CPSC617: Category Theory for Computer Science First Exercise Sheet

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This is due January 28th. Please attempt at least 10 questions ...

- (1) Prove that all maps in a preorder (regarded as a category) are bijic (that is both epic and monic) and that all sections and retractions are isomorphisms.
- (2) Prove that in any category \mathcal{F} , all of whose hom-sets are finite (i.e. it is enriched over finite sets), that
 - (a) \mathcal{F} need *not* be a finite category;
 - (b) Every monic endomorphism is an isomorphism;
 - (c) Every epic endomorphism is an isomorphism;
 - (d) The idempotent completion $Split(\mathcal{F})$ is finite set enriched whenever \mathcal{F} is;
 - (e) (Harder) For every endomorphims g there is a (smallest) $n \in \mathbb{N}$ such that g^n is an idempotent.
 - (f) (Even harder) In a finite set enriched category in which idempotents split every objects has a retraction to an object which is "fully retracted" (that is has no further non-trivial non-dientity retractions).
- (3) The category 2 is

$$1_A \bigcap A \xrightarrow{a} B \bigcap 1_B$$

What do the categories $\mathbf{2} + \mathbf{2}$ and $\mathbf{2} \times \mathbf{2}$ look like?

- (4) How many categories are there with 1,2, and 3 arrows?
- (5) Show that $Path(\mathcal{G})$, where \mathcal{G} is a directed graph, is a category and identify the monics, epics, sections, and retractions.
- (6) Here is an illustration of how two categories can have the same objects and maps but a completely different composition structure. Consider sets with relations but alter the composition to be:

$$RS = \{(x, z) | \forall y.(x, y) \in R \lor (y, z) \in S\}.$$

Prove that this forms a category (what are the identities?).

- (7) Consider the category of matrices over a rig:
 - (a) Prove that Mat(R), the category of matrices over a (non-commutative) rig R, is a category;
 - (b) (Harder) Prove that when the rig R has an involution $\overline{(\underline{\ })}: R \to R$ (where $\overline{0} = 0$, $\overline{x+y} = \overline{x} + \overline{y}$, $\overline{1} = 1$, and $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$) transposition given by

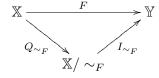
$$(_)^\dagger: \mathsf{Mat}(R)^\mathrm{op} o \mathsf{Mat}(R); A_{ij}: m o n \mapsto \overline{A}_{ji}: n o m$$

is a functor (in fact, a converse involution).

- (c) (Harder) Do all idempotents split in Mat(R)? Do they split when $R = \mathbb{R}$ is the field of real numbers? If this true for any field?
- (8) Show that in Sets:
 - (a) A map f is monic if an only if it is **injective** (f(x) = f(y)) implies x = y;
 - (b) A maps f is epic if and only if it is **surjective** (for every y in the codomain there is an x such that f(x) = y);
 - (c) All epics are retractions;
 - (d) Not all monics are sections;
 - (e) All bijics are isomorphisms.

Prove that the surjections and injections give a factorization system on Sets.

- (9) (Harder:) What are the monics in Rel?
- (10) Given any category \mathbb{X} and an object $A \in \mathbb{X}$ define $f \sim_A g$ for $f, g : X \to Y$ if and only if for every $x : A \to X$ it is the case that xf = xg. Show that \sim_A is a congruence and, furthermore, in $\mathbb{X}/\sim_A h \neq k : X \to Y$ implies there an $x : A \to X$ with $xh \neq xk$ (that is A always generates \mathbb{X}/\sim_A).
- (11) What are the monics in Cat? Show that every functor can be factorized as



where the first functor is full and bijective on objects while the second is faithful. Show that

- (a) This defines a factorization system on Cat;
- (b) (Harder) The \mathcal{E} -functors of this factorization do not include all epic functors;
- (c) (Harder) The \mathcal{M} -functors are not necessarily monic.
- (12) For idempotents in any category prove that:
 - (a) If an idempotent is either epic or monic then it is the identity map;

- (b) Prove that if rm = e, where e is an idempotent, r is epic, and m is monic, then the pair (r, m) provides a splitting for the idempotent e.
- (c) Give an example of two idempotents e_1 and e_2 such that neither e_1e_2 nor e_2e_1 are idempotents.
- (d) Show that if idempotents commute, $e_1e_2 = e_2e_1$, then the composite e_1e_2 is an idempotent.
- (e) The relation on idempotents $e \leq e'$ if and only if ee' = e is a preorder.
- (f) If e = sr and e' = s'r' are splittings and $e \le e'$ that there is a unique map α with $s = \alpha s'$.
- (13) $\mathsf{Sub}_{\mathcal{C}}(A)$, the category of subobjects of A, is defined for an object $A \in \mathcal{C}$, for any category \mathcal{C} is any category, to be the category:

Objects: monics $m: A' \to A$;

Maps: $f: m_1 \to m_2$ maps in C such that $f; m_2 = m_1$;

Identities: $1_{A'}: m \to m \text{ as in } \mathcal{C};$

Composition: As in C.

Prove that $Sub_{\mathcal{C}}(A)$ is a preorder.