CPSC617: Category Theory for Computer Science
Second Exercise Sheet

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Complete at least ten of the following questions.

(1) Describe how the following are natural tansformations between functors on Sets:

(a) Flattening a list of lists to a list,

(b) Appending two lists together,

(c) The first projection from the cartesian product,

(d) The diagonal map for the cartesian product,

(2) The **center** of a category consists of the natural endo-transformations on the identity functor. Prove that the center of any category is a commutative monoid. Is this a functor from categories to commutative monoids $\mathsf{Cat} \to \mathsf{CMon}$ (think about the situation for groups)?

(3) Prove carefully that Poset can be viewed as a Poset-enriched category.

(4) Prove carefully that the underlying functor from the category of categories to the category of directed graphs has a left adjoint.

(5) Let $\mathbb{R}$ be the real numbers viewed as a category by using the usual ordering; the integers may also be regarded as a category, $\mathbb{Z}$, by using the usual ordering and the usual inclusion of the integers into the reals is a functor. Prove that this functor has a left and right adjoint.

(6) A **Galois connection** is a contravariant adjunction between posets. That is $f \dashv g : P_1^{\mathrm{op}} \to P_2$.

(a) Show that $g(f(g(x))) = g(x)$ and $f(g(f(y))) = f(y)$ for any Galois connection and that the full subposets determined by $\{x | x = g(f(x))\} \subseteq P_1$ and $\{y | y = f(g(y))\} \subseteq P_2$ are (contra-)isomorphic.

(b) Show that any relation between sets $R \subseteq X \times Y$ induces a Galois connection $f_R \dashv g_R : P(X) \to P(Y)^{\mathrm{op}}$ by

$$f_R(X') = \{y \in Y | \forall x \in X'.xRy\} \qquad g_R(Y') = \{x \in X | \forall y \in Y'.xRy\}$$

(c) The original Galois connection works as follows: let F be a field extension of some field $K \subseteq F$ consider the automorphism group, G, of field isomorphisms α such that

$$\begin{tikzcd} & K \arrow[dl] \arrow[dr] & \\ F \arrow[rr, "\alpha"'] & & F \end{tikzcd}$$

then consider the relation $R \subset F \times G$ where $R(x, g) \Leftrightarrow x = g(x)$: that is a field element is related to a group element if the group element fixes the field element. Show that the group elements that fix a set of field elements is a subgroup and the field elements fixed by a get of group elements are a subfield ... discuss what this Galois correspondence tells one!

(7) Show that for any adjunction $(\eta, \epsilon) : F \dashv G : \mathbf{X} \to \mathbf{Y}$ the full subcategories $\mathbf{X}_\eta$, with objects X for which η_X is an isomorphism, $\mathbf{Y}_\epsilon$, with objects Y for which ϵ_Y is an isomorphism are equivalent.

(8) Show that in any adjunction the following are equivalent

- $\eta_{G(F(X))}$ is an isomorphism,
- $G(F(\eta_X))$ is an isomorphism,
- $\eta_{G(Y)}$ is an isomorphism,
- $\epsilon_{F(G(Y))}$ is an isomorphism,
- $F(G(\epsilon_Y))$ is an isomorphism,
- $\epsilon_{F(X)}$ is an isomorphism.

Call an adjunction satisfying any one of these conditions a "Galois adjunction" and conclude that the full subcategories of objects $\{G(F(X)) | x \in \mathbf{X}\}$ and $\{F(G(Y)) | Y \in \mathbf{Y}\}$ are equivalent.

(9) Let $f : X \to Y$ be any map of sets then prove that this induces a chain of adjoints $\exists_f \dashv f^* \dashv \forall_f : \mathcal{P}(X) \to \mathcal{P}(Y)$ where

- $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are the powersets (set of all subsets) of X and Y respectively,
- $\exists_f : \mathcal{P}(X) \to \mathcal{P}(Y); S \mapsto \{y \in Y | \exists x \in X \cdot f(x) = y \wedge x \in S\}$,
- $f^* : \mathcal{P}(Y) \to \mathcal{P}(X); T \mapsto \{x \in X | f(x) \in T\}$,
- $\forall_f : \mathcal{P}(X) \to \mathcal{P}(Y); S \mapsto \{y \in Y | \forall x \in X \cdot f(x) = y \Rightarrow x \in S\}$.

(10) Prove that the functor $A \times \_ : \mathsf{Sets} \to \mathsf{Sets}$ has a right adjoint (hint: think hom-set).

(11) The category of posets has an obvious inclusion into the category of preorders. Prove that this is a reflection (hint: how do you turn a preorder into an order?).

(12) Prove that the category of finite sets and relations is equivalent to the category of Boolean matrices:

Objects: Natural numbers $n \in \mathbb{N}$;

Maps: Boolean matrices $[b_{ij}]_{i=i,..,n}^{j=1..,m} : n \to m$;

Identity: The diagonal matrix;

Composition: Matrix multiplications with $\wedge$ as multiplication and $\vee$ as addition.

(13) Show that Rel, the category of sets and relations with the ordinary composition, is a poset enriched category (with $R \leq S$ iff $R \subseteq S$). This means it is a 2-category (whose hom objects are posets) and thus we may talk of adjoints. Prove that a relation is a left adjoint in Rel if and only if it is a function.