

# CPSC617: Category Theory for Computer Science

## Second Exercise Sheet

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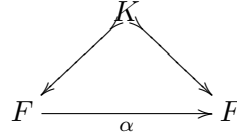
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Complete at least ten of the following questions.

- (1) Describe how the following are natural transformations between functors on **Sets**:
  - (a) Flattening a list of lists to a list,
  - (b) Appending two lists together,
  - (c) The first projection from the cartesian product,
  - (d) The diagonal map for the cartesian product,
- (2) The **center** of a category consists of the natural endo-transformations on the identity functor. Prove that the center of any category is a commutative monoid. Is this a functor from categories to commutative monoids  $\mathbf{Cat} \rightarrow \mathbf{CMon}$  (think about the situation for groups)?
- (3) Prove carefully that **Poset** can be viewed as a **Poset**-enriched category.
- (4) Prove carefully that the underlying functor from the category of categories to the category of directed graphs has a left adjoint.
- (5) Let  $\mathbb{R}$  be the real numbers viewed as a category by using the usual ordering; the integers may also be regarded as a category,  $\mathbb{Z}$ , by using the usual ordering and the usual inclusion of the integers into the reals is a functor. Prove that this functor has a left and right adjoint.
- (6) A **Galois connection** is a contravariant adjunction between posets. That is  $f \dashv g : P_1^{\text{op}} \rightarrow P_2$ .
  - (a) Show that  $g(f(g(x))) = g(x)$  and  $f(g(f(y))) = f(y)$  for any Galois connection and that the full subposets determined by  $\{x | x = g(f(x))\} \subseteq P_1$  and  $\{y | y = f(g(y))\} \subseteq P_2$  are (contra-)isomorphic.
  - (b) Show that any relation between sets  $R \subseteq X \times Y$  induces a Galois connection  $f_R \dashv g_R : P(X) \rightarrow P(Y)^{\text{op}}$  by

$$f_R(X') = \{y \in Y | \forall x \in X'. xRy\} \quad g_R(Y') = \{x \in X | \forall y \in Y'. xRy\}$$

- (c) The original Galois connection works as follows: let  $F$  be a field extension of some field  $K \subseteq F$  consider the automorphism group,  $G$ , of field isomorphisms  $\alpha$  such that



then consider the relation  $R \subset F \times G$  where  $R(x, g) \Leftrightarrow x = g(x)$ : that is a field element is related to a group element if the group element fixes the field element. Show that the group elements that fix a set of field elements is a subgroup and the field elements fixed by a get of group elements are a subfield ... discuss what this Galois correspondence tells one!

- (7) Show that for any adjunction  $(\eta, \epsilon) : F \dashv G : \mathbf{X} \rightarrow \mathbf{Y}$  the full subcategories  $\mathbf{X}_\eta$ , with objects  $X$  for which  $\eta_X$  is an isomorphism,  $\mathbf{Y}_\epsilon$ , with objects  $Y$  for which  $\epsilon_Y$  is an isomorphism are equivalent.

- (8) Show that in any adjunction the following are equivalent

- $\eta_{G(F(X))}$  is an isomorphism,
- $G(F(\eta_X))$  is an isomorphism,
- $\eta_{G(Y)}$  is an isomorphism,
- $\epsilon_{F(G(Y))}$  is an isomorphism,
- $F(G(\epsilon_Y))$  is an isomorphism,
- $\epsilon_{F(X)}$  is an isomorphism.

Call an adjunction satisfying any one of these conditions a “Galois adjunction” and conclude that the full subcategories of objects  $\{G(F(X)) | X \in \mathbf{X}\}$  and  $\{F(G(Y)) | Y \in \mathbf{Y}\}$  are equivalent.

- (9) Let  $f : X \rightarrow Y$  be any map of sets then prove that this induces a chain of adjoints  $\exists_f \dashv f^* \dashv \forall_f : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  where

- $\mathcal{P}(X)$  and  $\mathcal{P}(Y)$  are the powersets (set of all subsets) of  $X$  and  $Y$  respectively,
- $\exists_f : \mathcal{P}(X) \rightarrow \mathcal{P}(Y); S \mapsto \{y \in Y | \exists x \in X \cdot f(x) = y \wedge x \in S\}$ ,
- $f^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X); T \mapsto \{x \in X | f(x) \in T\}$ ,
- $\forall_f : \mathcal{P}(X) \rightarrow \mathcal{P}(Y); S \mapsto \{y \in Y | \forall x \in X \cdot f(x) = y \Rightarrow x \in S\}$ .

- (10) Prove that the functor  $A \times _ : \mathbf{Sets} \rightarrow \mathbf{Sets}$  has a right adjoint (hint: think hom-set).
- (11) The category of posets has an obvious inclusion into the category of preorders. Prove that this is a reflection (hint: how do you turn a preorder into an order?).
- (12) Prove that the category of finite sets and relations is equivalent to the category of Boolean matrices:

**Objects:** Natural numbers  $n \in \mathbb{N}$ ;

**Maps:** Boolean matrices  $[b_{ij}]_{i=1, \dots, n}^{j=1, \dots, m} : n \rightarrow m$ ;

**Identity:** The diagonal matrix;

**Composition:** Matrix multiplications with  $\wedge$  as multiplication and  $\vee$  as addition.

- (13) Show that  $\mathbf{Rel}$ , the category of sets and relations with the ordinary composition, is a poset enriched category (with  $R \leq S$  iff  $R \subseteq S$ ). This means it is a 2-category (whose hom objects are posets) and thus we may talk of adjoints. Prove that a relation is a left adjoint in  $\mathbf{Rel}$  if and only if it is a function.