CPSC617: Category Theory for Computer Science
First Exercise Sheet

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This is due January 28$^{\text{th}}$. Please attempt at least 10 questions ...

(1) Prove that all maps in a preorder (regarded as a category) are bijic (that is both epic and monic) and that all sections and retractions are isomorphisms.

(2) Prove that in any category $\mathcal{F}$, all of whose hom-sets are finite (i.e. it is enriched over finite sets), that

(a) $\mathcal{F}$ need *not* be a finite category;

(b) Every monic endomorphism is an isomorphism;

(c) Every epic endomorphism is an isomorphism;

(d) The idempotent completion $\mathsf{Split}(\mathcal{F})$ is finite set enriched whenever $\mathcal{F}$ is;

(e) (Harder) For every endomorphims g there is a (smallest) $n \in \mathbb{N}$ such that g^n is an idempotent.

(f) (Even harder) In a finite set enriched category in which idempotents split every objects has a retraction to an object which is "fully retracted" (that is has no further non-trivial – non-dientity – retractions).

(3) The category **2** is

$$\begin{tikzcd} A \arrow[loop left, "1_A"] \arrow[r, "a"] & B \arrow[loop right, "1_B"] \end{tikzcd}$$

What do the categories $\mathbf{2} + \mathbf{2}$ and $\mathbf{2} \times \mathbf{2}$ look like?

(4) How many categories are there with 1,2, and 3 arrows?

(5) Show that $\mathsf{Path}(\mathcal{G})$, where $\mathcal{G}$ is a directed graph, is a category and identify the monics, epics, sections, and retractions.

(6) Here is an illustration of how two categories can have the same objects and maps but a completely different composition structure. Consider sets with relations but alter the composition to be:

$$RS = \{(x,z) | \forall y.(x,y) \in R \vee (y,z) \in S\}.$$

Prove that this forms a category (what are the identities?).

(7) Consider the category of matrices over a rig:

(a) Prove that $\mathsf{Mat}(R)$, the category of matrices over a (non-commutative) rig R, is a category;

(b) (Harder) Prove that when the rig R has an involution $\overline{(\_)} : R \to R$ (where $\overline{0} = 0$, $\overline{x+y} = \overline{x} + \overline{y}$, $\overline{1} = 1$, and $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$) transposition given by

$$(\_)^{\dagger} : \mathsf{Mat}(R)^{\mathrm{op}} \to \mathsf{Mat}(R); A_{ij} : m \to n \mapsto \overline{A}_{ji} : n \to m$$

is a functor (in fact, a converse involution).

(c) (Harder) Do all idempotents split in $\mathsf{Mat}(R)$? Do they split when $R = \mathbb{R}$ is the field of real numbers? If this true for any field?

(8) Show that in Sets:

(a) A map f is monic if an only if it is **injective** ($f(x) = f(y)$ implies $x = y$);

(b) A maps f is epic if and only if it is **surjective** (for every y in the codomain there is an x such that $f(x) = y$);

(c) All epics are retractions;

(d) Not all monics are sections;

(e) All bijics are isomorphisms.

Prove that the surjections and injections give a factorization system on Sets.

(9) (Harder:) What are the monics in Rel?

(10) Given any category $\mathbb{X}$ and an object $A \in \mathbb{X}$ define $f \sim_A g$ for $f, g : X \to Y$ if and only if for every $x : A \to X$ it is the case that $xf = xg$. Show that $\sim_A$ is a congruence and, furthermore, in $\mathbb{X}/\sim_A$ $h \neq k : X \to Y$ implies there an $x : A \to X$ with $xh \neq xk$ (that is A always **generates** $\mathbb{X}/\sim_A$).

(11) What are the monics in Cat? Show that every functor can be factorized as

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "F"] \arrow[dr, "Q_{\sim_F}"'] & & \mathbb{Y} \\ & \mathbb{X}/\sim_F \arrow[ur, "I_{\sim_F}"'] & \end{tikzcd}$$

where the first functor is full and bijective on objects while the second is faithful. Show that

(a) This defines a factorization system on Cat;

(b) (Harder) The $\mathcal{E}$-functors of this factorization do not include all epic functors;

(c) (Harder) The $\mathcal{M}$-functors are not necessarily monic.

(12) For idempotents in any category prove that:

(a) If an idempotent is either epic or monic then it is the identity map;

(b) Prove that if $rm = e$, where e is an idempotent, r is epic, and m is monic, then the pair (r, m) provides a splitting for the idempotent e.

(c) Give an example of two idempotents e_1 and e_2 such that neither $e_1 e_2$ nor $e_2 e_1$ are idempotents.

(d) Show that if idempotents commute, $e_1 e_2 = e_2 e_1$, then the composite $e_1 e_2$ is an idempotent.

(e) The relation on idempotents $e \leq e'$ if and only if $ee' = e$ is a preorder.

(f) If $e = sr$ and $e' = s'r'$ are splittings and $e \leq e'$ that there is a unique map α with $s = \alpha s'$.

(13) $\mathsf{Sub}_{\mathcal{C}}(A)$, the category of subobjects of A, is defined for an object $A \in \mathcal{C}$, for any category $\mathcal{C}$ is any category, to be the category:

Objects: monics $m : A' \to A$;

Maps: $f : m_1 \to m_2$ maps in $\mathcal{C}$ such that $f; m_2 = m_1$;

Identities: $1_{A'} : m \to m$ as in $\mathcal{C}$;

Composition: As in $\mathcal{C}$.

Prove that $\mathsf{Sub}_{\mathcal{C}}(A)$ is a preorder.