathrm{Mt}(\mathbf{X})$ *above is an equivalence relation.*

Proof: We show the reflexivity, the symmetry, and the transitivity of $\sim$ as follows.

Reflexivity: For each map (f, Q), (f, Q) can be obtained from (f, Q) by 0 step $\simeq$ and so $(f, Q) \sim (f, Q)$.

Symmetry: It suffices to show that if $(f, Q) \simeq (g, R)$ then $(g, R) \simeq (f, Q)$, which is clear by the definition of $\simeq$.

Transitivity: Clearly, if $(f, Q) \sim (g, R)$ and $(g, R) \sim (h, S)$, then (g, R) and (h, S) can be obtained from (f, Q) and (g, R) by finite steps $\simeq$ respectively and so (h, S) can be obtained from (f, Q) by finite steps $\simeq$. Hence $(f, Q) \sim (h, S)$.

$\square$

Some properties of $\mathrm{Mt}(\mathbf{X})$ are summarized in the following lemma.

Lemma 4.3.2 *In the meet completion* $\mathrm{Mt}(\mathbf{X})$ *of a restriction category* $\mathbf{X}$*,*

(i) *if* $(h, k) \in Q$*, then* $(f, Q) = (f, Q \cup \{(\overline{h}, \overline{k})\})$*;*

(ii) *if* $\overline{h} \geq \overline{f}$*, then* $(f, Q \cup \{(h, h)\}) = (f, Q)$*;*

(iii) *let* $(h, k) \in Q$*. If* $h \leq u$ *and* $k \leq v$*, then* $(f, Q) = (f, Q \cup \{(u, v)\})$*;*

(iv) *for each restriction idempotent* e *that makes senses,* $(f, Q \cup Re) = (fe, Q \cup R)$*;*

(v) *if* $(f, f') \in Q$*, then* $(gf, Q) = (gf, Q \cup \{(gf, gf')\})$*;*

(vi) *if* $(f, f') \in Q$*, then* $(gf, Q \cup Rf) = (gf, Q \cup Rf \cup Rf')$*.*

Proof:

(i)

$$\begin{aligned}(f,Q) &= (f,Q\cup\{(h\overline{h},k\overline{k})\})\\ &= (f\overline{h}\,\overline{k},Q) \quad ([\mathbf{MC.6}])\\ &= (f\overline{h}\,\overline{k},Q\cup\{(1,1)\}) \quad ([\mathbf{MC.2}])\\ &= (f,Q\cup\{(\overline{h},\overline{k})\}) \quad ([\mathbf{MC.6}]).\end{aligned}$$

(ii) If $\overline{h}\geq\overline{f}$, then $\overline{h}\,\overline{f}=\overline{f}$ and so

$$\begin{aligned}(f,Q) &= (f\overline{f}\,\overline{h},Q\cup\{(1,1)\}) \quad ([\mathbf{MC.2}])\\ &= (f,Q\cup\{(1,1),(\overline{h},1)\}) \quad ([\mathbf{MC.6}])\\ &= (f,Q\cup\{(1,1),(h,h)\}) \quad ([\mathbf{MC.5}])\\ &= (f,Q\cup\{(h,h)\}) \quad ([\mathbf{MC.2}]).\end{aligned}$$

(iii) Since $h\leq u$ and $k\leq v$, $h=u\overline{h}$ and $k=v\overline{k}$. Then

$$\begin{aligned}(f,Q) &= (f,Q\cup\{(h,k)\})\\ &= (f,Q\cup\{(u\overline{h},v\overline{k})\})\\ &= (f\overline{h}\,\overline{k},Q\cup\{(u,v)\}) \quad ([\mathbf{MC.6}])\\ &= (f,Q\cup\{(u,v)\}) \quad ([\mathbf{MC.6}]).\end{aligned}$$

(iv) Assume $R = \{(s_1, t_1), \cdots, (s_n, t_n)\}$. Then

$$\begin{aligned}
(f, Q \cup Re) &= (f, Q \cup \{(s_1 e, t_1 e), \cdots, (s_n e, t_n e)) \\
&= (fe^n, Q \cup \{(s_1, t_1 e), \cdots, (s_n, t_n e)) \quad (\textbf{[MC.6]}) \\
&= (fe^n, Q \cup \{(t_1 e, s_1), \cdots, (t_n e, s_n)) \quad (\textbf{[MC.3]}) \\
&= (fe^{2n}, Q \cup \{(t_1, s_1), \cdots, (t_n, s_n)) \quad (\textbf{[MC.6]}) \\
&= (fe, Q \cup \{(t_1, s_1), \cdots, (t_n, s_n)) \quad (e^2 = e) \\
&= (fe, Q \cup \{(s_1, t_1), \cdots, (s_n, t_n)) \quad (\textbf{[MC.3]})) \\
&= (fe, Q \cup R).
\end{aligned}$$

(v) $(gf, Q) = (gf, Q \cup \{(f\overline{gf}, f')\}) = (gf, Q \cup \{(gf\overline{gf}, gf')\}) = (gf, Q \cup \{(gf, gf')\})$.

(vi) Assume that $R = \{(s_1, t_1), \cdots, (s_n, t_n)\}$. Then

$$\begin{aligned}
(gf, Q \cup Rf) &= (gf, Q \cup \{(s_1 f, t_1 f), \cdots, (s_n f, t_n f)\}) \\
&= (gf, Q \cup \{(s_1 f, t_1 f), \cdots, (s_n f, t_n f), (1, 1)\}) \\
&\qquad (\overline{1} \geq \overline{gf} \text{ and } \textbf{[MC.2]}) \\
&= (gf, Q \cup \{(s_1 f, t_1 f), \cdots, (s_n f, t_n f), (\overline{s_1 f}, 1), (\overline{t_1 f}, 1)\}) \\
&\qquad (\textbf{[MC.6]}) \\
&= (gf, Q \cup \{(s_1 f, t_1 f), \cdots, (s_n f, t_n f), (\overline{f}\ \overline{s_1 f}, 1), (\overline{f}\ \overline{t_1 f}, 1)\}) \\
&\qquad (\textbf{[MC.6]}) \\
&= (gf, Q \cup \{(s_1 f, t_1 f), \cdots, (s_n f, t_n f), (f\overline{s_1 f}, f), (f\overline{t_1 f}, f)\}) \\
&\qquad (\textbf{[MC.5]}) \\
&= (gf, Q \cup \{(s_1 f, t_1 f), \cdots, (s_n f, t_n f), (f\overline{s_1 f}, f'), (f\overline{t_1 f}, f')\}) \\
&\qquad ((f, f') \in Q \text{ and } \textbf{[MC.4]})
\end{aligned}$$

$$
\begin{aligned}
&= (gf, Q \cup \{(s_1f, t_1f), \cdots, (s_nf, t_nf), (s_1f, s_1f'), (t_1f, t_1f')\}) \\
&\qquad ([\mathbf{MC.5}]) \\
&= (gf, Q \cup \{(s_1f, t_1f), \cdots, (s_nf, t_nf), (s_1f', s_1f), (t_1f, t_1f')\}) \\
&\qquad ([\mathbf{MC.3}]) \\
&= (gf, Q \cup \{(s_1f, t_1f), \cdots, (s_nf, t_nf), (s_1f', s_1f), (t_1f, t_1f'), \\
&\quad (s_1f', t_1f')\}) \quad ([\mathbf{MC.4}]) \\
&= (gf, Q \cup \{(s_1f, t_1f), \cdots, (s_nf, t_nf), \\
&\quad (s_1f', t_1f')\}) \quad \text{(reserving the above process)} \\
&\cdots \text{ induction on } n \\
&= (gf, Q \cup \{(s_1f, t_1f), \cdots, (s_nf, t_nf), (s_1f', t_1f'), \\
&\quad \cdots, (s_nf', t_nf')\}) \\
&= (gf, Q \cup Rf \cup Rf'\}).
\end{aligned}
$$

□

Then we need to prove:

Lemma 4.3.3 *The composition, restriction, and meet defined above are well-defined.*

Proof:

1. **The Composition** $(g, R)(f, Q) = (gf, Q \cup Rf)$ **is well-defined.**

   We need to prove that if $(g, R) \sim (g_1, R_1)$ and $(f, Q) \sim (f_1, Q_1)$ then

   $$(g, R)(f, Q) \sim (g_1, R_1)(f_1, Q_1).$$

   It suffices to show that if $(g, R) \simeq (g_1, R_1)$ and $(f, Q) \simeq (f_1, Q_1)$ then

   $$(g, R)(f, Q) \sim (g, R)(f_1, Q_1) \;\; ((g, R) \text{ fixed})$$

and

$$(g,R)(f,Q) \sim (g_1,R_1)(f,Q) \ \ ((f,Q) \text{ fixed}).$$

Case 1. Fix (g,R).

[**MC.1**] If $(f,f') \in Q$, then $(f,Q) \simeq (f',Q)$. Note that

$$\overline{g} \cdot f\overline{gf} = \overline{g} \cdot \overline{g}f = \overline{g}f = f\overline{gf}.$$

Then

$$\begin{aligned}
(g,R)(f,Q) &= (gf, Q \cup Rf) \\
&= (gf, Q \cup Rf \cup \{(gf,gf')\}) \quad (\text{Lemma } 4.3.2(v)) \\
&= (gf, Q \cup Rf \cup Rf' \cup \{(gf,gf')\}) \quad (\text{Lemma } 4.3.2(vi)) \\
&= (gf', Q \cup Rf \cup Rf' \cup \{(gf,gf')\}) \quad ([\mathbf{MC.1}]) \\
&= (gf', Q \cup Rf' \cup \{(gf,gf')\}) \quad (\text{Lemma } 4.3.2(vi)) \\
&= (gf', Q \cup Rf') \quad (\text{Lemma } 4.3.2(v)) \\
&= (g,R)(f',Q).
\end{aligned}$$

[**MC.2**] Since

$$\begin{aligned}
(g,R)(f,Q) &= (gf, Q \cup Rf) \\
&= (gf, Q \cup Rf \cup \{(1,1)\}) \quad ([\mathbf{MC.2}]) \\
&= (g,R)(f, Q \cup \{(1,1)\}).
\end{aligned}$$

[MC.3] Since

$$\begin{aligned}(g,R)(f,Q\cup\{(h,k)\}) &= (gf,Q\cup Rf\cup\{(h,k)\})\\ &= (gf,Q\cup Rf\cup\{(k,h)\}) \quad (\textbf{[MC.3]})\\ &= (g,R)(f,Q\cup\{(k,h)\}).\end{aligned}$$

[MC.4] If $(h,k),(k,w)\in Q$, then $(f,Q)\sim(f,Q\cup\{(h,w)\})$ and

$$\begin{aligned}(g,R)(f,Q) &= (gf,Q\cup Rf)\\ &= (gf,Q\cup Rf\cup\{(h,w)\}) \quad (\textbf{[MC.4]})\\ &= (g,R)(f,Q\cup\{(h,w)\}).\end{aligned}$$

[MC.5] If $(h,k)\in Q$ and if $\overline{v}h=h$, then $(f,Q)\sim(f,Q\cup\{(vh,vk)\})$ and

$$\begin{aligned}(g,R)(f,Q) &= (gf,Q\cup Rf)\\ &= (gf,Q\cup Rf\cup\{(vh,vk)\}) \quad (\textbf{[MC.5]})\\ &= (g,R)(f,Q\cup\{(vh,vk)\}).\end{aligned}$$

[MC.6] If e is a restriction idempotent and $(h,k)\in Q$, then $(fe,Q)\sim(f,Q\cup\{(he,k)\})$ and

$$\begin{aligned}&(g,R)(fe,Q)\\ =\ &(gfe,Q\cup Rfe)\\ =\ &(gf,Q\cup Rfe\cup\{(he,k)) \quad ((h,k)\in Q \text{ and } \textbf{[MC.6]})\\ =\ &(gf,Q\cup Rf\cup\{(he,k)) \quad (\text{Lemma 4.3.2}(iv))\\ =\ &(g,R)(f,Q\cup\{(he,k)\}).\end{aligned}$$

Case 2. Fix (f, Q).

[MC.1] If $(g, g') \in R$, then $(g, R) \sim (g', R)$ and

$$\begin{aligned}(g, R)(f, Q) &= (gf, Q \cup Rf) \\ &= (g'f, Q \cup Rf) \quad ((gf, g'f) \in Rf) \\ &= (g', R)(f, Q).\end{aligned}$$

[MC.2] Since

$$\begin{aligned}(g, R)(f, Q) &= (gf, Q \cup Rf) \\ &= (gf, Q \cup Rf \cup \{(f, f)\}) \quad (\overline{f} \geq \overline{gf} \text{ and Lemma 4.3.2}(ii)) \\ &= (g, R \cup \{(1, 1)\})(f, Q).\end{aligned}$$

[MC.3] Since

$$\begin{aligned}(g, R \cup \{(t, s)\})(f, Q) &= (gf, Q \cup Rf \cup \{(tf, sf)\}) \\ &= (gf, Q \cup Rf \cup \{(sf, tf)\}) \quad ([\mathbf{MC.3}]) \\ &= (g, R \cup \{(s, t)\})(f, Q).\end{aligned}$$

[MC.4] If $(u, v), (v, w) \in R$, then $(g, R) \sim (g, R \cup \{(u, w)\})$ and

$$\begin{aligned}(g, R)(f, Q) &= (gf, Q \cup Rf) \\ &= (gf, Q \cup Rf \cup \{(uf, wf)\}) \\ &\qquad ((uf, vf), (vf, wf) \in Rf \text{ and } [\mathbf{MC.4}]) \\ &= (g, R \cup \{(u, w)\})(f, Q).\end{aligned}$$

[MC.5] If $(s,t) \in R$ and $\overline{v}s = s$, then $(g,R) \sim (g, R \cup \{(vs, vt)\})$ and

$$\begin{aligned} (g,R)(f,Q) &= (gf, Q \cup Rf) \\ &= (gf, Q \cup Rf \cup \{(vsf, vtf)\}) \quad (\overline{v}sf = sf \text{ and } [\mathbf{MC.5}])) \\ &= (g, R \cup \{(vs, vt)\})(f,Q). \end{aligned}$$

[MC.6] If $\overline{e} = e$ and $(s,t) \in R$, then $(ge, R) \sim (g, R \cup \{(se, t)\})$ and

$$\begin{aligned} (ge,R)(f,Q) &= (gef, Q \cup Rf) \\ &= (gf\overline{ef}, Q \cup Rf) \\ &= (gf, Q \cup Rf \cup \{(sf\overline{ef}, tf)\}) \\ &\qquad ((sf,tf) \in Rf \text{ and } [\mathbf{MC.6}])) \\ &= (gf, Q \cup Rf \cup \{(sef, tf)\}) \\ &= (g, R \cup \{(se,t)\})(f,Q). \end{aligned}$$

2. **Restriction $\overline{(f,Q)} = (\overline{f}, Q)$ is well-defined.**

[MC.1] If $(f,g) \in Q$, then $(f,Q) = (g,Q)$. We want to show that $\overline{(f,Q)} = \overline{(g,Q)}$. Note that

$$\begin{aligned} \overline{(f,Q)} &= (\overline{f}, Q) \\ &= (1, Q \cup \{(f\overline{f}, g)\}) \quad ((f,g) \in Q \text{ and } [\mathbf{MC.6}]) \\ &= (1, Q \cup \{(g\overline{g}, f)\}) \quad ([\mathbf{MC.3}]) \\ &= (\overline{g}, Q \cup \{(g,f)\}) \quad ([\mathbf{MC.6}]) \\ &= (\overline{g}, Q) \quad ((f,g) \in Q \ \text{ and } [\mathbf{MC.3}]) \\ &= \overline{(g,Q)}, \end{aligned}$$

as desired.

[MC.2] If $\overline{h} \geq \overline{f}$, then $(f, Q \cup \{(h,h)\}) \sim (f,Q)$. Since $\overline{h} \geq \overline{\overline{f}}$, we have

$$\overline{(f,Q)} = (\overline{f},Q) = (\overline{f}, Q \cup \{(h,h)\}) = \overline{(f, Q \cup \{(h,h)\})}.$$

[MC.3] If $(h,k) \in Q$, then $(f,Q) \sim (f, Q \cup \{(k,h)\})$ and

$$\overline{(f,Q)} = (\overline{f},Q) = (\overline{f}, Q \cup \{(k,h)\}) = \overline{(f, Q \cup \{(k,h)\})}.$$

[MC.4] If $(h,k),(k,w) \in Q$, then $(f,Q) \sim (f, Q \cup \{(h,w)\})$ and

$$\overline{(f,Q)} = (\overline{f},Q) = (\overline{f}, Q \cup \{(h,w)\}) = \overline{(f, Q \cup \{(h,w)\})}.$$

[MC.5] If $(h,k) \in Q$ and $\overline{v}h = h$, then $(f,Q) \sim (f, Q \cup \{(vh,vk)\})$ and

$$\overline{(f,Q)} = (\overline{f},Q) = (\overline{f}, Q \cup \{(vh,vk)\}) = \overline{(f, Q \cup \{(vh,vk)\})}.$$

[MC.6] If e is a restriction idempotent and $(h,k) \in Q$, then

$$(fe,Q) \sim (f, Q \cup \{(he,k)\})$$

and

$$\overline{(fe,Q)} = (\overline{fe},Q) = (\overline{f}e,Q) = (\overline{f}, Q \cup \{(he,k)\}) = \overline{(f, Q \cup \{(he,k)\})}.$$

3. **The meet** $(g,R) \wedge (f,Q) = (f, Q \cup R \cup \{(f,g)\})$ **is well-defined.**

Case 1. Fix (g,R).

[MC.1] If $(f,f') \in Q$, then $(f,Q) \sim (f',Q)$ and

$$\begin{aligned}
(f,Q) \wedge (g,R) &= (f, Q \cup R \cup \{(f,g)\}) \\
&= (f, Q \cup R \cup \{(f,g),(f',f)\}) \quad ((f,f') \in Q \text{ and } [\mathbf{MC.3}]) \\
&= (f, Q \cup R \cup \{(f,g),(f',f),(f',g)\}) \quad ([\mathbf{MC.4}]) \\
&= (f', Q \cup R \cup \{(f,g),(f',f),(f',g)\}) \quad ([\mathbf{MC.1}]) \\
&= (f', Q \cup R \cup \{(f,g),(f',g)\}) \quad ((f,f') \in Q \text{ and } [\mathbf{MC.3}]) \\
&= (f', Q \cup R \cup \{(f',g)\}) \\
&\qquad ((f,f'),(f',g) \in Q \cup R \cup \{(f',g)\} \text{ and } [\mathbf{MC.4}]) \\
&= (f',Q) \wedge (g,R).
\end{aligned}$$

[MC.2] Since

$$\begin{aligned}
(f,Q) \wedge (g,R) &= (f, Q \cup R \cup \{(f,g)\}) \\
&= (f, Q \cup R \cup \{(f,g),(1,1)\}) \quad ([\mathbf{MC.2}]) \\
&= (f, Q \cup \{(1,1)\}) \wedge (g,R).
\end{aligned}$$

[MC.3] Since

$$\begin{aligned}
(f, Q \cup \{(h,k)\}) \wedge (g,R) &= (f, Q \cup R \cup \{(f,g),(h,k)\}) \\
&= (f, Q \cup R \cup \{(f,g),(k,h)\}) \quad ([\mathbf{MC.3}]) \\
&= (f, Q \cup \{(k,h)\}) \wedge (g,R).
\end{aligned}$$

[MC.4] If $(h,k),(k,w) \in Q$, then $(f,Q) \sim (f, Q \cup \{(h,w)\})$ and

$$\begin{aligned}
(f,Q) \wedge (g,R) &= (f, Q \cup R \cup \{(f,g)\}) \\
&= (f, Q \cup R \cup \{(f,g),(h,w)\}) \quad ([\mathbf{MC.4}]) \\
&= (f, Q \cup \{(h,w)\}) \wedge (g,R).
\end{aligned}$$

[MC.5] If $(h,k) \in Q$ and if $\overline{v}h = h$, then $(f,Q) \sim (f, Q \cup \{(vh,vk)\})$ and

$$\begin{aligned} (f,Q) \wedge (g,R) &= (f, Q \cup R \cup \{(f,g)\}) \\ &= (f, Q \cup R \cup \{(f,g),(vh,vk)\}) \quad (\textbf{[MC.5]}) \\ &= (f, Q \cup \{(vh,vk)\}) \wedge (g,R). \end{aligned}$$

[MC.6] If e is a restriction idempotent and $(h,k) \in Q$, then

$$(fe,Q) \sim (f, Q \cup \{(he,k)\})$$

and

$$\begin{aligned} (fe,Q) \wedge (g,R) &= (fe, Q \cup R \cup \{(fe,g)\}) \\ &= (f, Q \cup R \cup \{(fe,g),(he,k)\}) \quad (\textbf{[MC.6]}) \\ &= (f, Q \cup R \cup \{(f,g),(he,k)\}) \\ &= (f, Q \cup \{(he,k)\}) \wedge (g,R). \end{aligned}$$

Case 2. Fix (f,Q).

[MC.1] If $(g,g') \in R$, then $(g,R) \sim (g',R)$ and

$$\begin{aligned} (f,Q) \wedge (g,R) &= (f, Q \cup R \cup \{(f,g)\}) \\ &= (f, Q \cup R \cup \{(f,g),(f,g'),(g',g)\}) \\ &\qquad ((g,g') \in R, \textbf{[MC.3], and [MC.4]}) \\ &= (f, Q \cup R \cup \{(f,g'),(g',g)\}) \quad (\textbf{[MC.4]}) \\ &= (f, Q \cup R \cup \{(f,g')\}) \quad (\textbf{[MC.3]}) \\ &= (f,Q) \wedge (g',R). \end{aligned}$$

[MC.2] Since

$$\begin{aligned}(f,Q)\wedge(g,R) &= (f,Q\cup R\cup\{(f,g)\})\\ &= (f,Q\cup R\cup\{(f,g),(1,1)\}) \quad ([\mathbf{MC.2}])\\ &= (f,Q)\wedge(g,R\cup\{(1,1)\}).\end{aligned}$$

[MC.3] Since

$$\begin{aligned}(f,Q)\wedge(g,R\cup\{(s,t)\}) &= (f,Q\cup R\cup\{(f,g),(s,t)\})\\ &= (f,Q\cup R\cup\{(f,g),(t,s)\}) \quad ([\mathbf{MC.3}])\\ &= (f,Q)\wedge(g,R\cup\{(t,s)\}).\end{aligned}$$

[MC.4] If $(u,v),(v,w)\in R$, then $(g,R)\sim(g,R\cup\{(u,w)\})$ and

$$\begin{aligned}(f,Q)\wedge(g,R) &= (f,Q\cup R\cup\{(f,g)\})\\ &= (f,Q\cup R\cup\{(f,g),(u,w)\}) \quad ([\mathbf{MC.4}])\\ &= (f,Q)\wedge(g,R\cup\{(u,w)\}).\end{aligned}$$

[MC.5] If $(s,t)\in R$ and $\overline{v}s=s$, then $(g,R)\sim(g,R\cup\{(vs,vt)\})$ and

$$\begin{aligned}(f,Q)\wedge(g,R) &= (f,Q\cup R\cup\{(f,g)\})\\ &= (f,Q\cup R\cup\{(f,g),(vs,vt)\}) \quad ([\mathbf{MC.5}])\\ &= (f,Q)\wedge(g,R\cup\{(vs,vt)\}).\end{aligned}$$

[MC.6] If $\overline{e} = e$ and $(s,t) \in R$, then $(ge, R) \sim (g, R \cup \{(se,t)\})$ and

$$\begin{aligned}
(f,Q) \wedge (ge,R) &= (f, Q \cup R \cup \{(f, ge)\}) \\
&= (g, Q \cup R \cup \{(f,ge),(se,t)\}) \quad (\textbf{[MC.6]}) \\
&= (f, Q \cup R \cup \{(ge,f),(se,t)\}) \quad (\textbf{[MC.3]}) \\
&= (f, Q \cup R \cup \{(g,f),(se^2,t)\}) \quad (\textbf{[MC.6]}) \\
&= (f, Q \cup R \cup \{(f,g),(se,t)\}) \quad (\textbf{[MC.3]}) \\
&= (f,Q) \wedge (g, R \cup \{(se,t)\}).
\end{aligned}$$

□

4.3.1 Mt($\mathbf{X}$) is a Restriction Category

We need to verify the identity law, associative law, and four restriction axioms. The identity law and associative law are verified as follows.

identity law: For any map $(f,Q) : X \to Y$, we have

$$(f,Q)(1_X, \emptyset) = (f, \emptyset \cup Q \cdot 1_X) = (f,Q)$$

and

$$(1_Y, \emptyset)(f,Q) = (f, Q \cup \emptyset f) = (f,Q).$$

So identity law holds true.

association law: For any maps $(f,Q) : X \to Y$, $(g,R) : Y \to Z$, and $(h,S) : Z \to A$,

$$\begin{aligned}
\Big((h,S)(g,R)\Big)(f,Q) &= (hg, R \cup Sg)(f,Q) \\
&= (hgf, Q \cup Rf \cup Sgf) \\
&= (h,S)(gf, Q \cup Rf) \\
&= (h,S)\Big((g,R)(f,Q)\Big).
\end{aligned}$$

So associative law holds true.

Hence Mt($\mathbf{X}$) is a category. To prove that Mt($\mathbf{X}$) is a restriction category, we verify the four restriction axioms as follows.

[**R.1**]

$$\begin{aligned}
(f,Q)\overline{(f,Q)} &= (f,Q)(\overline{f},Q)\\
&= (f, Q \cup Q\overline{f})\\
&= (f\overline{f}, Q) \quad \text{(Lemma 4.3.2}(iv))\\
&= (f,Q);
\end{aligned}$$

[**R.2**]

$$\begin{aligned}
\overline{(f,Q)}\,\overline{(g,R)} &= (\overline{f},Q)(\overline{g},R)\\
&= (\overline{f}\overline{g}, R \cup Q\overline{g})\\
&= (\overline{f}\overline{g}\,\overline{g}, R \cup Q) \quad \text{(Lemma 4.3.2}(iv))\\
&= (\overline{f}\overline{g}\overline{f}, Q \cup R)\\
&= (\overline{g}\overline{f}, Q \cup R\overline{f}) \quad \text{(Lemma 4.3.2}(iv))\\
&= \overline{(g,R)}\,\overline{(f,Q)};
\end{aligned}$$

[**R.3**]

$$\begin{aligned}
\overline{(g,R)\overline{(f,Q)}} &= (\overline{g\overline{f}}, Q \cup R\overline{f})\\
&= (\overline{g}\overline{f}, Q \cup R\overline{f})\\
&= (\overline{g},R)(\overline{f},Q)\\
&= \overline{(g,R)}\,\overline{(f,Q)};
\end{aligned}$$

[R.4]

$$\begin{aligned}
\overline{(g,R)}(f,Q) &= (\overline{g},R)(f,Q) \\
&= (\overline{g}f, Q\cup Rf) \\
&= (f\overline{gf}, Q\cup Rf) \\
&= (f\overline{gf}, Q\cup Rf\cup Q\overline{gf}) \quad \text{(Lemma 4.3.2}(iv)\text{)} \\
&= (f,Q)(\overline{gf}, Q\cup Rf) \\
&= (f,Q)\overline{(g,R)(f,Q)}.
\end{aligned}$$

4.3.2 Mt(**X**) is a Meet Restriction Category

For maps $(f,Q),(g,R):X\to Y$, note that

$$\begin{aligned}
& (f,Q)\leq (g,R) \\
\Leftrightarrow\ & (g,R)\overline{(f,Q)} = (f,Q) \\
\Leftrightarrow\ & (g\overline{f}, Q\cup R\overline{f}) = (f,Q).
\end{aligned}$$

The three meet axioms are showed as follows.

[M.1] For each map $(f,Q):X\to Y$ in Mt(**X**),

$$\begin{aligned}
(f,Q)\wedge(f,Q) &= (f, Q\cup Q\cup\{(f,f)\}) \\
&= (f,Q) \quad (\overline{f}\geq\overline{f} \text{ and Lemma 4.3.2}(ii)).
\end{aligned}$$

[M.2] For maps $(f,Q),(g,R):X\to Y$, since

$$\begin{aligned}
(f,Q)\overline{(f,Q\cup R\cup\{(f,g)\})} &= (f\overline{f}, Q\cup Q\overline{f}\wedge R\cup\{(f,g)\}) \\
&= (f, Q\cup R\cup\{(f,g)\}),
\end{aligned}$$

we have

$$(f,Q)\wedge(g,R)=(f,Q\cup R\cup\{(f,g)\})\leq(f,Q).$$

Similarly, $(f,Q)\wedge(g,R)\leq(g,R)$.

[M.3] For maps $(f,Q):X\to Y$, and $(g,R),(h,S):Y\to Z$,

$$\begin{aligned}\Big((g,R)\wedge(h,S)\Big)(f,Q) &= (g,S\cup R\cup\{(g,h)\})(f,Q)\\ &= (gf,Q\cup Sf\cup Rf\cup\{(gf,hf)\})\\ &= (gf,Q\cup Rf)\wedge(hf,Q\cup Sf)\\ &= \Big((g,R)(f,Q)\Big)\wedge\Big((h,S)(f,Q)\Big).\end{aligned}$$

Hence $\mathrm{Mt}(\mathbf{X})$ is a meet restriction category.

4.3.3 Mt(**X**) is a Free Meet Restriction Category

Let $\mathbf{mrCat_0}$ be the category of meet restriction categories and meet restriction functors between them. Then there is an obvious forgetful functor

$$U_{\mathrm{mt}}:\mathbf{mrCat}_0\to\mathbf{rCat}_0.$$

For each given restriction category $\mathbf{X}$, we have a restriction functor

$$J:\mathbf{X}\to U_{\mathrm{mt}}(\mathrm{Mt}(\mathbf{X}))$$

given by sending each map $f:X\to Y$ in $\mathbf{X}$ to a map $(f,\emptyset):X\to Y$ in $\mathrm{Mt}(\mathbf{X})$.

For each map $(f, Q) : X \to Y$ in $\mathrm{Mt}(\mathbf{X})$ with $Q = \{(f_1, f_1'), \cdots, (f_n, f_n')\}$, we have

$$\begin{aligned}
(f, Q) &= (f, \emptyset)(1_X, \{(f_1, f_1'), \cdots, (f_n, f_n')\}) \\
&= (f, \emptyset)\prod_{i=1}^{n}(1, \{(f_i, f_i')\}), \\
\overline{(f_i, \emptyset) \wedge (f_i', \emptyset)} &= \overline{(f_i, \{(f_i, f_i')\})} \\
&= (\overline{f_i}, \{(f_i, f_i')\}) \\
&= (1, \{(f_i\overline{f_i}, f_i')\}) \quad ([\mathbf{MC.6}]) \\
&= (1, \{(f_i, f_i')\}).
\end{aligned}$$

For a given restriction functor $F : \mathbf{X} \to U_{\mathrm{mt}}(\mathbf{Y})$ with a meet restriction category $\mathbf{Y}$, we define

$$F^* : \mathrm{Mt}(\mathbf{X}) \to \mathbf{Y}$$

by

$$\begin{aligned}
F^*(f, \emptyset) &= F(f) \\
F^*(f, Q) &= F(f)\prod_{i=1}^{n}\overline{F(f_i) \wedge F(f_i')}.
\end{aligned}$$

Then, for each $\mathbf{X}$-map $f : X \to Y$,

$$(U_{\mathrm{mt}}(F^*)J)(f) = U_{\mathrm{mt}}(F^*)(f, \emptyset) = F(f).$$

Suppose that $G : \mathrm{Mt}(\mathbf{X}) \to \mathbf{Y}$ is a meet restriction functor such that $U_{\mathrm{mt}}(G)J = F$. Then

$$U_{\mathrm{mt}}(G)(f, \emptyset) = (U_{\mathrm{mt}}(G)J)(f) = F(f) = U_{\mathrm{mt}}(F^*)(f, \emptyset)$$

and so, for any $\mathrm{Mt}(\mathbf{X})$-map $(f, Q) : X \to Y$ with $Q = \{(f_1, f_1'), \cdots, (f_n, f_n')\}$, we have

$$\begin{aligned}
U_{\mathrm{mt}}(G)(f, Q) &= U_{\mathrm{mt}}(G)\Big((f, \emptyset)\prod_{i=1}^{n}(1, \{(f_i, f_i')\})\Big) \\
&= U_{\mathrm{mt}}(G)(f, \emptyset)U_{\mathrm{mt}}(G)\Big(\prod_{i=1}^{n}(1, \{(f_i, f_i')\})\Big) \\
&= F(f)\prod_{i=1}^{n}U_{\mathrm{mt}}(G)(1, \{(f_i, f_i')\}) \\
&= F(f)\prod_{i=1}^{n}U_{\mathrm{mt}}(G)(\overline{(f_i, \emptyset) \wedge (f_i', \emptyset)}) \\
&= F(f)\prod_{i=1}^{n}\overline{U_{\mathrm{mt}}(G)(f_i, \emptyset) \wedge U_{\mathrm{mt}}(G)(f_i', \emptyset)} \\
&= F(f)\prod_{i=1}^{n}\overline{F(f_i) \wedge F(f_i')} \\
&= U_{\mathrm{mt}}(F^*)(f, Q).
\end{aligned}$$

Hence $G = F^*$ and therefore there is a unique meet restriction functor F^* suct that

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "J"] \arrow[dr, "F"'] & U_{\mathrm{mt}}(\mathrm{Mt}(\mathbf{X})) \arrow[d, "U_{\mathrm{mt}}(F^*)"] & \mathrm{Mt}(\mathbf{X}) \arrow[d, dotted, "\exists! F^*"] \\
& U_{\mathrm{mt}}(\mathbf{Y}) & \mathbf{Y}
\end{tikzcd}$$

commutes. Thus, there is a functor

$$F_{\mathrm{mt}} : \mathbf{rCat}_0 \to \mathbf{mrCat}_0$$

given by sending each restriction functor $F : \mathbf{X} \to \mathbf{Y}$ to a meet restriction functor $F_{\mathrm{mc}}(F) : \mathrm{Mt}(\mathbf{X}) \to \mathrm{Mt}(\mathbf{Y})$, where

$$F_{\mathrm{mt}}(F)(f, Q) = (F(f), F(Q))$$

and $\mathrm{Mt}(\mathbf{X})$ is free over a restriction category $\mathbf{X}$.

If $J(f) = (f, \emptyset) = (g, \emptyset) = J(g)$, then $(f, \emptyset) \sim (g, \emptyset)$. By Lemma 4.3.2, we have

$$(f, \emptyset) = (f, \{(h, h)|\overline{f} \leq \overline{h}\} \cup \{(s, t)|f \leq s, t\})$$

and

$$(g, \emptyset) = (g, \{(k,k)|\overline{g} \leq \overline{k}\} \cup \{(u,v)|g \leq u, v\}).$$

Hence $J(f) = J(g)$ if and only if $f = g$. So we have actually proved:

Theorem 4.3.4 $F_{\text{mt}} \dashv U_{\text{mt}} : \mathbf{mrCat}_0 \to \mathbf{rCat}_0$ *is an adjoint pair with a faithful unit so that* $\text{Mt}(\mathbf{X})$ *is a free meet restriction category over a restriction category* $\mathbf{X}$*.*

4.3.4 Meet Completion for Inverse Categories

In the last subsection, we provided the meet completion $\text{Mt}(\mathbf{X})$ for restriction categories. Recall that an inverse semigroup is *meet complete* if every non-empty subset has a meet ([31], p.27). Similar to join completion for inverse semigroups, it is well-known that each inverse semigroup can be embedded in a meet complete inverse semigroup, called *meet completion* ([31], p.34). This subsection is intended to show the relationship between the meet completion for restriction categories and the meet completion for inverse categories.

First, we recall the meet completion for inverse semigroups, described in [31] (pp.34-36). Let S be an inverse semigroup and X a non-empty subset of S. Write

$$X \uparrow = \{s \in S | x \leq s \text{ for some } x \in X\}.$$

A subset X is called *up closed* if $X = X \uparrow$. A non-empty up closed subset A of S is called a *coset* if $A = AA^{(-1)}A$. For each non-empty subset $X \subseteq S$, define

$$\langle X \rangle = \bigcap \{\text{cosets } A | X \subseteq A\}.$$

One can describe the meet completion $K(S)$ for an inverse semigroup S with

composition: $A \otimes B = \langle AB \rangle$.

The category $\mathbf{invSgrp}_0$ of all inverse semigroups and homomorphisms between them, has the subcategory $\mathbf{minvSgrp}_0$ consisting of all meet complete inverse semigroups.

There is the forgetful functor $U_{\mathrm{mci}} : \mathbf{minvSgrp_0} \to \mathbf{invSgrp_0}$, forgetting the meets. U_{mci} has the left adjoint F_{mci} given by the meet completion $K(\ )$:

Theorem 4.3.5 *$F_{\mathrm{mci}} \dashv U_{\mathrm{mci}} : \mathbf{minvSgrp_0} \to \mathbf{invSgrp_0}$ is an adjoint pair with a faithful unit so that the meet completion $K(S)$ is a free meet inverse semigroup over an inverse semigroup S.*

PROOF: Theorems 28, 29, 30 [31], pp.35-36. □

Remark 4.3.6 The meet completion for inverse semigroups can be fitted to inverse categories. Let $\mathbf{I}$ be an inverse category. The meet completion $K(\mathbf{I})$ can be described as the following meet complete inverse category with:

objects: $X \in \mathbf{I}$;

maps: a map $U : X \to Y$ is given by a coset $U \subseteq \mathrm{map}_{\mathbf{I}}(X, Y)$;

identities: $1_X = \{1_X\}$;

composition: for any maps $U : X \to Y$ and $V : Y \to Z$ in $K(\mathbf{I})$, $VU = \langle VU \rangle$;

restriction: $\overline{U} = \langle U^{(-1)} U \rangle$;

meet: $\wedge_{i\in\Gamma} U_i = \bigcap_{i\in\Gamma} U_i$.

Let $\mathbf{mcinvCat}_0$ be the subcategory of $\mathbf{invCat}_0$, consisting of all *meet complete inverse categories*, analogous to meet complete inverse semigroups. Similarly, the obvious forgetful functor $U_{\mathrm{mc}} : \mathbf{mcinvCat}_0 \to \mathbf{invCat}_0$ has the left adjoint given by the meet completion $K(\mathbf{I})$:

$F_{\mathrm{mc}} \dashv U_{\mathrm{mc}} : \mathbf{mcinvCat}_0 \to \mathbf{invCat}_0$ is an adjoint pair with a faithful unit so that $K(\mathbf{I})$ is a free meet complete inverse category over an inverse category $\mathbf{I}$.

Let us be back to the meet completion $\mathrm{Mt}(\mathbf{X})$ for restriction categories proved in the last subsection. As each inverse category $\mathbf{I}$ is a restriction category, we have a meet restriction category Mt(**I**) that turns out to be a meet inverse category, where an inverse category is called a *meet inverse category* if there is a meet operation on parallel maps in such that the three meet axioms in the definition of a meet restriction category (Definition 4.1.1) are satisfied. To see this, we characterize idempotents in Mt(**I**) by the following lemma.

Lemma 4.3.7 *An* Mt(**I**)*-map* $(f, Q) : X \to Y$ *is an idempotent if and only if there exists a restriction idempotent* e *in* **I** *such that* $(f, Q) = (e, P)$*.*

PROOF: Clearly, if e is a restriction idempotent in **I**, then $(e, P)^2 = (e^2, P \cup Pe) = (e, P)$ and so each (e, P) is an idempotent in Mt(**I**).

Conversely, if $(f, Q)^2 = (f^2, Q \cup Qf) = (f, Q)$, then after finite steps $\simeq$ there are restriction idempotents e_1 and e_2 such that

$$(f, Q)^2 = (f, T \cup \{(fe_1, f^2e_2)\}) = (f, Q).$$

By [**MC.5**], $(f, T \cup \{(fe_1, f^2e_2)\}) = (f, T \cup \{(\overline{f}e_1, fe_2)\}) = (\overline{f}, T \cup \{(\overline{f}e_1, fe_2)\})$, as desired. □

Now we are ready to show that Mt(**I**) is a meet inverse category.

Lemma 4.3.8 *Let* **I** *be an inverse category. Then* Mt(**I**) *is a meet inverse category.*

PROOF: As Mt(**I**) is a meet restriction category, it suffices to show each Mt(**I**)-map has a least one regular-inverse and idempotents with the same domain in Mt(**I**) commute.

For each Mt(**I**)-map $(f, Q) : X \to Y$,

$$\begin{aligned} (f, Q)(f^{(-1)}, Qf^{(-1)})(f, Q) &= (ff^{(-1)}, Qf^{(-1)})(f, Q) \\ &= (ff^{(-1)}f, Q \cup Qf^{(-1)}f) \\ &= (f, Q) \quad ([\mathbf{MC.6}]). \end{aligned}$$

So (f,Q) has a regular inverse $(f^{(-1)}, Qf^{(-1)})$.

By Lemma 4.3.7, each idempotent in $\mathrm{Mt}(\mathbf{I})$ has the form of (e,P) with a restriction idempotent e. For any idempotents with the same domain $(e_1,Q_1),(e_2,Q_2) : X \to X$, here e_1 and e_2 are restriction idempotents in $\mathbf{I}$, we have

$$\begin{aligned} (e_1,Q_1)(e_2,Q_2) &= (e_1e_2, Q_2 \cup Q_1e_2) \\ &= (e_1e_2, Q_2e_2 \cup Q_1) \quad ([\mathbf{MC.6}]) \\ &= (e_2,Q_2)(e_1,Q_1). \end{aligned}$$

Hence idempotents with the same domain in $\mathrm{Mt}(\mathbf{I})$ commute. Thus, $\mathrm{Mt}(\mathbf{I})$ is a meet inverse category. ☐

By Lemma 4.3.8, $F_{\mathrm{mt}}(\mathbf{invCat}_0)$ is a subcategory of $\mathbf{minvCat_0}$. So there is an adjunction $F_{\mathrm{mt}} \dashv U_{\mathrm{mt}} : \mathbf{minvCat}_0 \to \mathbf{invCat}_0$ given by restricting adjunction $F_{\mathrm{mt}} \dashv U_{\mathrm{mt}} : \mathbf{mrCat}_0 \to \mathbf{rCat}_0$ to $\mathbf{invCat}_0$. We have:

Theorem 4.3.9 *There is a commutative adjunction diagram:*

$$\begin{tikzcd}
\mathbf{invCat}_0 \arrow[r, shift left, "F_{\mathrm{mti}}"] \arrow[r, phantom, "\perp"] \arrow[d, hook] & \mathbf{minvCat}_0 \arrow[l, shift left, "U_{\mathrm{mti}}"] \arrow[d, hook] \\
\mathbf{rCat}_0 \arrow[r, shift left, "F_{\mathrm{mt}}"] \arrow[r, phantom, "\perp"] & \mathbf{mrCat}_0 \arrow[l, shift left, "U_{\mathrm{mt}}"]
\end{tikzcd}$$

For a given inverse category $\mathbf{I}$, both $\mathrm{Mt}(\mathbf{I})$ and $K(\mathbf{I})$ share some properties as reflected in the following two propositions.

Proposition 4.3.10 *For an inverse category* $\mathbf{I}$*, if* $(f,Q) : X \to Y$ *is an* $\mathrm{Mt}(\mathbf{I})$*-map with* $(h,k) \in Q$*, then*

1. $h \leq u$ *and* $k \leq v$ *implies* $(f,Q) = (f, Q \cup \{(u,v)\})$,
2. $(f,Q) = (f, Q \cup \{(1, h^{(-1)}k)\}) = (f, Q \cup \{(k^{(-1)}h, 1)\})$.

Proof:

1. By Lemma 4.3.2(*iii*).

2. As $h = hh^{(-1)}h = \overline{h^{(-1)}}h$,

$$\begin{aligned} (f, Q) &= (f, Q \cup \{(h^{(-1)}h, h^{(-1)}k)\}) \quad ([\mathbf{MC.5}]) \\ &= (fh^{(-1)}h, Q \cup \{(1, h^{(-1)}k)\}) \quad ([\mathbf{MC.6}]) \\ &= (f, Q \cup \{(1, h^{(-1)}hh^{(-1)}k)\}) \quad ([\mathbf{MC.6}]) \\ &= (f, Q \cup \{(1, h^{(-1)}k)\}). \end{aligned}$$

Similarly, $(f, Q) = (f, Q \cup \{(k^{(-1)}h, 1)\})$.

$\square$

So, for an inverse category $\mathbf{I}$ each parallel pair (h, k) can be turned into the form of $(1, h^{(-1)}k)$ or $(k^{(-1)}h, 1)$.

Proposition 4.3.11 *For an inverse category* $\mathbf{I}$ *and an* $\mathrm{Mt}(\mathbf{I})$*-map* (f, Q)*, if*

$$(1, h), (1, k), (1, l) \in Q$$

and h, k, l *have the same domain, then*

1. $(1, h^{(-1)}) \in Q$;

2. $(h, hk), (1, hk) \in Q$;

3. $(1, hk^{(-1)}l) \in Q$.

Proof:

1. As

$$\begin{aligned}
(1,h) \in Q &\Leftrightarrow (h,1) \in Q \\
&\Leftrightarrow (\overline{h^{(-1)}}h,1) \in Q \\
&\Rightarrow (h^{(-1)}h,h^{(-1)}) \in Q \\
&\Rightarrow (1,h^{(-1)}hh^{(-1)}) \in Q \\
&\Leftrightarrow (1,h^{(-1)}) \in Q.
\end{aligned}$$

2. Since

$$\begin{aligned}
(1,h),(1,k) \in Q &\Rightarrow (1,\overline{h}),(1,k) \in Q \\
&\Rightarrow (\overline{h},1),(1,k) \in Q \\
&\Rightarrow (\overline{h},k) \in Q \\
&\Rightarrow (h,hk) \in Q.
\end{aligned}$$

Clearly, $(1,h),(h,hk) \in Q$ imply $(1,hk) \in Q$.

3. As $(1,h),(1,k^{(-1)}),(1,l) \in Q$.

□

4.3.5 Interactions with Cartesianess

In Chapter 2, we studied cartesian objects in restriction categories and how to add partial products to a restriction category freely. In this subsection, we shall study how meet completion interacts with cartesianess.

Recall that in a restriction category, a partial terminal object is an object 1 satisfying for each object X there is a total map $!_X : X \to 1$ such that for any map $f : X \to 1$,

$f =!_X\overline{f}$:

$$\begin{tikzcd} X \arrow[loop left, "\overline{f}"] \arrow[r, "!_X"] \arrow[r, bend right, "f"'] & 1 \end{tikzcd}$$

For each pair of objects X and Y in a restriction category, a binary partial product of X and Y is an object $X \times Y$ with two total maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ such that for any pair of maps $f : Z \to X$ and $g : Z \to Y$ there is a unique map $\langle f, g\rangle : Z \to X \times Y$ such that

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dotted, "\langle f,g\rangle"] \arrow[d, phantom, xshift=-2em, "\geq"] \arrow[d, phantom, xshift=2em, "\leq"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

A restriction category is *cartesian* if it has a partial terminal object and binary partial products.

Assume that $\mathbf{X}$ is a cartesian restriction category with a partial terminal object $(1, !)$ and a binary partial product $(X \times Y, \pi_X, \pi_Y)$ for each pair of $\mathbf{X}$-objects X and Y. We are wondering if the unit $J : \mathbf{X} \to \mathrm{Mt}(\mathbf{X})$, given by sending $f : X \to Y$ to $(f, \emptyset) : X \to Y$, preserves the partial terminal object $(1, !)$ and the binary partial product $(X \times Y, \pi_X, \pi_Y)$.

Claim 1. $(1, (!, \emptyset))$ *is a partial terminal object in* $\mathrm{Mt}(\mathbf{X})$ *so that the unit* J *preserves partial terminal objects.*

In fact, for each $\mathrm{Mt}(\mathbf{X})$-object X, there is a total $\mathrm{Mt}(\mathbf{X})$-map $(!_X, \emptyset) : X \to 1$ such that for any $\mathrm{Mt}(\mathbf{X})$-map $(f, Q) : X \to 1$

$$(!_X, \emptyset)\overline{(f, Q)} = (!_X, \emptyset)(\overline{f}, Q) = (!_X\overline{f}, Q) = (f, Q).$$

Clearly, $\overline{(!_X, \emptyset)} = (\overline{!_X}, \emptyset) = (!_X, \emptyset)$. So $(!_X, \emptyset)$ is total in $\mathrm{Mt}(\mathbf{X})$. Hence $(1, (!, \emptyset))$ is a partial terminal object in $\mathrm{Mt}(\mathbf{X})$.

Claim 2. $(X \times Y, (\pi_X, \emptyset), (\pi_Y, \emptyset))$ *is not a partial product of* X *and* Y *in* $\mathrm{Mt}(\mathbf{X})$ *so that* J *does not preserve binary partial products.*

For each pair Mt($\mathbf{X}$)-maps $(f,Q) : Z \to X$ and $(g,R) : Z \to Y$, the Mt($\mathbf{X}$)-map $(\langle f,g\rangle, Q \cup R) : Z \to X \times Y$ satisfies

$$(\pi_X,\emptyset)((\langle f,g\rangle,Q\cup R) = (\pi_X\langle f,g\rangle,Q\cup R) = (f\overline{g},R\cup Q\overline{g}) = (f,Q)\overline{(g,R)} \leq (f,Q)$$

and

$$(\pi_Y,\emptyset)((\langle f,g\rangle,Q\cup R) = (\pi_Y\langle f,g\rangle,Q\cup R) = (g\overline{f},Q\cup R\overline{f}) = (g,R)\overline{(f,Q)} \leq (g,R) :$$

$$\begin{tikzcd}
& Z \arrow[ld, "{(f,Q)}"'] \arrow[rd, "{(g,R)}"] \arrow[d, dotted, "{(\langle f,g\rangle,Q\cup R)}"] & \\
X & X \times Y \arrow[l, "{(\pi_X,\emptyset)}"] \arrow[r, "{(\pi_Y,\emptyset)}"'] \arrow[lu, phantom, "\geq"] \arrow[ru, phantom, "\leq"] & Y
\end{tikzcd}$$

However, the uniqueness of the Mt($\mathbf{X}$)-map $(\langle f,g\rangle, Q \cup R)$ is a problem as $(f,Q) = (f',Q)$ and $(g,R) = (g',R)$ do not imply $(\langle f,g\rangle, Q\cup R) = (\langle f',g'\rangle, Q\cup R)$ generally. For example, given $\mathbf{X}$-maps f, f', g, g' with the same domain and codomain, suppose that $f \neq f'$ and $g \neq g'$. Then

$$(f,\{(f,f'),(f',f)\}) = (f',\{(f,f'),(f',f)\})$$

and

$$(g,\{(g,g'),(g',g)\}) = (g',\{(g,g'),(g',g)\})$$

but

$$(\langle f,g\rangle,\{(f,f'),(f',f),(g,g'),(g',g)\}) \neq (\langle f',g'\rangle,\{(f,f'),(f',f),(g,g'),(g',g)\}).$$

So, we have:

Proposition 4.3.12 *The unit* $J : \mathbf{X} \to \mathrm{Mt}(\mathbf{X})$*, given by sending* $f : X \to Y$ *to* $(f,\emptyset) : X \to Y$*, preserves partial terminal objects but does not preserve binary partial products generally.*

But, we have:

Proposition 4.3.13 *Given a cartesian restriction category* $\mathbf{X}$*, then the unit* $J : \mathbf{X} \to \mathrm{Mt}(\mathbf{X})$*, given by sending* $f : X \to Y$ *to* $(f, \emptyset) : X \to Y$*, preserves binary partial products if and only if the maps in* $\mathrm{Mt}(\mathbf{X})$*, defined in Section 4.3, satisfies one more equivalent step* $\sim$*:*

[MC.7] $(f,Q) \sim (f',Q')$ *and* $(g,R) \sim (g',R')$ *imply* $(\langle f,g\rangle, Q\cup R) \sim (\langle f',g'\rangle, Q'\cup R')$.

Proof: (The Sketch). First of all, it is easy to show that $\sim$, after adding [**MC.7**], is an equivalence relation as we did in Lemma 4.3.1. Then, it is routine to verify that the composition, restriction, and meet defined in Section 4.3 are well defined after adding [**MC.7**], as in Lemma 4.3.3. Now we want to show that $J(X \times Y, \pi_X, \pi_Y) = (X \times Y, (\pi_X, \emptyset), (\pi_Y, \emptyset))$ is a partial product of X and Y in $\mathrm{Mt}(\mathbf{X})$ with adding [**MC.7**].

For any $\mathrm{Mt}(\mathbf{X})$-maps $(f,Q) : Z \to X$ and $(g,R) : Z \to Y$, as we did in Claim 2 above, we have the following diagram:

$$\begin{tikzcd}
 & Z \arrow[dl, "{(f,Q)}"'] \arrow[dr, "{(g,R)}"] \arrow[d, dotted, "{(\langle f,g\rangle,Q\cup R)}"] & \\
X & X \times Y \arrow[l, "{(\pi_X,\emptyset)}"] \arrow[r, "{(\pi_Y,\emptyset)}"'] \arrow[ul, phantom, "\geq"] \arrow[ur, phantom, "\leq"] & Y
\end{tikzcd}$$

Let $(h,S) : Z \to X \times Y$ be an $\mathrm{Mt}(\mathbf{X})$-map such that

$$\begin{tikzcd}
 & Z \arrow[dl, "{(f,Q)}"'] \arrow[dr, "{(g,R)}"] \arrow[d, dotted, "{(h,S)}"] & \\
X & X \times Y \arrow[l, "{(\pi_X,\emptyset)}"] \arrow[r, "{(\pi_Y,\emptyset)}"'] \arrow[ul, phantom, "\geq"] \arrow[ur, phantom, "\leq"] & Y
\end{tikzcd}$$

Then

$$(\pi_X, \emptyset)(h,S) = (\pi_X h, S) = (f,Q)\overline{(g,R)} = (f,Q)(\overline{g},R) = (f\overline{g}, Q \cup R)$$

and

$$(\pi_Y, \emptyset)(h,S) = (\pi_Y h, S) = (g,R)\overline{(f,Q)} = (g,R)(\overline{f},Q) = (g\overline{f}, Q \cup R)$$

and so

$$(\pi_X h, S) \sim (f\overline{g}, Q \cup R) \text{ and } (\pi_Y h, S) \sim (g\overline{f}, Q \cup R).$$

By [**MC.7**], we have

$$(h,S)=(\langle\pi_X h,\pi_Y h\rangle,S\cup S)=(\langle f\overline{g},g\overline{f}\rangle,Q\cup R\cup Q\cup R)=(\langle f,g\rangle,Q\cup R).$$

Hence the uniqueness of $(\langle f,g\rangle,Q\cup R)$ follows and therefore $(X\times Y,(\pi_X,\emptyset),(\pi_Y,\emptyset))$ is a partial product of X and Y in $\mathrm{Mt}(\mathbf{X})$.

Conversely, let $(X\times Y,\pi_X,\pi_Y)$ be a partial product of X and Y in $\mathbf{X}$. If the unit $J:\mathbf{X}\to\mathrm{Mt}(\mathbf{X})$ preserves binary partial products, then $(X\times Y,(\pi_X,\emptyset),(\pi_Y,\emptyset))$ is a partial product of X and Y in $\mathrm{Mt}(\mathbf{X})$. Suppose that $(f,Q)\sim(f',Q')$ and $(g,R)\sim(g',R')$. Then, as we did in Claim 2 above, we have the following diagram

$$\begin{tikzcd}
 & Z \arrow[ld, "{(f,Q)}"'] \arrow[rd, "{(g,R)}"] \arrow[d, dotted, "{(\langle f,g\rangle,Q\cup R)}"', "{(\langle f',g'\rangle,Q'\cup R')}"] \arrow[ld, phantom, shift right=4, "\geq"] \arrow[rd, phantom, shift left=4, "\leq"] & \\
X & X\times Y \arrow[l, "{(\pi_X,\emptyset)}"] \arrow[r, "{(\pi_Y,\emptyset)}"'] & Y
\end{tikzcd}$$

and so $(\langle f,g\rangle,Q\cup R)\sim(\langle f',g'\rangle,Q'\cup R')$. ☐

Chapter 5

Range Categories

In [10], we introduced the notion of range categories that axiomatized both domain and image of a partial map and showed that the category of all range categories with split restrictions is equivalent to the category of all $\mathcal{M}$-stable factorization systems so that each range category $\mathbf{X}$ can be embedded fully and faithfully into the partial maps category of some $\mathcal{M}$-stable factorization system after splitting restriction idempotents of $\mathbf{X}$. In [43], Schein embedded each type 3 function system, which can be viewed as a range category with one object, satisfying certain condition ([**RR.6**] below), into the category of partial functions $\mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$. This chapter is devoted to studying the Schein's representation theorem for type 3 function systems in range categories.

5.1 Introduction to Range Categories

In this section, we shall give a brief introduction to range categories and collect some results on range categories, which we shall use in this chapter later on.

5.1.1 Range Category Basics

Let $f : X \to Y$ be a partial map in $\mathbf{Par}(\mathbf{Set}, \mathrm{Monics}_{\mathbf{Set}})$. We define a partial map $\widehat{f} : Y \to Y$ by

$$\widehat{f}(y) = \begin{cases} y & \text{if } \exists x\ f(x) = y, \\ \uparrow & \text{otherwise.} \end{cases}$$

Obviously $\widehat{f}$ describes the *range* of f and satisfies the following four conditions:

[RR.1] $\overline{\widehat{f}} = \widehat{f}$ for each map f,

[RR.2] $\widehat{f}f = f$ for each map f,

[RR.3] $\widehat{\overline{g}f} = \overline{g}\widehat{f}$ for all maps f, g with $\mathrm{codom}(f) = \mathrm{dom}(g)$,

[RR.4] $\widehat{g\widehat{f}} = \widehat{gf}$ for all maps f, g with $\mathrm{codom}(f) = \mathrm{dom}(g)$.

Definition 5.1.1 *A* range structure *on a restriction category* **C** *is an assignment of a map* $\widehat{f} : Y \to Y$ *in* **C** *to each map* $f : X \to Y$ *such that the four range axioms* [**RR**.**1**], [**RR**.**2**], [**RR**.**3**], *and* [**RR**.**4**] *mentioned above are satisfied. A restriction category with a range structure is called* a range category.

Here are some examples of range categories.

Example 5.1.2 1. Any category is a range category with trivial restriction structure and trivial range structure given by

$$\overline{f} = 1_X \text{ and } \widehat{f} = 1_Y,$$

for any map $f : X \to Y$.

2. $\mathbf{Par}(\mathbf{Set}, \mathrm{Monics}_{\mathbf{Set}})$ is a restriction category with restriction given by

$$\overline{f}(x) = \begin{cases} x & \text{whenever} \quad \downarrow f(x), \\ \uparrow & \text{otherwise.} \end{cases}$$

for each map $f : X \to Y$. It is also a range category with the range structure given by

$$\widehat{f}(y) = \begin{cases} y & \text{if} \;\; \exists x \; f(x) = y, \\ \uparrow & \text{otherwise.} \end{cases}$$

We denote the range category of sets and partial functions by

$$\mathbf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}}).$$

3. Any inverse semigroup with an identity can be regarded as the one object range category with the restriction and the range given by $\overline{x} = x^{(-1)}x$ and $\widehat{x} = xx^{(-1)}$ (Similar to Example 1.6.6.4).

4. More generally, each inverse category is a range category with the restriction and the range given by $\overline{f} = f^{(-1)}f$ and $\widehat{f} = ff^{(-1)}$. The four range axioms are checked as follows:

   [RR.1] $\overline{\widehat{f}} = \overline{ff^{(-1)}} = (ff^{(-1)})^{(-1)}ff^{(-1)} = ff^{(-1)}ff^{(-1)} = ff^{(-1)} = \widehat{f}$;

   [RR.2] $\widehat{f}f = ff^{(-1)}f = f$;

   [RR.3]

$$\begin{aligned}
\widehat{\overline{g}f} &= \widehat{g^{(-1)}gf} \\
&= g^{(-1)}gf(g^{(-1)}gf)^{(-1)} \\
&= g^{(-1)}gff^{(-1)}g^{(-1)}g \\
&= g^{(-1)}gg^{(-1)}gff^{(-1)} \\
&= g^{(-1)}gff^{(-1)} \\
&= \overline{g}\widehat{f};
\end{aligned}$$

   [RR.4]

$$\begin{aligned}
\widehat{g\widehat{f}} &= \widehat{gff^{(-1)}} \\
&= gff^{(-1)}(gff^{(-1)})^{(-1)} \\
&= gff^{(-1)}ff^{(-1)}g^{(-1)} \\
&= gff^{(-1)}g^{(-1)} \\
&= (gf)(gf)^{(-1)} \\
&= \widehat{gf}.
\end{aligned}$$

Some basic properties of range categories are recorded in the following lemma, which are easy to verify.

Lemma 5.1.3 *In a range category,*

(*i*) $\widehat{g}\widehat{f} = \widehat{f}\widehat{g}$ *if* codom(f) = codom(g)*;*

(*ii*) $\widehat{f}\overline{g} = \overline{g}\widehat{f}$ *if* dom(g) = codom(f)*;*

(*iii*) $\widehat{\widehat{g}f} = \widehat{g}\widehat{f}$ *if* codom(f) = codom(g)*;*

(*iv*) $\widehat{f} = 1$ *if f is epic. In particular,* $\widehat{1} = 1$*;*

(*v*) $(\widehat{f})^2 = \widehat{f}$ *for each map f;*

(*vi*) $\widehat{\widehat{f}} = \widehat{f}$ *for each map f;*

(*vii*) $\widehat{\overline{f}} = \overline{f}$ *for each map f;*

(*viii*) $\widehat{gf}\widehat{g} = \widehat{gf}$ *if* codom(f) = dom(g)*;*

(*ix*) $\widehat{\widehat{g}\widehat{f}} = \widehat{g}\widehat{f}$ *if* codom(f) = codom(g).

A functor $F : \mathbf{C} \to \mathbf{D}$ between two range categories is called *a range functor* if $F(\overline{f}) = \overline{F(f)}$ and $F(\widehat{f}) = \widehat{F(f)}$ for each map f in $\mathbf{C}$. A natural transformation between two range functors is called a *range transformation* if its components are total.

Range categories and range functors form a category, denoted by $\mathbf{rrCat}_0$. There is an evident forgetful functor $U_{rr} : \mathbf{rrCat}_0 \to \mathbf{Cat}_0$, which forgets restriction and range structures. Range categories, range functors between them, and range natural transformations form a 2-category, denoted by $\mathbf{rrCat}$. Again, there is an evident forgetful 2-functor $U_{rr} : \mathbf{rrCat} \to \mathbf{Cat}$. $\mathbf{rrCat}$ has an important full 2-subcategory, comprising those objects with split restriction, denoted by $\mathbf{rrCat}_s$. The underlying category of $\mathbf{rrCat}_s$ is denoted by $\mathbf{rrCat}_{s0}$.

5.1.2 Splitting Restriction Idempotents

Given a range category $\mathbf{C}$, as in Subsection 1.6.4, one can split restriction idempotents of $\mathbf{C}$ to form $\mathsf{Split}(\mathbf{C})$ that is a range category. Similar to Proposition 1.6.11, we have:

Proposition 5.1.4 ([10], Proposition 2.1.4) *If* $\mathbf{C}$ *is a range category, so is* $\mathsf{Split}(\mathbf{C})$*, but with a split restriction structure given by the restriction in the category* $\mathbf{C}$*.*

Similar to Proposition 1.6.13, one has:

Proposition 5.1.5 *There is an adjunction with a full and faithful unit*

$$\eta_{\mathbf{C}} : \mathbf{C} \to E(\mathsf{Split}(\mathbf{C}))$$

given by sending $f : X \to Y$ *to* $f : 1_X \to 1_Y$*:*

$$\begin{tikzcd}
\mathbf{rrCat}_{s0} \arrow[r, hook, bend right, "E"'] \arrow[r, phantom, "\perp"] & \mathbf{rrCat}_0 \arrow[l, bend right, "\mathsf{Split}"']
\end{tikzcd}$$

where E *is the inclusion.*

5.1.3 $\mathcal{M}$-Stable Factorization Systems and Range Categories

In Subsection 1.6.5, we have seen that each $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ gives rise to a restriction category $\mathsf{Par}(\mathbf{C}, \mathcal{M})$. It is natural to ask when $\mathsf{Par}(\mathbf{C}, \mathcal{M})$ becomes a range category. As in [10], $\mathcal{M}$-stable factorization systems provide a possible answer.

Let $\mathbf{C}$ be a category and let $\mathcal{A}$ be a set of maps in $\mathbf{C}$, along which pullbacks exist. A factorization system $(\mathcal{E}, \mathcal{M})$ of $\mathbf{C}$ is said to be *stable along* $\mathcal{A}$*-maps* if for any $a \in \mathcal{A}$ and any $(\mathcal{E}, \mathcal{M})$-factorization $f = m_f e_f$, $f' = m'_f e'_f$ is a pullback of $f = m_f e_f$ along a, then $f' = m'_f e'_f$ is the $(\mathcal{E}, \mathcal{M})$-factorization of f'.

Recall that a category is *regular* if each map has a kernel pair and each kernel pair has a coequalizer and if regular epics are stable. The algebraic and monadic categories over **Set**, including Ω-algebras, are regular [3]. Any regular category admits the $(\mathrm{RegEpi}, \mathrm{Mon})$-factorization system which is stable [3].

But $\mathbf{Set}_{\mathrm{ffib}}$, defined in Example 1.6.16, does not admit any $(\mathcal{E},\mathcal{M})$-factorization system, where

$$\mathcal{M} = \{\text{injections } i : A \hookrightarrow B \mid |B \setminus i(A)| < +\infty\}.$$

Factorization systems are related to other categorical notions, such as fibrations ([23], [38]). localizations ([3], [44]), torsion theory [38], and Eilenberg-Moore algebras [26]. Stability of factorization systems can be characterized using fibrations as follows.

Let $(\mathcal{E},\mathcal{M})$ be a factorization system in a given category $\mathbf{C}$. Recall that $(\mathcal{E},\mathcal{M})$ gives rise to a bifibration $\partial : \mathcal{M} \to \mathbf{C}$ that is a subbifibration of the basic codomain fibration $\delta : \mathbf{C}^2 \to \mathbf{C}$:

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, hook] \arrow[dr, "\partial"'] & & \mathbf{C}^2 \arrow[dl, "\delta"] \\
& \mathbf{C} &
\end{tikzcd}$$

Recall also that a bifibration $P : \mathbf{B} \to \mathbf{C}$ is said to satisfy *Beck-Chevalley condition* if for each pullback square in $\mathbf{C}$:

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "u"'] & B \arrow[d, "g"] \\
C \arrow[r, "f"] & D
\end{tikzcd}$$

the canonical natural transformation $v_!u^* \Rightarrow g^*f_!$ is an isomorphism. The pullback stability of factorization systems has been characterized using Beck-Chevalley condition by Hughes and Jacobs as in the following proposition.

Proposition 5.1.6 ([23]) *The bifibration $\partial : \mathcal{M} \to \mathbf{C}$ induced by a factorization system $(\mathcal{E},\mathcal{M})$ in $\mathbf{C}$ satisfies Beck-Chevalley condition if and only if $\mathcal{E}$ is stable.*

To characterize $\mathcal{M}$-pullback stability of factorization systems, we introduce:

Definition 5.1.7 *Let $(\mathcal{E},\mathcal{M})$ be a factorization system in a given category $\mathbf{C}$. The bifibration $\partial : \mathcal{M} \to \mathbf{C}$ is said to satisfy* range pre-Beck-Chevalley condition *if for each* *pullback square in* $\mathbf{C}$ *in which* $e \in \mathcal{E}$ *and* $m \in \mathcal{M}$*:*

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "u"'] & B \arrow[d, "m"] \\
C \arrow[r, "e"] & D
\end{tikzcd}$$

the canonical natural transformation $v_! u^* \Rightarrow m^* e_!$ *is an isomorphism.*

Using the similar idea as in Proposition 5.1.6, we have:

Proposition 5.1.8 *The bifibration* $\partial : \mathcal{M} \to \mathbf{C}$ *induced by a factorization system* $(\mathcal{E}, \mathcal{M})$ *in* $\mathbf{C}$ *satisfies range pre-Beck-Chevalley condition if and only if* $\mathcal{E}$ *is* $\mathcal{M}$*-stable.*

Proof: Suppose that $\mathcal{E}$ is $\mathcal{M}$-stable. Given a pullback square:

$$\begin{tikzcd}
A \arrow[r, "v"] \arrow[d, "u"'] & B \arrow[d, "m"] \\
C \arrow[r, "e"] & D
\end{tikzcd}$$

in $\mathbf{C}$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$ and for each $\mathcal{M}$-map $x : X \to C$, suppose that the left and the right squares are pullback squares and $ex = e_!(x)e_1$ is an $(\mathcal{E}, \mathcal{M})$-factorization. Then there is a unique map e_2 such that $m^* e_! x \cdot e_2 = v \cdot u^* x$ and the outermost square commutes:

$$\begin{tikzcd}
\bullet \arrow[rrr, dotted, "e_2"] \arrow[dr, "u^*x"] \arrow[ddd, "m_1"'] & & & \bullet \arrow[dl, "m^*e_!x"'] \arrow[ddd, "m_2"] \\
& A \arrow[r, "v"] \arrow[d, "u"'] & B \arrow[d, "m"] & \\
& C \arrow[r, "e"] & D & \\
X \arrow[ur, "x"'] \arrow[rrr, "e_1"] & & & \bullet \arrow[ul, "e_!x"]
\end{tikzcd}$$

Obviously, by pullback composition and cancellation rules, the outermost square is a pullback square and so $m_1, m_2 \in \mathcal{M}$ and $e_1, e_2 \in \mathcal{M}$. Hence $m^* e_! x \cdot e_2$ is indeed an $(\mathcal{E}, \mathcal{M})$-factorization of $v \cdot u^* x$ and therefore $v_! u^* \cong m^* e_!$.

Conversely, assume that $\partial : \mathcal{M} \to \mathbf{C}$ satisfies range pre-Beck-Chevalley condition. Given an $\mathcal{E}$-map $e : C \to D$ and an $\mathcal{M}$-map $m : B \to D$, we form a pullback square

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "u"'] & B \arrow[d, "m"] \\ C \arrow[r, "e"] & D \end{tikzcd}$$

in $\mathbf{C}$. Then $v_! u^* \cong m^* e_!$ and so

$$v_!(1_A) \cong v_!(u^*(1_C)) \cong m^*(e_!(1_C)) \cong m^*(1_D) \cong 1_B.$$

Hence v has $(\mathcal{E}, \mathcal{M})$-factorization $1_B \cdot v$ and therefore $v \in \mathcal{E}$. Thus, $\mathcal{E}$ is $\mathcal{M}$-stable. $\square$

Suppose that $\mathcal{M}$**StabFac** is with

objects: $\mathcal{M}$-stable factorization systems $(\mathbf{C}, \mathcal{E}, \mathcal{M})$, where $\mathbf{C}$ is a category such that $\mathbf{C}$ has an $(\mathcal{E}, \mathcal{M})$-factorization system which is stable along $\mathcal{M}$-maps with $\mathcal{M} \subseteq \{\text{monics in } \mathbf{C}\}$, and $\mathbf{C}$ has pullbacks along $\mathcal{M}$-maps;

maps: $(\mathcal{E}, \mathcal{M})$-functors. A $(\mathcal{E}, \mathcal{M})$*-functor* $F : (\mathbf{C}, \mathcal{E}, \mathcal{M}) \to (\mathbf{C}', \mathcal{E}', \mathcal{M}')$ is a functor $F : \mathbf{C} \to \mathbf{C}'$ such that $F(\mathcal{E}) \subseteq \mathcal{E}'$, $F(\mathcal{M}) \subseteq \mathcal{M}'$, and F preserves pullbacks along $\mathcal{M}$-maps;

composition: as the composition of functors;

identities: $1_{(\mathbf{C}, \mathcal{E}, \mathcal{M})} = 1_{\mathbf{C}}$;

2-cells: $\mathcal{M}$-cartesian natural transformations. A natural transformation $\alpha : F \to G$ between $(\mathcal{E}, \mathcal{M})$-functors: $F, G : (\mathbf{C}, \mathcal{E}, \mathcal{M}) \to (\mathbf{C}', \mathcal{E}', \mathcal{M}')$ is $\mathcal{M}$*-cartesian* if for each $m : A \to B$ in $\mathcal{M}$,

$$\begin{tikzcd} F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(m)"'] & G(A) \arrow[d, "G(m)"] \\ F(B) \arrow[r, "\alpha_B"] & G(B) \end{tikzcd}$$

is a pullback diagram.

Then, $\mathcal{M}$**StabFac** is a 2-category. Its underlying category is denoted, as usual, by $\mathcal{M}\mathbf{StabFac}_0$.

Theorem 5.1.9 ([10], Theorem 4.3) *Let* $\mathbf{C}$ *be a category with an* $(\mathcal{E},\mathcal{M})$*-factorization system which is stable along* $\mathcal{M}$*-maps with* $\mathcal{M} \subseteq \{\text{monics in } \mathbf{C}\}$. *If* $\mathbf{C}$ *has pullbacks along* $\mathcal{M}$*-maps, then* $\mathsf{Par}(\mathbf{C},\mathcal{M})$ *is a range category with the split restriction structure given by* $\overline{(m,f)} = (m,m)$ *and the range structure given by* $\widehat{(m,f)} = (m_f, m_f)$*, where* m_f *is determined by the* $(\mathcal{E},\mathcal{M})$*-factorization of* f*:* $f = m_f e_f$ *with* $e_f \in \mathcal{E}$ *and* $m_f \in \mathcal{M}$*. Furthermore, a map is total in* $\mathsf{Par}(\mathbf{C},\mathcal{M})$ *if and only if it is total as a partial map.*

The range category induced by the $\mathcal{M}$-stable factorization system $(\mathbf{C},\mathcal{E},\mathcal{M})$ in Theorem 5.1.9 will be denoted by $\mathsf{Par}(\mathbf{C},\mathcal{E},\mathcal{M})$. As explained in [10], we have:

Proposition 5.1.10 *There is a 2-functor* $\mathsf{Par} : \mathcal{M}\mathbf{StabFac} \to \mathbf{rrCat}$ *taking*

$$F : (\mathbf{C},\mathcal{E},\mathcal{M}) \to (\mathbf{C}',\mathcal{E}',\mathcal{M}')$$

to

$$\mathsf{Par}(F) : \mathsf{Par}(\mathbf{C},\mathcal{E},\mathcal{M}) \to \mathsf{Par}(\mathbf{C}',\mathcal{E}',\mathcal{M}')$$

given by sending (m,f) *to* $(F(m),F(f))$.

5.1.4 Completeness of Range Categories

Let $\mathbf{D}$ be a range category with split restriction. Then consider the following:

$$\mathcal{E}_{\mathbf{D}} = \{f : X \to Y \,\text{in}\, \mathsf{Total}(\mathbf{D}) \mid \widehat{f} = 1_Y\}$$

and

$$\mathcal{M}_{\mathbf{D}} = \{m : X \to Y \,\text{in}\, \mathsf{Total}(\mathbf{D}) \mid \exists\; r : Y \to X \text{ in } \mathbf{D},\; rm = 1_X \text{ and } \overline{r} = mr\}.$$

We have:

Theorem 5.1.11 ([10], Theorem 4.4) *If* $\mathbf{D}$ *is a range category with split restriction, then* $\mathsf{Total}(\mathbf{D})$ *admits the* $(\mathcal{E}_\mathbf{D}, \mathcal{M}_\mathbf{D})$*-factorization system which is stable along* $\mathcal{M}_\mathbf{D}$*-maps and* $\mathsf{Total}(\mathbf{D})$ *has pullbacks along* $\mathcal{M}_\mathbf{D}$*-maps, where* $\mathcal{E}_\mathbf{D}$ *and* $\mathcal{M}_\mathbf{D}$ *are as above with*

$$\mathcal{M}_\mathbf{D} \subseteq \{\text{monics in } \mathsf{Total}(\mathbf{D})\}.$$

More precisely, for each map f *in* $\mathsf{Total}(\mathbf{D})$*, the* $(\mathcal{E}_\mathbf{D}, \mathcal{M}_\mathbf{D})$*-factorization of* f *is*

$$f = m_f \cdot r_f f$$

where $\widehat{f} = m_f r_f$ *with* $r_f m_f = 1$.

If $F : \mathbf{C} \to \mathbf{D}$ is a range functor between two range categories with split restriction, then we have a functor $\mathsf{Total}(F) : \mathsf{Total}(\mathbf{C}) \to \mathsf{Total}(\mathbf{D})$ by restricting F to $\mathsf{Total}(\mathbf{C})$. The construction of pullbacks in $\mathsf{Total}(\mathbf{D})$ (see Lemma 1.6.21) yields that $\mathsf{Total}(F)$ preserves pullbacks along $\mathcal{M}_\mathbf{C}$-maps. Obviously, $\mathsf{Total}(F)\mathcal{E}_\mathbf{C} \subseteq \mathcal{E}_\mathbf{D}$, and $\mathsf{Total}(F)\mathcal{M}_\mathbf{C} \subseteq \mathcal{M}_\mathbf{D}$. Hence, we have a functor

$$\mathsf{Total}(F) : (\mathsf{Total}(\mathbf{C}), \mathcal{E}_\mathbf{C}, \mathcal{M}_\mathbf{C}) \to (\mathsf{Total}(\mathbf{D}), \mathcal{E}_\mathbf{D}, \mathcal{M}_\mathbf{D})$$

and therefore a functor $\mathsf{Total} : \mathbf{rrCat}_s \to \mathcal{M}\mathbf{StabFac}$ given by:

$$\begin{tikzcd}
\mathbf{C} \arrow[dd, "F"'] & \mapsto & (\mathsf{Total}(\mathbf{C}), \mathcal{E}_\mathbf{C}, \mathcal{M}_\mathbf{C}) \arrow[dd, "\mathsf{Total}(F)"] \\
 & \mapsto & \\
\mathbf{D} & \mapsto & (\mathsf{Total}(\mathbf{D}), \mathcal{E}_\mathbf{D}, \mathcal{M}_\mathbf{D})
\end{tikzcd}$$

Range categories with split restrictions are essentially the same as $\mathcal{M}$-stable factorization systems.

Theorem 5.1.12 ([10], Theorem 4.5) *The* 2*-functors* Total *and* Par *give an equivalence of* 2*-categories between* $\mathbf{rrCat}_s$ *and* $\mathcal{M}\mathbf{StabFac}$.

By Proposition 5.1.5 and Theorem 5.1.12, immediately one has:

Theorem 5.1.13 (Completeness of Range Categories [10]) *Every range category embeds via a full and faithful range preserving functor into a range category of the form* $\mathsf{Par}(\mathbf{C}, \mathcal{E}, \mathcal{M})$*, where* $\mathbf{C}$ *has the* $\mathcal{M}$*-stable factorization system* $(\mathcal{E}, \mathcal{M})$*.*

Similar to Corollary 1.6.24, we have the following corollary.

Corollary 5.1.14 *For given* $\mathcal{M}$*-stable factorization systems* $(\mathbf{C}, \mathcal{E}, \mathcal{M})$ *and* $(\mathbf{C}', \mathcal{E}', \mathcal{M}')$*, the following are equivalent:*

(*i*) $\mathsf{Par}(\mathbf{C}, \mathcal{E}, \mathcal{M}) \approx \mathsf{Par}(\mathbf{C}', \mathcal{E}', \mathcal{M}')$ *in* $\mathbf{rrCat}$*;*

(*ii*) $(\mathbf{C}, \mathcal{E}, \mathcal{M}) \approx (\mathbf{C}', \mathcal{E}', \mathcal{M}')$ *in* $\mathcal{M}\mathbf{StabFac}$*;*

(*iii*) *There are category equivalences* $F : \mathbf{C} \to \mathbf{C}'$ *and* $G : \mathbf{C}' \to \mathbf{C}$ *such that* $F(\mathcal{M}) \subseteq \mathcal{M}'$*,* $F(\mathcal{E}) \subseteq \mathcal{E}'$*,* $G(\mathcal{M}') \subseteq \mathcal{M}$*, and* $G(\mathcal{E}') \subseteq \mathcal{E}$*.*

PROOF: "(*i*) $\Leftrightarrow$ (*ii*):" As Par and $\mathcal{M}\mathsf{Total}$ are part of equivalences between $\mathbf{rCat}_s$ and $\mathcal{M}\mathbf{Cat}$, it is clear.

"(*ii*) $\Rightarrow$ (*iii*):" Assume that $F : (\mathbf{C}, \mathcal{E}, \mathcal{M}) \to (\mathbf{C}', \mathcal{E}', \mathcal{M}')$ and $G : (\mathbf{C}', \mathcal{E}', \mathcal{M}') \to (\mathbf{C}, \mathcal{E}, \mathcal{M})$ are such that $GF \approx 1_{(\mathbf{C}, \mathcal{E}, \mathcal{M})}$ and $FG \approx 1_{(\mathbf{C}'\mathcal{E}', \mathcal{M}')}$. Then, obviously, $GF \approx 1_{\mathbf{C}}$ and $FG \approx 1_{\mathbf{C}'}$ with $F(\mathcal{M}) \subseteq \mathcal{M}'$, $F(\mathcal{E}) \subseteq \mathcal{E}'$, $G(\mathcal{M}') \subseteq \mathcal{M}$, and $G(\mathcal{E}') \subseteq \mathcal{E}$.

"(*iii*) $\Rightarrow$ (*ii*):" Clearly, F and G give rise to $(\mathcal{E}, \mathcal{M})$-functors such that $GF \approx 1_{(\mathbf{C}, \mathcal{E}, \mathcal{M})}$ and $FG \approx 1_{(\mathbf{C}', \mathcal{E}', \mathcal{M}')}$. $\square$

To split idempotents in inverse categories, we need the following lemma.

Lemma 5.1.15 *For a given map* $f : X \to Y$ *in an inverse category* $\mathbf{I}$*,*

(*i*) *the following are equivalent:*

(*a*) $\overline{f} = 1_X$ $(\widehat{f} = 1_Y)$*;*

(*b*) f *is a split monic (epic);*

(c) *f is a restriction monic (epic).*

(ii) $\overline{f} = 1_X$ *and* $\widehat{f} = 1_Y$ *if and only if* f *is an isomorphism.*

PROOF:

(i) We shall prove the “monic” case as the “epic” case is similar.

“$(a) \Rightarrow (b)$:” $\overline{f} = f^{(-1)}f = 1_X$ implies clearly that f is a split monic.

“$(b) \Rightarrow (a)$” and “$(c) \Rightarrow (a)$:” Since each monic is total.

“$(a) \Rightarrow (c)$:” $\overline{f} = 1_X$ implies $f^{(-1)}f = 1_X$. But

$$\overline{ff^{(-1)}} = (ff^{(-1)})^{(-1)}(ff^{(-1)}) = ff^{(-1)}ff^{(-1)} = ff^{(-1)}.$$

So f is a restriction monic.

(ii) By *(i)*.

$\square$

Given an inverse category $\mathbf{I}$, by Lemma 1.6.12 $\mathsf{Split}(\mathbf{I})$ is again an inverse category with the split restriction so that by Example 5.1.2 (4) $\mathsf{Split}(\mathbf{I})$ can be viewed as a range category with the split restriction. Now, applying Theorem 5.1.12 we have an equivalence of range categories:

$$\mathsf{Split}(\mathbf{I}) \approx \mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{I})), \mathcal{E}_{\mathsf{Split}(\mathbf{I})}, \mathcal{M}_{\mathsf{Split}(\mathbf{I})}),$$

where, by Lemma 5.1.15,

$$\mathcal{E}_{\mathsf{Split}(\mathbf{I})} = \{\text{isomorphisms in } \mathsf{Split}(\mathbf{I})\}$$

and

$$\mathcal{M}_{\mathsf{Split}(\mathbf{I})} = \{\text{split monics in } \mathsf{Split}(\mathbf{I})\}.$$

Clearly, Yoneda embedding

$$\mathcal{Y} : (\mathsf{Total}(\mathsf{Split}(\mathbf{I})), \mathcal{E}_{\mathsf{Split}(\mathbf{I})}, \mathcal{M}_{\mathsf{Split}(\mathbf{I})})$$

$$\to (\text{SplitMonics}(\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{I}))^{\mathrm{op}}}), \{\text{Isomorphisms}\}, \{\text{Split monics}\})$$

is an $(\mathcal{E}, \mathcal{M})$-functor and

$$\mathsf{Par}(\mathcal{Y}) : \mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{I})), \mathcal{E}_{\mathsf{Split}(\mathbf{I})}, \mathcal{M}_{\mathsf{Split}(\mathbf{I})})$$

$$\to \mathsf{Par}(\text{SplitMonics}(\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{I}))^{\mathrm{op}}}), \{\text{Isomorphisms}\}, \{\text{Split monics}\})$$

is a full and faithful range functor. Thus, clearly, we have a full and faithful range functor:

$$\mathbf{I} \hookrightarrow \mathsf{Split}(\mathbf{I}) \approx \mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{I})), \mathcal{E}_{\mathsf{Split}(\mathbf{I})}, \mathcal{M}_{\mathsf{Split}(\mathbf{I})})$$

$$\to \mathsf{Par}(\text{SplitMonics}(\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{I}))^{\mathrm{op}}}), \{\text{Isomorphisms}\}, \{\text{Split monics}\}).$$

5.2 Schein's Representation Theorem for Range Categories

Let $\mathcal{P}(X)$ be the set of partial functions on a set X. As described in [24], $\mathcal{P}(X)$ has some interesting operations $R, L, I, J, \wedge, \curlywedge$. For example, the operation R can be defined by the following four R-axioms:

(R1) $xR(x) = x$,

(R2) $R(x)R(y) = R(y)R(x)$,

(R3) $R(R(x)) = R(x)$, and

(R4) $R(xy)R(y) = R(xy)$.

R is called *twisted* if

(R5) $R(x)y = yR(xy)$.

L can be defined dually by the following four L-axioms:

(L1) $L(x)x = x$,

(L2) $L(x)L(y) = L(y)L(x)$,

(L3) $L(L(x)) = L(x)$, and

(L4) $L(x)L(xy) = L(xy)$.

Recall that a semigroup with extra operations is *representable* if it is isomorphic to a subalgebra of $\mathcal{P}(X)$ endowed with some subset Λ of the set of operations $\{R, L, I, J, \wedge, \curlywedge\}$. Schweizer and Sklar ([39], [40], [41], [42]) introduced the notion of a *type* 2 *function system*, which one can show is equivalent to an $\{R, L\}$-semigroup in which R is twisted and L satisfies the law:

(L5) $L(xy) = L(xL(y))$.

A type 2 function system is called a *type* 3 *function system* if it satisfies Schein's condition:

(L6) $xz = yz \Rightarrow xL(z) = yL(z)$.

Schein proved that each type 3 function system is representable ([43], [24]).

Clearly, each type 3 function system can be viewed as a range category with one object. In this section, we shall fit Schein's proof for representing $\{R, L\}$-semigroups to range categories. Throughout this section, $\mathbf{C}$ is a range category.

5.2.1 Permissible Arrow Chains

Let us first observe when an assignment $\mathcal{T} : \mathbf{C} \to \mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$ is a range functor.

Lemma 5.2.1 *Let* $\mathcal{T} : \mathbf{C} \to \mathsf{Par}(\mathbf{Set}, \text{Epics}_{\mathbf{Set}}, \text{Monics}_{\mathbf{Set}})$ *be given by sending* $\mathbf{C}$*-map* $f : X \to Y$ *to to the partial function*

$$\begin{tikzcd}
& \mathrm{d}(f) \arrow[dl, hook'] \arrow[dr, "T_f"] & \\
\mathrm{pd}(f) & & \mathrm{c}(f)
\end{tikzcd}$$

where $\mathrm{c}(f) \subseteq \mathrm{pd}(f)$*. If*

(S1) *for each* $\mathbf{C}$*-object* X*,* $T_{1_X} : \mathrm{d}(1_X) \to \mathrm{c}(1_X)$ *is* $1_{\mathrm{d}(1_X)}$*,*

(S2) *for each pair of composable* $\mathbf{C}$*-maps* $f : X \to Y$ *and* $g : Y \to Z$*,* $T_f^{-1}(\mathrm{d}(g)) = \mathrm{d}(gf)$ *and* $T_g T_f = T_{gf}$*,*

(S3) *for each* $\mathbf{C}$*-map* $f : X \to Y$*,* $T_{\overline{f}} : \mathrm{d}(\overline{f}) \to \mathrm{c}(\overline{f})$ *is the inclusion* $\mathrm{d}(f) \hookrightarrow \mathrm{pd}(f)$*, and*

(S4) *for each* $\mathbf{C}$*-map* $f : X \to Y$*,* $T_{\widehat{f}} : \mathrm{d}(\widehat{f}) \to \mathrm{c}(\widehat{f})$ *is the inclusion* $T_f(\mathrm{d}(f)) \hookrightarrow \mathrm{c}(f)$*,*

then $\mathcal{T}$ *is a range functor.*

PROOF: From (**S1**) and (**S2**), $\mathcal{T}$ is a functor. (**S3**) implies that $\mathcal{T}$ is a restriction functor while (**S4**) implies that $\mathcal{T}$ preserves the range $\widehat{f}$. ☐

According to Schein [43], a *permissible arrow chain (pac)* in $\mathbf{C}$ is an odd length (the number of maps) tuple $\alpha = \langle f_0, g_1, f_1, \cdots, g_i, f_i, \cdots \rangle$ of $\mathbf{C}$-maps such that

[pac.1] $\widehat{f_i} = \widehat{g_i}$ and $\overline{f_i} = \overline{g_{i+1}}$ that can be presented graphically:

$$\begin{tikzcd}
& \bullet \arrow[dl, "f_0"'] \arrow[dr, "g_1"] & & \bullet \arrow[dl, "f_1"'] & \cdots & \bullet \arrow[dr, "g_i"] & & \bullet \arrow[dl, "f_i"'] & \cdots \\
A & & \bullet & & & & \bullet & &
\end{tikzcd}$$

for all integers i that make senses.

A pac $\langle f_0, g_1, f_1, \cdots, g_i, f_i, \cdots \rangle$ in $\mathbf{C}$ is *reduced (rpac)* if

[rpac] for all integers i that make senses and all $\mathbf{C}$-maps x, $xg_{i+1} \neq f_i$.

Clearly, given a pac(rpac) $\langle f_0, g_1, f_1, \cdots \rangle$ in $\mathbf{C}$, a part $\langle f_{i-1}, g_i, \cdots \rangle$ of $\langle f_0, g_1, f_1, \cdots \rangle$ is again a pac(rpac).

Example 5.2.2 1. Let f be a $\mathbf{C}$-map. Then $\langle f \rangle$ is a rpac.

2. For each $\mathbf{C}$-map $f : A \to B$ that is not a section and $\overline{f} = 1_A$ (for example, a monic that is not a section) and each $\mathbf{C}$-map $g : A \to B$, $\langle 1_A, f, \widehat{f} \rangle$ is a rpac and $\langle g, 1_A, 1_A \rangle$ is always not a rpac.

Parallel to (**L6**), we assume further that the range category $\mathbf{C}$ satisfies

[**RR**.**6**] $xh = yh \Rightarrow x\widehat{h} = y\widehat{h}$.

Example 5.2.3 1. Each type 3 function system is a one-object category that satisfies [**RR**.**6**].

2. Each inverse category satisfies [**RR**.**6**] as $gf = hf \Rightarrow gff^{(-1)} = hff^{(-1)} \Leftrightarrow g\widehat{f} = h\widehat{f}$.

3. We will see shortly that $\mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$ also always satisfies [**RR**.**6**] (in Proposition 5.3.1) but

$$\mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{Set})), \mathcal{E}_{\mathsf{Split}(\mathbf{Set})}, \mathcal{M}_{\mathsf{Split}(\mathbf{Set})})$$

does not satisfy [**RR**.**6**] (in Example 5.3.9).

Let A be an object in $\mathbf{C}$ and let

$$\mathbf{pac}(\mathbf{C}) = \{\text{pacs in } \mathbf{C}\}, \ \mathbf{pac}(A) = \{\text{pacs } \langle f_0, g_1, f_1, \cdots \rangle \text{ in } \mathbf{C} \mid \mathrm{cod}(f_0) = A\},$$

and

$$\mathbf{rpac}(\mathbf{C}) = \{\text{rpacs in } \mathbf{C}\}, \ \mathbf{rpac}(A) = \{\text{rpacs } \langle f_0, g_1, f_1, \cdots \rangle \text{ in } \mathbf{C} \mid \mathrm{cod}(f_0) = A\}.$$

As Schein did in [43], $\langle f_0, g_1, f_1, \cdots \rangle < h$ if $\overline{h} f_0 = f_0$ and define $h\langle f_0, g_1, f_1, \cdots \rangle = \langle hf_0, g_1, f_1, \cdots \rangle$ when hf_0 is composable.

In particular, for each $\alpha \in \mathbf{rpac}(A)$ and each total map m with $\mathrm{dom}(m) = A$, $\alpha < m$ always.

For a given $\alpha = \langle f_0, g_1, f_1, \cdots \rangle \in \mathbf{pac}(\mathbf{C})$ and a $\mathbf{C}$-map h such that $h\alpha$ is composable, $h\alpha = \langle hf_0, g_1, f_1, \cdots \rangle$ may not be in $\mathbf{pac}(\mathbf{C})$ since $\overline{hf_o}$ may not equal to $\overline{g_1}$. But, if $\alpha = \langle f_0, g_1, f_1, \cdots \rangle < h$, then $h\alpha = h\langle f_0, g_1, f_1, \cdots \rangle = \langle hf_0, g_1, f_1, \cdots \rangle$ is a pac again since

$$\overline{hf_0} = \overline{\overline{h}f_0} = \overline{f_0} = \overline{g_1}.$$

So we have:

Lemma 5.2.4 *Given a pac α, if $\alpha < h$, then $h\alpha$ is also a pac.*

If $\alpha = \langle f_0, g_1, f_1, \cdots \rangle \in \mathbf{rpac}(\mathbf{C})$ and $\alpha < h$, then the pac $h\alpha$ may not be a rpac. In this case, we can use the following reduction operation rd to reduce $h\langle f_0, g_1, f_1, \cdots \rangle$ into a rpac.

$\langle f_0, g_1, f_1, \cdots \rangle \in \mathbf{pac}(\mathbf{C})$ is said to be *reduced* to $\langle xf_1, g_2, f_2, \cdots \rangle$ if $f_0 = xg_1$ for some map x, written as $\mathrm{rd}\langle f_0, g_1, f_1, \cdots \rangle = \langle xf_1, g_2, f_2, \cdots \rangle$.

Given a rpac α with $\alpha < h$, we can apply the reduction operation rd to $h\alpha = \langle hf_0, g_1, f_1, \cdots \rangle$ k-steps anywhere rd is applicable from the beginning until rd can not be applied further so that we obtain a rpac $\mathrm{rd}^k(h\alpha)$ and the rpac $\mathrm{rd}^k(h\alpha)$ is denoted by $\mathrm{red}(h\alpha)$.

5.2.2 Technical Lemmas

Here are some technical results on permissible arrow chains.

Lemma 5.2.5 *If $\langle f_0, g_1, f_1, \cdots \rangle \in \mathbf{pac}(\mathbf{C})$ and $f_0 = xg_1$, then $\beta = \langle f_1, g_1, \cdots \rangle < x$ and so $\langle xf_1, g_1, \cdots \rangle \in \mathbf{pac}(\mathbf{C})$. Moreover, if $\alpha \in \mathbf{rpac}(\mathbf{C})$ and $\alpha < h$, then $\mathrm{red}(h\alpha) \in \mathbf{rpac}(\mathbf{C})$.*

Proof: Since

$$\begin{aligned}
\overline{xf_1} &= \overline{x\widehat{f_1}f_1} \ \ ([\mathbf{RR.2}]) \\
&= \overline{x\widehat{g_1}f_1} \ \ (\widehat{f_1} = \widehat{g_1}) \\
&= \overline{\overline{x}\widehat{g_1}f_1} \ \ (\overline{gf} = \overline{\overline{g}f}) \\
&= \overline{\widehat{\overline{x}g_1}f_1} \ \ ([\mathbf{RR.3}]) \\
&= \overline{\widehat{g_1\overline{xg_1}}f_1} \ \ ([\mathbf{RR.4}]) \\
&= \overline{\widehat{g_1\overline{f_0}}f_1} \ \ (xg_1 = f_0) \\
&= \overline{\widehat{g_1\overline{g_1}}f_1} \ \ (\overline{f_0} = \overline{g_1}) \\
&= \overline{\widehat{g_1}f_1} \ \ ([\mathbf{R.1}]) \\
&= \overline{\widehat{f_1}f_1} \ \ (\widehat{f_1} = \widehat{g_1}) \\
&= \overline{f_1},
\end{aligned}$$

we have $\overline{x}f_1 = f_1\overline{xf_1} = f_1\overline{f_1} = f_1$. Then $\beta = \langle f_1, g_2, \cdots \rangle < x$ and so $\langle xf_1, g_2, \cdots \rangle \in$ **pac**(**C**).

If $\alpha \in$ **rpac**(**C**) and $\alpha < h$, then red$(h\alpha) \in$ **pac**(**C**) and satisfies [**rpac**] and so red$(h\alpha) \in$ **rpac**(**C**). ☐

Lemma 5.2.6 *$\alpha < f$ if and only if $\overline{f}\alpha = \alpha$.*

Proof: Assume that $\alpha = \langle f_0, g_1, f_1, \cdots \rangle$. Then $\alpha < f$ if and only if $\overline{f}f_0 = f_0 \Leftrightarrow \overline{\overline{f}}f_0 = f_0 \Leftrightarrow \overline{f}\alpha = \alpha$. ☐

Lemma 5.2.7 *Given a pac α, reduction on α is well-defined.*

Proof: Assume $\alpha = \langle f_0, g_1, f_1, \cdots \rangle$. If $f_0 = xg_1 = yg_1$, then, by [**RR.6**], we have $x\widehat{g_1} = y\widehat{g_1}$ and so $x\widehat{f_1} = y\widehat{f_1}$. Hence $xf_1 = yf_1$ and therefore $\langle xf_1, g_2, \cdots \rangle = \langle yf_1, g_2, \cdots \rangle$, as desired. ☐

Lemma 5.2.8 *For each pac* α *with* $\alpha < \overline{f}$, $\mathrm{red}(\overline{f}\alpha) = \mathrm{red}(\alpha)$.

PROOF: Since

$$\alpha < \overline{f} \Leftrightarrow \overline{f}\alpha = \alpha,$$

we have $\mathrm{red}(\overline{f}\alpha) = \mathrm{red}(\alpha)$. □

Lemma 5.2.9 *Given a pac* α*, then* $\alpha < h$ *if and only if* $\mathrm{red}(\alpha) < h$.

PROOF: Assume that $\alpha = \langle f_0, g_1, f_1, \cdots \rangle$ is reduced to $\beta = \langle xf_1, g_2, f_2, \cdots \rangle$ with $f_0 = xg_1$. Note that

$$\begin{aligned} \alpha < h &\Leftrightarrow \overline{h}f_0 = f_0 \\ &\Leftrightarrow \overline{h}xg_1 = xg_1 \\ &\Leftrightarrow \overline{h}x\widehat{g_1} = x\widehat{g_1} \quad ([\mathbf{RR}.\mathbf{6}]) \\ &\Leftrightarrow \overline{h}x\widehat{f_1} = x\widehat{f_1} \\ &\Leftrightarrow \overline{h}xf_1 = xf_1 \quad ([\mathbf{RR}.\mathbf{6}]) \\ &\Leftrightarrow \beta < h. \end{aligned}$$

□

Lemma 5.2.10 *Given a pac* $\alpha = \langle f_0, g_1, f_1, \cdots \rangle$*, if* $\alpha < h_1$ *and* $\alpha < h_2$*, then* $h_1\alpha = h_2\alpha \Leftrightarrow \mathrm{red}(h_1\alpha) = \mathrm{red}(h_2\alpha)$.

PROOF: "⇒": Obviously.

"⇐": Assume that after being reduced $k-1$ steps, we have

$$\mathrm{red}(h_1\alpha) = \langle x_kf_k, g_{k+1}, f_{k+1}, \cdots \rangle = \mathrm{red}(h_2\alpha).$$

Then

$$\begin{aligned} h_1 f_0 &= x_1 g_1, \\ x_1 f_1 &= x_2 g_2, \\ &\vdots \\ x_{k-1} f_{k-1} &= x_k g_k, \\ h_2 f_0 &= y_1 g_1, \\ y_1 f_1 &= y_2 g_2, \\ &\vdots \\ y_{k-1} f_{k-1} &= y_k g_k, \\ x_k f_k &= y_k f_k. \end{aligned}$$

for some x_i, y_j, $1 \leq i, j \leq k$. By **[RR.6]** $x_k f_k = y_k f_k$ implies $x_k \widehat{f_k} = y_k \widehat{f_k}$. That is $x_k \widehat{g_k} = y_k \widehat{g_k}$. Hence $x_k g_k = y_k g_k$. Since $x_{k-1} f_{k-1} = x_k g_k$ and $y_{k-1} f_{k-1} = y_k g_k$, $x_{k-1} f_{k-1} = y_{k-1} f_{k-1}$. Continuing in this way, we have $x_1 g_1 = y_1 g_1$. Thus, $h_1 f_0 = h_2 f_0$. Therefore, $h_1 \alpha = h_2 \alpha$. □

Lemma 5.2.11 *If a pac $\alpha < f$, then* $\mathrm{red}(f\alpha) = \mathrm{red}(f \cdot \mathrm{red}(\alpha))$.

PROOF: It suffices to prove that $f\alpha$ can be reduced to $f \cdot \mathrm{red}(\alpha)$. Assume $\alpha = \langle f_0, g_1, f_1, \cdots \rangle$ and $\mathrm{red}(\alpha) = \langle x_k f_k, g_{k+1}, f_{k+1}, \cdots \rangle$. Then

$$\begin{aligned} f_0 &= x_1 g_1, \\ x_1 f_1 &= x_2 g_1, \\ &\vdots \\ x_{k-1} f_{k-1} &= x_k g_k. \end{aligned}$$

Composing f from the left, we have

$$\begin{aligned} ff_0 &= fx_1g_1, \\ fx_1f_1 &= fx_2g_1, \\ &\vdots \\ fx_{k-1}f_{k-1} &= fx_kg_k. \end{aligned}$$

This means that $f\alpha$ can be reduced to $f \cdot \mathrm{red}(\alpha)$, as desired. ☐

Lemma 5.2.12 *If a pac $\alpha < g$ and $\alpha < h$, then $g\alpha = h\alpha \Leftrightarrow \mathrm{red}(g \cdot \mathrm{red}(\alpha)) = \mathrm{red}(h \cdot \mathrm{red}(\alpha))$.*

Proof: Since

$$\begin{aligned} g\alpha = h\alpha &\Leftrightarrow \mathrm{red}(g\alpha) = \mathrm{red}(h\alpha) \ \text{(Lemma 5.2.10)} \\ &\Leftrightarrow \mathrm{red}(g \cdot \mathrm{red}(\alpha)) = \mathrm{red}(h \cdot \mathrm{red}(\alpha)) \ \text{(Lemma 5.2.11)}. \end{aligned}$$

☐

Lemma 5.2.13 *If a pac $\alpha < g$ and $\alpha < h$, then $g\alpha = h\alpha \Leftrightarrow g \cdot \mathrm{red}(\alpha) = h \cdot \mathrm{red}(\alpha)$. In particular, $g\alpha = \alpha \Leftrightarrow g \cdot \mathrm{red}f\alpha) = \mathrm{red}(\alpha)$.*

Proof: Since

$$\begin{aligned} g\alpha = h\alpha &\Leftrightarrow \mathrm{red}(g \cdot \mathrm{red}(\alpha)) = \mathrm{red}(h \cdot \mathrm{red}(\alpha)) \ \text{(Lemma 5.2.12)} \\ &\Leftrightarrow g \cdot \mathrm{red}(\alpha) = h \cdot \mathrm{red}(\alpha) \ \text{(Lemma 5.2.10)}. \end{aligned}$$

☐

Lemma 5.2.14 *Assume that* $\beta = \langle u_0, v_1, u_1, \cdots \rangle \in \mathbf{rpac}(\mathbf{C})$ *and* f *is a* $\mathbf{C}$*-map. If* $\beta < \widehat{f}$ *and* $\alpha = \langle \overline{\widehat{u_0}f}, \widehat{u_0}f, \beta \rangle$*, then*

- (i) $\alpha \in \mathbf{pac}(\mathbf{C})$,
- (ii) $\alpha < f$,
- (iii) $\rho_f(\mathrm{red}(\alpha)) = \beta$.

Proof:

- (i) $\beta < \widehat{f}$ gives $\overline{\widehat{f}}u_0 = u_0$. Since $\overline{\overline{\widehat{u_0}f}} = \overline{\widehat{u_0}f}$ and $\widehat{\widehat{u_0}f} = \widehat{\widehat{u_0}\widehat{f}} = \widehat{\widehat{f}\widehat{u_0}} = \widehat{\widehat{f}u_0} = \widehat{u_0}$, we have $\alpha \in \mathbf{pac}(\mathbf{C})$.
- (ii) $\overline{f}\;\;\overline{\widehat{u_0}f} = \overline{\widehat{u_0}f}$ implies $\alpha < f$.
- (iii) Since

$$\begin{aligned}
\rho_f(\mathrm{red}(\alpha)) &= \mathrm{red}(f \cdot \mathrm{red}(\alpha)) \\
&= \mathrm{red}(f\alpha) \;\; \text{(Lemma 5.2.13)} \\
&= \mathrm{red}(\langle f\overline{\widehat{u_0}f}, \widehat{u_0}f, \beta \rangle) \\
&= \mathrm{red}(\langle \widehat{u_0}f, \widehat{u_0}f, \beta \rangle) \\
&= \beta.
\end{aligned}$$

□

5.2.3 Schein's Representation for Range Categories

Define $\mathcal{S} : \mathbf{C} \to \mathsf{Par}(\mathbf{Sets}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$ by sending $f : A \to B$ to a partial map $\xi_f = (i, \rho_f) : \mathbf{rpac}(A) \to \mathbf{rpac}(B)$, where $|\rho_f| = \{\alpha \in \mathbf{rpac}(A) : \alpha < f\}$ and $\rho_f(\alpha) = \mathrm{red}(f\alpha)$:

$$\begin{tikzcd}
 & {|\rho_f|} \arrow[dl, hook', "i"'] \arrow[dr, "\rho_f"] & \\
\mathbf{rpac}(A) & & \mathbf{rpac}(B)
\end{tikzcd}$$

That is, with Schein's representation, $\mathrm{d}(f) = |\rho_f|$, $\mathrm{pd}(f) = \mathbf{rpac}(A)$, $\mathrm{c}(f) = \mathbf{rpac}(B)$, and $T_f = \rho_f$.

Note that, to have Schein's representation $\mathcal{S}$ well-defined, one must assume that the range category $\mathbf{C}$ satisfies [**RR**.**6**]. We have:

Lemma 5.2.15 *$\mathcal{S}$ is a range functor.*

Proof: By Lemma 5.2.1, we need to verify conditions (**S1**)-(**S4**).

(S1) For each $\mathbf{C}$-object A and each $\alpha = \langle f_0, g_1, f_1, \cdots \rangle \in \mathbf{rpac}(A)$, $\overline{1_A} f_0 = f_0$ implies $|\rho_{1_A}| = \mathbf{rpac}(A)$ and $\rho_{1_A}(\alpha) = \mathrm{red}(1_A \alpha) = \mathrm{red}(\alpha) = \alpha$ implies $\rho_{1_A} = T_{1_A}$. Hence $T_{1_A} = \rho_{1_A} = 1_{\mathcal{S}(A)}$, that is $\mathcal{S}$ preserves identities.

(S2) For any $\mathbf{C}$-maps $f : A \to B$ and $g : B \to C$, we need to check $\rho_f^{-1}(|\rho_g|) = |\rho_{gf}|$ and $\rho_g \rho_f = \rho_{gf}$. For any $\alpha = \langle f_0, g_1, f_1, \cdots \rangle \in \rho_f^{-1}(\{\beta \in \mathbf{rpac}(B), \beta < g\}) \subseteq |\rho_f|$, $\rho_f(\alpha) = \mathrm{red}(f\alpha) \in \{\beta \in \mathbf{rpac}(B), \beta < g\}$. Hence $\mathrm{red}(f\alpha) < g$ and $\alpha < f$. Thus, $\overline{g} \cdot \mathrm{red}(f\alpha) = \mathrm{red}(f\alpha)$ that implies $\overline{g}(f\alpha) = f\alpha$ and $\overline{f}\alpha = \alpha$. It follows that $\overline{g}(ff_0) = ff_0$ and $\overline{f}f_0 = f_0$. Then

$$\begin{aligned} \overline{gf}f_0 &= \overline{\overline{g}f}f_0 \\ &= f_0\overline{\overline{g}ff_0} \\ &= f_0\overline{ff_0} \\ &= \overline{f}f_0 \\ &= f_0, \end{aligned}$$

and so $\alpha < gf$. Hence $\alpha \in |\rho_{gf}|$.

Conversely, for any $\alpha = \langle f_0, g_1, f_1, \cdots \rangle \in \{\alpha \in \mathbf{rpac}(A), \alpha < gf\}$, $\alpha < gf$. Then

$$\overline{gf}f_0 = f_0$$

and so

$$\overline{f} f_0 = \overline{f}\ \ \overline{gf} f_0 = \overline{gf} f_0 = f_0$$

and

$$\overline{g} f f_0 = f \overline{gf} f_0 = f f_0.$$

Hence $\alpha < f$ and $\mathrm{red}(f\alpha) < g$ since, by Lemma 5.2.9, $f\alpha < g \Leftrightarrow \mathrm{red}(f\alpha) < g$ and therefore $\alpha \in \rho_f^{-1}(\{\beta \in \mathbf{rpac}(B), \beta < g\})$. Thus,

$$\rho_f^{-1}(\{\beta \in \mathbf{rpac}(B), \beta < g\}) = \{\alpha \in \mathbf{rpac}(A), \alpha < gf\}$$

which means $\rho_f^{-1}(|\rho_g|) = |\rho_{gf}|$.

For any $\alpha \in |\rho_{gf}|$, we have

$$\begin{aligned}
(\rho_g \rho_f)(\alpha) &= \rho_g(\mathrm{red}(f\alpha)) \\
&= \mathrm{red}(g \cdot (\mathrm{red}(f\alpha))) \\
&= \mathrm{red}(gf\alpha) \ \ \text{(Lemma 5.2.13)} \\
&= \rho_{gf}(\alpha).
\end{aligned}$$

Then $\rho_g \rho_f = \rho_{gf}$.

(S3) For each $\mathbf{C}$-map $f : A \to B$ and each $\alpha \in |\rho_{\overline{f}}| = \{\alpha \in \mathbf{rpac}(A), \alpha < \overline{f}\}$,

$$\rho_{\overline{f}}(\alpha) = \mathrm{red}(\overline{f}\alpha) = \mathrm{red}(\alpha) = \alpha.$$

Then $\rho_{\overline{f}}$ is the inclusion. Since

$$\begin{aligned}
\alpha \in |\rho_{\overline{f}}| &\Leftrightarrow \alpha < \overline{f} \\
&\Leftrightarrow \overline{\overline{f}}\alpha = \alpha \\
&\Leftrightarrow \alpha < f \\
&\Leftrightarrow \alpha \in |\rho_f|,
\end{aligned}$$

we have $|\rho_f| = |\rho_{\overline{f}}|$. Then $\rho_{\overline{f}} : |\rho_{\overline{f}}| \to \mathbf{rpac}(A)$ is the inclusion $|\rho_{\overline{f}}| \subseteq \mathbf{rpac}(A)$.

(S4) For each **C**-map $f : A \to B$ and each $\alpha \in |\rho_{\widehat{f}}| = \{\alpha \in \mathbf{rpac}(A), \alpha < \widehat{f}\}$,

$$\rho_{\widehat{f}}(\alpha) = \mathrm{red}(\widehat{f}\alpha) = \mathrm{red}(\alpha) = \alpha.$$

Then $\rho_{\widehat{f}}$ is the inclusion.

Let $\rho_f = m_{\rho_f} e_{\rho_f}$ be $(\{\text{surjections}\}, \{\text{injections}\})$-factorization of ρ_f in **Set**. It suffices to prove that

$$|m_{\rho_f}| = \rho_f(|\rho_f|) = |\rho_{\widehat{f}}|.$$

For any $\beta \in |m_{\rho_f}| = \rho_f(|\rho_f|)$, there is a rpac $\alpha = \langle f_0, \cdots \rangle \in |\rho_f|$ such that $\xi_f(\alpha) = \mathrm{red}(f\alpha) = \beta$. Since $f\alpha = \widehat{f}(f\alpha)$, $\mathrm{red}(f \cdot \alpha) = \widehat{f} \cdot \mathrm{red}(f\alpha)$ and so $\beta = \widehat{f}\beta$. Hence $\beta = \widehat{f}\beta = \overline{\widehat{f}}\beta$ and therefore $\beta < \widehat{f}$. Thus, $\beta \in |\rho_{\widehat{f}}|$.

Conversely, if $\beta = \langle u_0, \cdots \rangle \in |\rho_{\widehat{f}}|$. then $\beta < \widehat{f}$. By Lemma 5.2.14, there is a rpac α such that $\alpha \in |\rho_f|$ and $\rho_f(\alpha) = \beta$. Thus, $\beta \in \rho_f(|\rho_f|)$, as desired.

□

Lemma 5.2.16 *If* **C** *satisfies* [**RR**.**6**], *then* $\mathcal{S} : \mathbf{C} \to \mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$ *is a faithful functor.*

Proof: By Lemma 5.2.15, it suffices to prove that $\mathcal{S}$ is faithful. For any **C**-objects A, B, any maps $f, g \in \hom_{\mathbf{C}}(A, B)$ such that $f \neq g$, we distinguish the following cases:

(1) $\overline{f} = \overline{g}$. we have $\langle \overline{f} \rangle < f, g$ and so $\rho_f(\langle \overline{f} \rangle) = \mathrm{red}(\langle f\overline{f} \rangle) = \langle f \rangle$ and $\rho_g(\langle \overline{f} \rangle) = \mathrm{red}(\langle g\overline{f} \rangle) = \langle g \rangle \neq \langle f \rangle$. Then $\xi_f \neq \xi_g$.

(2) $\overline{f} \neq \overline{g}$. In this case, we distinguish two cases:

- *(i)* $\overline{f}\overline{g} = \overline{f}$, then $\overline{f}\overline{g} \neq \overline{g}$ and so $\langle \overline{g} \rangle \in |\rho_g| \setminus |\rho_f|$.
- *(ii)* $\overline{f}\overline{g} \neq \overline{f}$, then $\langle \overline{f} \rangle \in |\rho_f| \setminus |\rho_g|$.

Hence $|\rho_g| \neq |\rho_f|$ and therefore $\xi_f \neq \xi_g$.

In all cases, $\xi_f \neq \xi_g$ and so $\mathcal{S}$ is faithful. $\square$

By Lemma 5.2.16, immediately we have:

Theorem 5.2.17 *Given a range category* $\mathbf{X}$, $\mathbf{X}$ *satisfies* [$\mathbf{RR.6}$] *if and only if there is a faithful functor*

$$\mathcal{T} : \mathbf{X} \to \mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}}).$$

Proof: "$\Leftarrow$": If there is a faithful functor $\mathcal{T} : \mathbf{X} \to \mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$, then, as explained in Example 5.2.3, $\mathbf{X}$ must satisfy [$\mathbf{RR.6}$] since $\mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$ does.

"$\Rightarrow$": By Lemma 5.2.16. $\square$

However, Schein's representation $\mathcal{S}$ is not full as shown in the following

Example 5.2.18 (*i*) Let $\mathbf{1}$ be the range category with one object 1 and one map 1_1 and with trivial restriction and trivial range. Then $\mathbf{1}$ satisfies [$\mathbf{RR.6}$] and $\mathbf{rpac}(1) = \{1_1\}$. Since 1_1 is total, $|\rho_{1_1}| = \mathbf{rpac}(1)$ and $\rho_{1_1} = 1_{\mathbf{rpac}(1)}$. That is, $\mathcal{S}(1_1) = 1_{\mathbf{rpac}(1)}$. But there is another partial map $(\emptyset, \emptyset) : \mathbf{rpac}(1) \to \mathbf{rpac}(1)$. Hence $\mathcal{S} : \mathbf{1} \to \mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$ is not full.

(*ii*) Since $\mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$ satisfies [$\mathbf{RR.6}$], there is a faithful Schein's representation

$$\mathcal{S} : \mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}}) \to \mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$$

sending each partial map

$$\begin{tikzcd} (m,f) = & & X' \arrow[dl, tail, "m"'] \arrow[dr, "f"] & \\ & X & & Y \end{tikzcd}$$

to

$$\begin{tikzcd}
& {|\rho_{(m,f)}|} \arrow[dl, hook'] \arrow[dr, "\rho_{(m,f)}"] & \\
\mathbf{rpac}(X) & & \mathbf{rpac}(Y)
\end{tikzcd}$$

Let $X = \{a\}$, $Y = \{1, 2\}$. Consider

$$\mathcal{S}|_{\hom(X,Y)} : \hom_{\mathsf{Par}(\mathbf{Set},\mathrm{Epics}_{\mathbf{Set}},\mathrm{Monics}_{\mathbf{Set}})}$$

$$\rightarrow \hom_{\mathsf{Par}(\mathbf{Set},\mathrm{Epics}_{\mathbf{Set}},\mathrm{Monics}_{\mathbf{Set}})}(\mathbf{rpac}(\mathbf{X}), \mathbf{rpac}(\mathbf{Y}))$$

and a partial map

$$(i,t) = \begin{tikzcd}
& \{1_X : X \to X\} \arrow[dl, hook', "i"'] \arrow[dr, "t"] & \\
\mathbf{rpac}(X) & & \mathbf{rpac}(Y)
\end{tikzcd}$$

in $\hom_{\mathsf{Par}(\mathbf{Set},\mathrm{Epics}_{\mathbf{Set}},\mathrm{Monics}_{\mathbf{Set}})}(\mathbf{rpac}(\mathbf{X}), \mathbf{rpac}(\mathbf{Y}))$, where t is given by sending 1_X to $1_{\{1,2\}}$. If $\mathcal{S}|_{\hom(X,Y)}$ were surjective, then there were a partial map

$$(m,f) = \begin{tikzcd}
& X' \arrow[dl, hook', "m"'] \arrow[dr, "f"] & \\
\{a\} & & \{1,2\}
\end{tikzcd}$$

such that

$$\mathcal{S}(m,f) = \begin{tikzcd}
& {|\rho_{(m,f)}|} \arrow[dl, hook'] \arrow[dr, "\rho_{(m,f)}"] & \\
\mathbf{rpac}(X) & & \mathbf{rpac}(Y)
\end{tikzcd} = (i,t)$$

but this is not possible since $t \neq \rho_{(m,f)}$ by noting that $|\rho_{(m,f)}| = \{1_X\}$ implies $\rho_{(m,f)}(1_X) = (m, f) \neq 1_{\{1,2\}} = t(1_X)$. Hence $\mathcal{S}$ is not full.

5.2.4 Interactions of Schein's Representation with Meets

Recall that, by Proposition 4.2.2, for an $\mathcal{M}$-category $(\mathbf{X}, \mathcal{M})$, if $\mathbf{X}$ has equalizers and regular monics of $\mathbf{X}$ are in $\mathcal{M}$, then $\mathsf{Par}(\mathbf{X}, \mathcal{M})$ is a meet restriction category with the meet give by the equalizer described in the proof of Proposition 4.2.2. If $\mathbf{C}$ is a meet range category (a range category with a meet operation) satisfying [**RR.6**], then one has Schein's representation

$$\mathcal{S} : \mathbf{C} \to \mathsf{par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}}).$$

Now it is natural to ask if $\mathcal{S}$ preserves meets. We have:

Proposition 5.2.19 *If $\mathbf{C}$ is a meet range category satisfying* [**RR.6**]*, then Schein's representation $\mathcal{S}$ on $\mathbf{C}$ preserves the meet of $\mathbf{C}$.*

PROOF: It suffices to prove that $\mathcal{S}(f \wedge g) = \mathcal{S}(f) \wedge \mathcal{S}(g)$ for each pair of parallel $\mathbf{C}$-maps $f, g : A \to B$.

Recall that $\mathcal{S}(f \wedge g)$, $\mathcal{S}(f)$, and $\mathcal{S}(g)$ are given by the following partial maps respectively:

$$\begin{tikzcd}
& {|\rho_{f\wedge g}|} \arrow[dl, hook] \arrow[dr, "\rho_{f\wedge g}"] & \\
\mathbf{rpac}(A) & & \mathbf{rpac}(B),
\end{tikzcd}$$

$$\begin{tikzcd}
& {|\rho_f|} \arrow[dl, hook] \arrow[dr, "\rho_f"] & \\
\mathbf{rpac}(A) & & \mathbf{rpac}(B),
\end{tikzcd}$$

and

$$\begin{tikzcd}
& {|\rho_g|} \arrow[dl, hook] \arrow[dr, "\rho_g"] & \\
\mathbf{rpac}(A) & & \mathbf{rpac}(B)
\end{tikzcd}$$

$\mathcal{S}(f) \wedge \mathcal{S}(g)$ turns out to be the following partial map

$$\begin{tikzcd}
& E \arrow[dl, hook] \arrow[dr, "\rho_f"] & \\
\mathbf{rpac}(A) & & \mathbf{rpac}(B)
\end{tikzcd}$$

where E is given by the following equalizer diagram

$$\begin{tikzcd} E \arrow[r, hook] & {|\rho_f| \cap |\rho_g|} \arrow[r, shift left, "\rho_f"] \arrow[r, shift right, "\rho_g"'] & \mathbf{rpac}(B) \end{tikzcd}$$

so that

$$E = \{\alpha \in \mathbf{rpac}(A) \mid \alpha < f, \alpha < g, f\alpha = g\alpha\}.$$

For each $\alpha \in E$, we have

$$\overline{f}\alpha = \alpha, \overline{g}\alpha = \alpha, f\alpha = g\alpha.$$

Then

$$(f \wedge g)\alpha = f\alpha \wedge g\alpha = f\alpha = g\alpha$$

and so

$$\overline{f \wedge g}\alpha = \alpha\overline{(f \wedge g)\alpha} = \alpha\overline{f\alpha} = \overline{f}\alpha = \alpha.$$

Hence $\alpha \in |\rho_{f\wedge g}|$.

Conversely, if $\alpha \in |\rho_{f\wedge g}|$ then $\overline{f \wedge g}\alpha = \alpha$ and so

$$f\alpha = f\overline{f \wedge g}\alpha = g\overline{f \wedge g}\alpha = g\alpha,$$

$$\overline{f}\alpha = \overline{f}\ \overline{f \wedge g}\alpha = \overline{f \wedge g}\alpha = \alpha,$$

and

$$\overline{g}\alpha = \overline{g}\overline{f \wedge g}\alpha = \overline{f \wedge g}\alpha = \alpha.$$

Thus, $\alpha \in E$. So $|\rho_{f\wedge g}| = E$.

For each $\alpha \in |\rho_{f\wedge g}| = E$, clearly,

$$(f \wedge g)\alpha = f\overline{f \wedge g}\alpha = f\alpha = g\alpha.$$

That is,

$$\rho_{f\wedge g}|_{|\rho_{f\wedge g}|} = \rho_f|_E = \rho_g|_E.$$

It follows that $\mathcal{S}(f \wedge g) = \mathcal{S}(f) \wedge \mathcal{S}(g)$, as desired. □

5.3 Partial Map Categories and Condition [**RR**.**6**]

To understand why we need the condition [**RR**.**6**] in Lemma 5.2.16, we consider an $\mathcal{M}$-stable factorization system $(\mathcal{E},\mathcal{M})$ in a category $\mathbf{D}$, namely, $\mathcal{E}$ is $\mathcal{M}$-stable. Recall that if $(\mathcal{E},\mathcal{M})$ is $\mathcal{M}$-stable then $\mathsf{Par}(\mathbf{D},\mathcal{E},\mathcal{M})$ is a split range category by Theorem 5.1.9. We have:

Proposition 5.3.1 *Let* $\mathbf{D}$ *be a category and* $\mathcal{M} \subseteq \mathrm{Monics}_{\mathbf{D}}$. *If* $\mathbf{D}$ *admits an* $\mathcal{M}$*-stable factorization system* $(\mathcal{E},\mathcal{M})$*, then* $\mathsf{Par}(\mathbf{D},\mathcal{E},\mathcal{M})$ *satisfies* [**RR**.**6**] *if and only if* $\mathcal{E} \subseteq \mathrm{Epics}_{\mathbf{D}}$.

Proof: "if" part: Given an $\mathcal{M}$-stable factorization system $(\mathcal{E},\mathcal{M})$ such that $\mathcal{M} \subseteq \mathrm{Monics}_{\mathbf{D}}$ and $\mathcal{E} \subseteq \mathrm{Epics}_{\mathbf{D}}$ in a category $\mathbf{D}$, let us check out if $\mathsf{Par}(\mathbf{D},\mathcal{E},\mathcal{M})$ satisfies [**RR**.**6**]. For any partial maps $(i,h),(j,x),(k,y)$ such that $(j,x)(i,h) = (k,y)(i,h)$, let $h = m_h e_h$ be the $(\mathcal{E},\mathcal{M})$-factorization of h. Looking at the following diagrams:

$$\begin{tikzcd}
 & & D_{xh} \arrow[r, dotted, "e'_h"] \arrow[dl, hook, "j''"'] \arrow[dr, "h'"] & E' \arrow[dl, dotted, hook, "j'"'] \arrow[d, dotted, "m'_h"] & \\
 & D_h \arrow[r, dotted, "e_h"] \arrow[dl, hook, "i"'] \arrow[dr, "h"] & E \arrow[d, dotted, "m_h"] & D_x \arrow[dl, hook, "j"'] \arrow[dr, "x"] & \\
A & & B & & C
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & D_{yh} \arrow[r, dotted, "e''_h"] \arrow[dl, hook, "k''"'] \arrow[dr, "h''"] & E'' \arrow[dl, dotted, hook, "k'"'] \arrow[d, dotted, "m''_h"] & \\
 & D_h \arrow[r, dotted, "e_h"] \arrow[dl, hook, "i"'] \arrow[dr, "h"] & E \arrow[d, dotted, "m_h"] & D_y \arrow[dl, hook, "k"'] \arrow[dr, "y"] & \\
A & & B & & C
\end{tikzcd}$$

in which each square is a pullback diagram, we have

$$xh' = xm'_h e'_h = yh'' = ym''_h e''_h$$

and

$$ij'' = ik''.$$

Since i is monic, $j'' = k''$. Hence

$$j'e'_h = e_h j'' = e_h k'' = k'e''_h$$

and therefore, by the uniqueness of the factorization $j'e'_h = k'e''_h$, we get $e'_h \cong e''_h$. It follows that $xm'_h = ym''_h$, which means that $(j,x)\widehat{(i,h)} = (k,y)\widehat{(i,h)}$. So $\mathsf{Par}(\mathbf{D},\mathcal{E},\mathcal{M})$ satisfies [**RR.6**].

"only if" part: For each $\mathcal{E}$-map $e : X \to Y$ with $xe = ye$, we have

$$(1_Y, x)(1_X, e) = (1_Y, y)(1_X, e)$$

in $\mathsf{Par}(\mathbf{D},\mathcal{E},\mathcal{M})$ by the following diagram

$$\begin{tikzcd}
 & & X \arrow[dl, hook', "1_X"'] \arrow[dr, "e"] & & \\
 & X \arrow[dl, hook', "1_X"'] \arrow[dr, "e"] & & Y \arrow[dl, hook', "1_Y"'] \arrow[dr, shift left, "x"] \arrow[dr, shift right, "y"'] & \\
X & & Y & & Z
\end{tikzcd}$$

By [**RR.6**], we have $(1_Y, x)\widehat{(1_X, e)} = (1_Y, y)\widehat{(1_X, e)}$. Since e has the $(\mathcal{E},\mathcal{M})$-factorization $e = 1_Y \cdot e$, $(1_Y, x)\widehat{(1_X, e)} = (1_Y, y)\widehat{(1_X, e)}$ means that $(1_Y, x)(1_Y, 1_Y) = (1_Y, y)(1_Y, 1_Y)$:

$$\begin{tikzcd}
 & & Y \arrow[dl, hook', "1_Y"'] \arrow[dr, "1_Y"] & & \\
 & Y \arrow[dl, hook', "1_Y"'] \arrow[dr, "1_Y"] & & Y \arrow[dl, hook', "1_Y"'] \arrow[dr, shift left, "x"] \arrow[dr, shift right, "y"'] & \\
Y & & Y & & Z
\end{tikzcd}$$

Hence $x = y$ and therefore $e \in \text{Epics}_{\mathbf{D}}$. $\square$

Recall that a factorization system $(\mathcal{E},\mathcal{M})$ of a category is *proper* if each $\mathcal{E}$-map is epic and each $\mathcal{M}$-map is monic. So, by Proposition 5.3.1, we have:

Corollary 5.3.2 *If* $\mathbf{D}$ *admits an* $\mathcal{M}$*-stable factorization system* $(\mathcal{E}, \mathcal{M})$ *with* $\mathcal{M} \subseteq \mathrm{Monics}_{\mathbf{D}}$, *then* $\mathsf{Par}(\mathbf{D}, \mathcal{E}, \mathcal{M})$ *satisfies* [**RR**.**6**] *if and only if* $(\mathcal{E}, \mathcal{M})$ *is proper.*

When a range category $\mathbf{D}$ satisfies [**RR**.**6**], epics in $\mathbf{D}$ can be characterized by the range.

Lemma 5.3.3 *If a range category* $\mathbf{D}$ *satisfies* [**RR**.**6**], *then*

(i) *a map* f *is an epic* $\Leftrightarrow \widehat{f} = 1$;

(ii) $\mathrm{Epics}_{\mathbf{D}} = \{f | \widehat{f} = 1\}$.

PROOF:

(i) "$\Rightarrow$": $\widehat{f} f = f = 1 \cdot f$ and f is an epic imply $\widehat{f} = 1$.

"$\Leftarrow$": Assume $\widehat{f} = 1$. If $xf = yf$, then, by [**RR**.**6**], $x = x\widehat{f} = y\widehat{f} = y$ and so f is an epic.

(ii) By (i).

□

For epics in a partial map category, we have:

Lemma 5.3.4 *Suppose that a given category* $\mathbf{D}$ *admits an* $\mathcal{M}$*-stable factorization system* $(\mathcal{E}, \mathcal{M})$ *such that* $\mathcal{M} \subseteq \mathrm{Monics}_{\mathbf{D}}$ *and* $\mathcal{E} \subseteq \mathrm{Epics}_{\mathbf{D}}$. *If* $(m, f) : A \to B$ *is a partial map in* $\mathsf{Par}(\mathbf{D}, \mathcal{E}, \mathcal{M})$, *then the following are equivalent:*

(i) $\widehat{(m, f)} = 1_B$;

(ii) (m, f) *is an epic;*

(iii) $f \in \mathcal{E}$.

PROOF: By Proposition 5.3.1, $\mathsf{Par}(\mathbf{D}, \mathcal{E}, \mathcal{M})$ satisfies [**RR.6**]. By Lemma 5.3.3, clearly $(i) \Leftrightarrow (ii)$.

$(i) \Rightarrow (iii)$: Assume that $f = m_f e_f$ with $e_f \in \mathcal{E}$ and $m_f \in \mathcal{M}$. Then, by (i),

$$\widehat{(m,f)} = (m_f, m_f) = 1_B,$$

and so m_f is an isomorphism that belongs to $\mathcal{E}$. Hence $f = m_f e_f \in \mathcal{E}$.

$(iii) \Rightarrow (i)$: If $f \in \mathcal{E}$, then $f = 1 \cdot f$ is an $(\mathcal{E}, \mathcal{M})$-factorization of f and so

$$\widehat{(m,f)} = (1,1) = 1_B.$$

□

For a range category with the split restriction, we have:

Lemma 5.3.5 *For a given split range category* $\mathbf{D}$*, then the following are equivalent:*

- (i) $\mathbf{D}$ *satisfies* [**RR.6**]*;*
- (ii) $\mathsf{Par}(\mathsf{Total}(\mathbf{D}), \mathcal{E}_{\mathbf{D}}, \mathcal{M}_{\mathbf{D}})$ *satisfies* [**RR.6**]*;*
- (iii) $\mathcal{E}_{\mathbf{D}} \subseteq \mathrm{Epics}_{\mathsf{Total}(\mathbf{D}))}$.

PROOF: Since $\mathbf{D} \approx \mathsf{Par}(\mathsf{Total}(\mathbf{D}), \mathcal{E}_{\mathbf{D}}, \mathcal{M}_{\mathbf{D}})$ and Proposition 5.3.1. □

For the range categories obtained by splitting restriction idempotents from range categories, we have:

Lemma 5.3.6 *Given a range category* $\mathbf{C}$*, then* $\mathbf{C}$ *satisfies* [**RR.6**] *if and only if* $\mathsf{Split}(\mathbf{C})$ *satisfies* [**RR.6**].

PROOF: Assume that $\mathbf{C}$ satisfies [**RR.6**]. Let $f : e_1 \to e_2$ and $x, y : e_2 \to e_3$ be maps in $\mathsf{Split}(\mathbf{C})$ such that $xf = yf$. Then $xf = yf$ in $\mathbf{C}$ and so, by [**RR.6**], $x\widehat{f} = y\widehat{f}$. Hence $\mathsf{Split}(\mathbf{C})$ satisfies [**RR.6**].

Conversely, suppose $f : X \to Y$ and $x, y : Y \to Z$ and $\mathbf{C}$-maps such that $xf = yf$. Then we have $\mathsf{Split}(\mathbf{C})$-maps $f : 1_X \to 1_Y$ and $x, y : 1_Y \to 1_Z$ such that $xf = yf$ and so $x\widehat{f} = y\widehat{f}$ since $\mathsf{Split}(\mathbf{C})$ satisfies $[\mathbf{RR.6}]$. Hence $\mathbf{C}$ satisfies $[\mathbf{RR.6}]$. $\square$

If $\mathbf{C}$ is a range category, then, by Proposition 5.1.4 $\mathsf{Split}(\mathbf{C})$ is a split range category and by Theorem 5.1.11 $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$ admits the $(\mathcal{E}_{\mathsf{Split}(\mathbf{C})}, \mathcal{M}_{\mathsf{Split}(\mathbf{C})})$-factorization system which is stable along $\mathcal{M}_{\mathsf{Split}(\mathbf{C})}$-maps and has pullbacks along $\mathcal{M}_{\mathsf{Split}(\mathbf{C})}$-maps, where

$$\mathcal{E}_{\mathsf{Split}(\mathbf{C})} = \{f : e_1 \to e_2 \,\text{in}\, \mathsf{Total}(\mathsf{Split}(\mathbf{C})) \,|\, \widehat{f} = 1_{e_2}\}$$

and

$$\mathcal{M}_{\mathsf{Split}(\mathbf{C})} =$$

$$\{m : e_1 \to e_2 \,\text{in}\, \mathsf{Total}(\mathsf{Split}(\mathbf{C})) \,|\, \exists\; r : e_2 \to e_1 \text{ in } \mathsf{Split}(\mathbf{C}),\; rm = 1_{e_1} \text{ and } \overline{mr} = mr\}.$$

We shall denote the $\mathcal{M}_{\mathsf{Split}(\mathbf{C})}$-stable factorization system

$$(\mathsf{Total}(\mathsf{Split}(\mathbf{C})), \mathcal{E}_{\mathsf{Split}(\mathbf{C})}, \mathcal{M}_{\mathsf{Split}(\mathbf{C})})$$

by $\mathsf{K}(\mathbf{C})$.

Recall that for a given map $f : e_1 \to e_2$ in $\mathsf{Total}(\mathsf{Split}(\mathbf{C}))$, $f = \widehat{f} \cdot f$ is the $(\mathcal{E}_{\mathsf{Split}(\mathbf{C})}, \mathcal{M}_{\mathsf{Split}(\mathbf{C})})$-factorization of f, where $f : e_1 \to \widehat{f}$ and $\widehat{f} : \widehat{f} \to e_2$ are the $\mathcal{E}_{\mathsf{Split}(\mathbf{C})}$-map and the $\mathcal{M}_{\mathsf{Split}(\mathbf{C})}$-map of f, respectively.

We are now wondering when the partial maps category $\mathsf{Par}(\mathsf{K}(\mathbf{C}))$ satisfies $[\mathbf{RR.6}]$.

Proposition 5.3.7 *Given a range category* $\mathbf{C}$*, the following are equivalent:*

- *(i)* $\mathbf{C}$ *satisfies* $[\mathbf{RR.6}]$*;*
- *(ii)* $\mathsf{Split}(\mathbf{C})$ *satisfies* $[\mathbf{RR.6}]$*;*
- *(iii)* $\mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{C})), \mathcal{E}_{\mathsf{Split}(\mathbf{C})}, \mathcal{M}_{\mathsf{Split}(\mathbf{C})})$ *satisfies* $[\mathbf{RR.6}]$*;*
- *(iv)* $\mathcal{E}_{\mathbf{C}} \subseteq \mathrm{Epics}_{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))}$.

Proof: By Lemmas 5.3.5 and 5.3.6. □

By Proposition 5.3.7, we immediately have the completeness of range categories satisfying [**RR**.**6**] in partial map categories.

Theorem 5.3.8 *Given a range category* $\mathbf{X}$*,* $\mathbf{X}$ *satisfies* [**RR**.**6**] *if and only if* $\mathbf{X}$ *embeds via a full and faithful range preserving functor into a range category of the form* $\mathsf{Par}(\mathbf{Y}, \mathcal{E}, \mathcal{M})$ *with* $\mathcal{E} \subseteq \mathrm{Epics}_{\mathbf{Y}}$ *and* $\mathcal{M} \subseteq \mathrm{Monics}_{\mathbf{Y}}$*.*

But not every partial map category satisfies [**RR**.**6**] by the following example.

Example 5.3.9 In general, for a given range category $\mathbf{C}$, $\mathsf{Par}(\mathbf{C}, \mathcal{E}_{\mathbf{C}}, \mathcal{M}_{\mathbf{C}})$ may not satisfy [**RR**.**6**]. Let $\mathbf{S}$ be a category with at least one map f that is not epic. (**Set** is such a category!) Then $\mathbf{S}$ can be regarded as a trivial split range category with the inclusion

$$\mathbf{S} \hookrightarrow \mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{S})), \mathcal{E}_{\mathsf{Split}(\mathbf{S})}, \mathcal{M}_{\mathsf{Split}(\mathbf{S})}).$$

Since f is not epic, there are $\mathbf{S}$-maps x, y such that $xf = yf$ but $x \neq y$, that is $x\widehat{f} \neq y\widehat{f}$. Hence $\mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{S})), \mathcal{E}_{\mathsf{Split}(\mathbf{S})}, \mathcal{M}_{\mathsf{Split}(\mathbf{S}))}$ does not satisfy [**RR**.**6**]. This does not contradict to Propositions 5.3.1 and 5.3.7 since $\mathcal{E}_{\mathbf{S}} \nsubseteq \mathrm{Epics}_{\mathsf{Total}(\mathsf{Split}(\mathbf{S}))}$.

Recall that, by Proposition 5.1.5, there is an adjunction:

$$\begin{tikzcd}
\mathbf{rrCat}_{s0} \arrow[r, hook, shift right=1.5, "E"'] \arrow[r, phantom, "\perp"] & \mathbf{rrCat}_0 \arrow[l, shift right=1.5, "\mathsf{Split}"']
\end{tikzcd}$$

with a full and faithful unit $\eta_{\mathbf{C}} : \mathbf{C} \to E(\mathsf{Split}(\mathbf{C}))$ given by sending $f : X \to Y$ to $f : 1_X \to 1_Y$, where E is the inclusion. So, for each split range category $\mathbf{Y}$ and each range functor $F : \mathbf{X} \to E(\mathbf{Y})$, there is a unique range functor $F^\sharp : \mathsf{Split}(\mathbf{X}) \to \mathbf{Y}$ such that

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\eta_{\mathbf{X}}"] \arrow[dr, "F"'] & E(\mathsf{Split}(\mathbf{X})) \arrow[d, "E(F^\sharp)"] & \mathsf{Split}(\mathbf{X}) \arrow[d, dotted, "\exists! F^\sharp"] \\
& E(Y) & \mathbf{Y}
\end{tikzcd}$$

commutes, where $F^\sharp$ is given by mapping $f : (e_1 : X \to X) \to (e_2 : Y \to Y)$ to $F(f) : F(X) \to F(Y)$. Obviously, F is full (faithful) if and only if $F^\sharp$ is full (faithful).

By Proposition 5.3.7, if a range category $\mathbf{C}$ satisfies [$\mathbf{RR}.\mathbf{6}$], so do both $\mathsf{Split}(\mathbf{C})$ and $\mathsf{Par}(\mathsf{K}(\mathbf{C}))$ and $\mathcal{E}_{\mathbf{C}} \subseteq \mathrm{Epics}_{\mathsf{Total}(\mathsf{Split}(\mathbf{C}))}$. Hence we have:

Proposition 5.3.10 *There is the following adjunction situation:*

$$\begin{tikzcd}
\mathcal{M}\mathbf{StabFac}_{0+[\mathcal{E}\subseteq\{\text{epics}\}]} \arrow[r, shift right, "\mathsf{Par}"'] \arrow[r, phantom, "\approx"] \arrow[d, hook] & \mathbf{rrCat}_{s0+[\mathbf{RR}.\mathbf{6}]} \arrow[l, shift right, "\mathsf{Total}"'] \arrow[r, hook, shift right, "E"'] \arrow[r, phantom, "\perp"] \arrow[d, hook] & \mathbf{rrCat}_{0+[\mathbf{RR}.\mathbf{6}]} \arrow[l, shift right, "\mathsf{Split}"'] \arrow[d, hook] \\
\mathcal{M}\mathbf{StabFac}_0 \arrow[r, shift right, "\mathsf{Par}"'] \arrow[r, phantom, "\approx"] & \mathbf{rrCat}_{s0} \arrow[l, shift right, "\mathsf{Total}"'] \arrow[r, hook, shift right, "E"'] \arrow[r, phantom, "\perp"] & \mathbf{rrCat}_0 \arrow[l, shift right, "\mathsf{Split}"']
\end{tikzcd}$$

where $\mathbf{rrCat}_{0+[\mathbf{RR}.\mathbf{6}]}$ *(*$\mathbf{rrCat}_{s0+[\mathbf{RR}.\mathbf{6}]}$*) is the subcategory of* $\mathbf{rrCat}_0$ *(*$\mathbf{rrCat}_{s0}$*), comprising of those objects satisfying the condition* [$\mathbf{RR}.\mathbf{6}$]*, respectively, and* $\mathcal{M}\mathbf{StabFac}_{0+[\mathcal{E}\subseteq\{\text{epics}\}]}$ *is the subcategory of* $\mathcal{M}\mathbf{StabFac}_0$*, with* $\mathcal{M}$*-stable factorization systems* $(\mathbf{C}, \mathcal{E}, \mathcal{M})$ *such that* $\mathcal{E}$*-maps are epic in* $\mathbf{C}$*, as objects.*

From the top and bottom rows in Proposition 5.3.10, we have adjunctions:

$$\mathsf{Total} \cdot \mathsf{Split} \dashv E \cdot \mathsf{Par} : \mathcal{M}\mathbf{StabFac}_{0+[\mathcal{E}\subseteq\{\text{epics}\}]} \to \mathbf{rrCat}_{0+[\mathbf{RR}.\mathbf{6}]}$$

and

$$\mathsf{Total} \cdot \mathsf{Split} \dashv E \cdot \mathsf{Par} : \mathcal{M}\mathbf{StabFac}_0 \to \mathbf{rrCat}_0.$$

So, for each $\mathcal{M}$-stable factorization system $(\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}})$ and each range functor

$$F : \mathbf{C} \to E(\mathsf{Par}(\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}})),$$

there is a unique range functor $F^\sharp : \mathsf{Split}(\mathbf{C}) \to \mathsf{Par}(\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}})$ and a unique $(\mathcal{E}, \mathcal{M})$-functor $F^* : \mathsf{K}(\mathbf{C}) \to (\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}})$ such that

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\eta_{\mathbf{C}}"] \arrow[rrd, "F"'] & E\mathsf{Split}(\mathbf{C}) \arrow[r, "\approx"] \arrow[rd, "E(F^\sharp)"] & E\mathsf{Par}(\mathsf{K}(\mathbf{C})) \arrow[d, "E\mathsf{Par}(F^*)"] & \mathsf{Split}(\mathbf{C}) \arrow[d, dotted, "\exists! F^\sharp"] & \mathsf{K}(\mathbf{C}) \arrow[d, dotted, "\exists! F^*"] \\
& & E\mathsf{Par}(\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}}) & \mathsf{Par}(\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}}) & (\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}})
\end{tikzcd}$$

commutes, where $\mathsf{K}(\mathbf{C}) = (\mathsf{Total}(\mathsf{Split}(\mathbf{C})), \mathcal{E}_{\mathsf{Split}(\mathbf{C})}, \mathcal{M}_{\mathsf{Split}(\mathbf{C})})$. Obviously, we have:

Proposition 5.3.11 *For any $\mathcal{M}$-stable factorization system $(\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}})$ and any range functor $F : \mathbf{C} \to E(\mathsf{Par}(\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}}))$, there is a unique range functor $F^\sharp : \mathsf{Split}(\mathbf{C}) \to \mathsf{Par}(\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}})$ and a unique $(\mathcal{E}, \mathcal{M})$-functor $F^* : \mathsf{K}(\mathbf{C}) \to (\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}})$ such that*

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\eta_{\mathbf{C}}"] \arrow[drr, "F"'] & E\mathsf{Split}(\mathbf{C}) \arrow[r, "\approx"] \arrow[dr, "E(F^\sharp)"'] & E\mathsf{Par}(\mathsf{K}(\mathbf{C})) \arrow[d, "E\mathsf{Par}(F^*)"] & \mathsf{Split}(\mathbf{C}) \arrow[d, dotted, "\exists ! F^\sharp"] & \mathsf{K}(\mathbf{C}) \arrow[d, dotted, "\exists ! F^*"] \\
 & & E\mathsf{Par}(\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}}) & \mathsf{Par}(\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}}) & (\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}})
\end{tikzcd}$$

commutes and F is faithful (full) if and only if F^* is faithful (full), where

$$\mathsf{K}(\mathbf{C}) = (\mathsf{Total}(\mathsf{Split}(\mathbf{C})), \mathcal{E}_{\mathsf{Split}(\mathbf{C})}, \mathcal{M}_{\mathsf{Split}(\mathbf{C})}).$$

By Proposition 5.3.11, finding a representation from a range category $\mathbf{X}$ to

$$\mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$$

is equivalent to finding a range functor

$$F^\sharp : \mathsf{Split}(\mathbf{X}) \to \mathsf{Par}(\mathbf{X}, \mathcal{E}_{\mathbf{X}}, \mathcal{M}_{\mathbf{X}})$$

and is equivalent to finding an $(\mathcal{E}, \mathcal{M})$-functor

$$F^* : \mathsf{K}(\mathbf{X}) \to (\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}}).$$

Note that $\mathsf{Split}(\mathbf{X})$ is a range category with a split restriction while $\mathsf{K}(\mathbf{X})$ is a total range category.

For a given a range category $\mathbf{X}$ satisfying [**RR.6**], applying Schein's representation $\mathcal{S} : \mathbf{X} \to \mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$ to Proposition 5.3.11, there is a unique $(\mathcal{E}, \mathcal{M})$-functor $\mathcal{S}^*$ such that

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "E(\approx)\eta_{\mathbf{X}}"] \arrow[dr, "\mathcal{S}"'] & E\mathsf{Par}(\mathsf{K}(\mathbf{X})) \arrow[d, "E\mathsf{Par}(\mathcal{S}^*)"] & \mathsf{K}(\mathbf{X}) \arrow[d, dotted, "\exists ! \mathcal{S}^*"] \\
 & E\mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}}) & (\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})
\end{tikzcd}$$

commutes. The description of $\mathcal{S}^*$ is as follows.

Firstly, there is a unique range functor $\mathcal{S}^\sharp : \mathsf{Split}(\mathbf{X}) \to \mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})$ such that

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\eta_{\mathbf{X}}"] \arrow[dr, "\mathcal{S}"'] & E(\mathsf{Split}(\mathbf{X})) \arrow[d, "E(\mathcal{S}^\sharp)"] & \mathsf{Split}(\mathbf{X}) \arrow[d, dotted, "\exists ! \mathcal{S}^\sharp"] \\
& E(\mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})) & \mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})
\end{tikzcd}$$

commutes, where $\mathcal{S}^\sharp$ is given by mapping $f : (e_1 : X \to X) \to (e_2 : Y \to Y)$ to $\mathcal{S}(f) : \mathcal{S}(X) \to \mathcal{S}(Y)$ that is

$$\begin{tikzcd}
& {|\rho_f|} \arrow[dl, hook', "i"'] \arrow[dr, "\rho_f"] & \\
\mathbf{rpac}(X) & & \mathbf{rpac}(Y).
\end{tikzcd}$$

Since $\Phi : \mathsf{Split}(\mathbf{X}) \to \mathsf{Par}(\mathcal{M}\mathsf{Total}(\mathsf{Split}(\mathbf{X})))$, given by sending $f : (e_1 : X \to X) \to (e_2 : Y \to Y)$ to

$$\begin{tikzcd}
& \overline{f} \arrow[dl, hook', "\overline{f}"'] \arrow[dr, "f"] & \\
e_1 & & e_2,
\end{tikzcd}$$

is an equivalence of categories, $\mathsf{par}(\mathcal{S})$ is given by sending

$$\begin{tikzcd}
& e_1' \arrow[dl, hook', "m"'] \arrow[dr, "f"] & \\
e_1 & & e_2
\end{tikzcd}$$

in $\mathsf{Par}(\mathsf{K}(\mathbf{X}))$ to $\mathcal{S}^\sharp(fr)$, where $rm = 1_{e_1'}$ and $\overline{mr} = mr$. Hence there is a faithful $(\mathcal{E}, \mathcal{M})$-functor:

$$\mathcal{S}^* : \mathsf{K}(\mathbf{X}) \to (\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}}),$$

given by sending $f : e_1 \to e_2$ to $\rho_f : \mathbf{rpac}(e_1) \to \mathbf{rpac}(e_2)$, such that

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\eta_{\mathbf{X}}"] \arrow[drr, "\mathcal{S}"'] & E\mathsf{Split}(\mathbf{X}) \arrow[r, "E(\approx)"] \arrow[dr, "E(\mathcal{S}^\sharp)"] & E\mathsf{Par}(\mathsf{K}(\mathbf{X})) \arrow[d, "E\mathsf{Par}(\mathcal{S}^*)"] & \mathsf{K}(\mathbf{X}) \arrow[d, dotted, "\exists ! \mathcal{S}^*"] \\
& & E\mathsf{Par}(\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}}) & (\mathbf{Set}, \mathrm{Epics}_{\mathbf{Set}}, \mathrm{Monics}_{\mathbf{Set}})
\end{tikzcd}$$

commutes.

Chapter 6

Conclusions and Further Work

In this thesis, we studied certain structures: partial products, joins, meets, and ranges over restriction categories. In this chapter, we provide some concluding remarks and considerations for further work.

6.1 Main Results

The main results we obtained in this thesis are summarized as follows.

1. **Cartesian Restriction Categories**

   We produced a free partial product structure for restriction categories by fitting restriction structures to product completion. The result is summarized in Theorem 2.2.2.

2. **Join Restriction Categories and $\mathcal{M}$-adhesive Categories**

   We introduced the notion of join restriction categories (Definition 3.1.7) and described a free join structure, called join completion, to restriction categories (Theorem 3.1.12), which was linked to the join completion given in inverse semigroups by providing adjunctions among restriction categories, join restriction categories, inverse categories, and join inverse categories (Theorem 3.1.34). To answer when a partial map category is a join restriction category, we defined $\mathcal{M}$-adhesive categories and $\mathcal{M}$-gaps (Definitions 3.2.7 and 3.2.8) and proved the characterization theorem of partial map categories with joins (Theorem 3.3.6) and the completeness of join restriction categories (Theorem 3.3.7). We also showed that $\mathcal{M}$-gaps can give a join completion (Proposition 3.3.11).

3. **Meet Restriction Categories**

   We introduced the notion of meet restriction categories (Definition 4.1.1), showed the completeness of meet restriction categories in partial map categories (Theorem 4.2.3), and provided a free meet restriction structure for restriction categories (Theorem 4.3.4). We also studied when the free meet structure we provided preserves finite partial products (Proposition 4.3.12).

4. **Range Categories**

   We generalized Schein's representation theorem for type 3 function systems to range categories satisfying [**RR.6**] (Theorem 5.2.17) and studied when a partial map category satisfies Schein's condition [**RR.6**] (Propositions 5.3.1 and 5.3.7).

6.2 Further Work

We list here some possible directions for future work and some questions to which we would like to know the answers.

1. In this thesis, we have studied partial products, joins, meets, ranges on restriction categories. However, one of the objectives was to study how these structures interact with computability. Some work in this direction has been done, see, for example, [6, 12, 13, 45], but there remains much to do.

2. As mentioned in Subsection 5.1.3, factorization systems are related to other categorical notions, such as fibrations. localizations, torsion theory, and Eilenberg-Moore algebras. In [9], we have constructed free restriction categories using certain free fibrations. However, how ranges (= $\mathcal{M}$-stable factorization systems in some senses) interact with these categorical notions should be further studied.

3. By Proposition 5.1.5 and Theorem 5.1.12, each range category $\mathbf{X}$ can be fully and faithfully embedded, in a restriction and range preserving manner, into a partial map category:

   $$\mathbf{X} \overset{\eta_{\mathbf{X}}}{\to} \mathsf{Split}(\mathbf{X}) \approx \mathsf{Par}(\mathsf{Total}(\mathsf{Split}(\mathbf{X})), \mathcal{E}_{\mathsf{Split}(\mathbf{X})}, \mathcal{M}_{\mathsf{Split}(\mathbf{X})}).$$

   The Yoneda embedding $\mathcal{Y}$ can further map $\mathsf{Total}(\mathsf{Split}(\mathbf{X}))$ to its presheaf category $\mathbf{Set}^{\mathsf{Total}(\mathsf{Split}(\mathbf{X}))^{\mathrm{op}}}$. On the other hand, there is a coproduct functor from the presheaf category to $\mathbf{Set}$. We conjecture that this way would lead to at least a faithful range functor from each given range category satisfying [**RR.6**] to the category of sets and partial functions, but the details need to be filled out.

4. Subsection 4.3.5 studied meets' interactions with partial products. However, there is much work still needed to be done on how partial products, joins, meets, and ranges interact with each other and even with other mathematical structures.

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