Natural Numbers Objects and Quasitoposes

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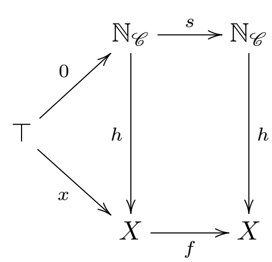
1 Natural Numbers Objects

If toposes are to categories that are "set-like" and simultaneously "generalized spaces," then we will need a notion of the Axiom of Infinity that can be interpreted internally to a category in order to see when we can have "infinite generalized spaces" and "infinite objects" inside our topos. The formulation we use was discovered by Lawvere in [3] and uses the natural numbers to give a notion of an internalization of the Axiom of Infinity. We follow this notion, as not only does it give us a notion of being infinite in a topos, but it also lets us describe when things are internally finite and locally finite in a relevant sense.

Remark 1.1. As a generic statement, most of the arguments in the proofs here are those presented in [2], save for where they differ.

Definition 1.2. Assume that $\mathscr{C}$ is a category with terminal object $\top$. The category $\mathscr{C}$ has a *natural numbers object* (abbreviated NNO) $(\mathbb{N}_{\mathscr{C}}, 0 : \top \to \mathbb{N}_{\mathscr{C}}, s : \mathbb{N}_{\mathscr{C}} \to \mathbb{N}_{\mathscr{C}})$ if there exists an object $\mathbb{N}_{\mathscr{C}}$ and morphisms $0 : \top \to \mathbb{N}_{\mathscr{C}}$ and $s : \mathbb{N}_{\mathscr{C}} \to \mathbb{N}_{\mathscr{C}}$ such that given any global point $x : \top \to X$ and any endomorphism $f : X \to X$ in $\mathscr{C}$, there exists a unique morphism $h : \mathbb{N}_{\mathscr{C}} \to X$ such that the diagram



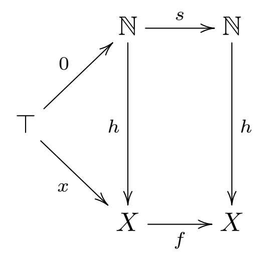
commutes in $\mathscr{C}$.

Note that if a category $\mathscr{C}$ has a natural numbers object $(\mathbb{N}_{\mathscr{C}}, 0, s)$, then it is unique up to unique isomorphism. We will see this in detail later, but intuitively we can see this by virtue of the fact that it is defined by a universal property.

Remark 1.3. In what follows in this section, we will always (unless we explicitly say otherwise) assume that our category $\mathscr{C}$ has a terminal object. We will also say that we "assume a category has *an* NNO" instead of the more literal "assume a category has *a* NNO" for linguistic and speaking-the-sentence-out-loud reasons. Finally, we will, if no confusion is likely to result, write our NNOs as $\mathbb{N}$ instead of $\mathbb{N}_{\mathscr{C}}$, as implicitly $\mathbb{N}$ will be the NNO inside our category of interest.

Remark 1.4. We will often abbreviate the instanciation of an NNO from the triple $(\mathbb{N}, 0, s)$ to simply writing $\mathbb{N}$ and then using the 0 and s morphisms as necessary.

Remark 1.5. We should think of the morphism $0 : \top \to \mathbb{N}$ as picking out the relevant 0 global element of the natural numbers, and the morphism $s : \mathbb{N} \to \mathbb{N}$ as the successor morphism. In this way, the universal property defining an NNO is simply saying that we can do recursion internally to our category. Explicitly, the diagram



states the categorical versions of the statements "$h(0) = x$" and "for any $n \in \mathbb{N}$, $f(h(n)) = h(n+1)$."

Example 1.6. In the category **Set**, the natural numbers object is the usual Peano-arithmetic-constructed set $\mathbb{N}$ of natural numbers. Given a set X with a point $x : \top \to X$ and endomorphism $f : X \to X$, the unique function $h : \mathbb{N} \to X$ is given by

$$h(n) = \begin{cases} x & \text{if } n = 0; \\ f^n(x) & \text{if } n \geq 1. \end{cases}$$

Example 1.7. In the category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, for any locally small category $\mathscr{C}$, the natural numbers object is the constant presheaf at $\mathbb{N}$. That is, $\mathbb{N}_{[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]} =: \underline{\mathbb{N}}$ is the functor defined by sending $U \mapsto \mathbb{N}$ and $f \mapsto \mathrm{id}_{\mathbb{N}}$ for all $X \in \mathscr{C}_0^{\mathrm{op}}$ and for all $f \in \mathscr{C}_1^{\mathrm{op}}$. If $\mathscr{C}$ is instead a Grothendieck topos (of sheaves on some base category $\mathscr{X}$), then the natural numbers object of $\mathscr{C}$ is the sheafification of the constant presheaf $\underline{\mathbb{N}}$, i.e.,

$$\mathbb{N}_{\mathscr{C}} = (\underline{\mathbb{N}})^{++},$$

where the $(-)^{++}$ functor is the sheafification functor from $[\mathscr{X}^{\mathrm{op}}, \mathbf{Set}] \to \mathscr{C}$.

Example 1.8. Not every Cartesian closed category need have an NNO, even if $\mathscr{C}$ is a topos. For instance, if $\mathscr{C} = \mathbf{FinSet}$ is the topos of finite sets, then $\mathscr{C}$ has no natural numbers object.

We now give some key colimit descriptions of the NNO due to Freyd. The first description we give will be in categories with either binary products or coproducts, and the second will be a description that is true in any category. For the proof of the coproduct characterization, however, we will need to use some theory of endofunctor algebras; details on this can be found in Appendix A, which itself explains the difference between T-algebras and $\mathbb{T}$-algebras, for an endofunctor T and monad $\mathbb{T}$. The take away of this will be that we can characterize the NNO of a category $\mathscr{C}$ with a terminal object and binary coproducts as a universal T-algebra. This will be developed carefully below, and the reader uninterested in this demonstration may safely skip it.

For the time being, assume that $\mathbb{N}$ is the NNO of the category $\mathscr{C}$ and assume that $\mathscr{C}$ has binary coproducts. Define the endofunctor $T : \mathscr{C} \to \mathscr{C}$ by

$$TX := \top \coprod X$$

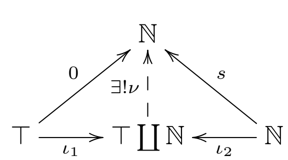
on objects X, and for morphisms f,

$$Tf := [\mathrm{id}_\top, f].$$

We will show that the NNO is the initial T-algebra, which will also give one of our colimit descriptions of the NNO.

Lemma 1.9. *The NNO $\mathbb{N}$ is the initial object in the category $\mathscr{C}^T$.*

Proof. First note that since there are morphisms $\top \xrightarrow{0} \mathbb{N}$ and $\mathbb{N} \xrightarrow{s} \mathbb{N}$ in $\mathscr{C}$, there is a unique morphism ν making the diagram

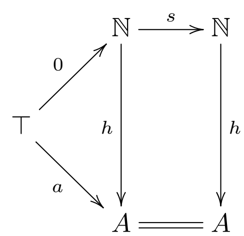


commute. Since $\top \coprod \mathbb{N} = T\,\mathbb{N}$, the map ν makes $\mathbb{N}$ into a T-algebra. It also follows that $\nu = [0, s]$, where $[0, s]$ is the comparison map.

Now assume that (A, α) is a T-algebra. Note that there then is a global point

$$\begin{array}{c} \top \xrightarrow{\iota_1} \top \coprod A = TA \xrightarrow{\alpha} A \\ \hline\hline \top \xrightarrow{a} A \end{array}$$

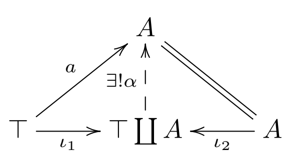
in $\mathscr{C}$. There then exists a unique morphism $h : \mathbb{N} \to A$ making the diagram



commute in $\mathscr{C}$. To see that $h : \mathbb{N} \to A$ is a T-algebra morphism, we simply need to check that $\alpha \circ Th \stackrel{?}{=} h \circ \nu$. However, we first observe that:

$$\begin{array}{c} T\,\mathbb{N} \xrightarrow{h \circ \nu} A \\ \hline T\,\mathbb{N} \xrightarrow{\nu} \mathbb{N} \xrightarrow{h} A \\ \hline \top \coprod \mathbb{N} \xrightarrow{[0,s]} \mathbb{N} \xrightarrow{h} A \\ \hline \top \coprod \mathbb{N} \xrightarrow{[h \circ 0, h \circ s]} A \\ \hline \top \coprod \mathbb{N} \xrightarrow{[a,h]} A \end{array}$$

On the other hand, we first notice that α is realized from the diagram



which implies that $\alpha = [a, \mathrm{id}_A]$. Now we derive that:

$$\begin{array}{c} T\,\mathbb{N} \xrightarrow{\alpha \circ Th} A \\ \hline T\,\mathbb{N} \xrightarrow{Th} TA \xrightarrow{\alpha} \\ \hline \top \coprod \mathbb{N} \xrightarrow{\mathrm{id}_\top \coprod h} \top \coprod A \xrightarrow{\alpha} A \\ \hline \top \coprod \mathbb{N} \xrightarrow{\mathrm{id}_\top \coprod h} \top \coprod A \xrightarrow{[a, \mathrm{id}_A]} A \\ \hline \top \coprod \mathbb{N} \xrightarrow{[a \circ \mathrm{id}_\top, \mathrm{id}_A \circ h]} A \\ \hline \top \coprod \mathbb{N} \xrightarrow{[a,h]} A \end{array}$$

This shows that

$$\alpha \circ Th = [a, h] = h \circ \nu,$$

and hence proves that h is a T-algebra morphism. That h is unique is readily seen and follows from the universal property defining $\mathbb{N}$. This concludes the proof that $\mathbb{N}$ is the initial T-algebra. □

This brings us to Freyd's colimit description of the NNO. Of particular interest to us will be the case in which $\mathscr{C}$ is a topos and hence all descriptions hold.

Proposition 1.10. *Let $\mathscr{C}$ be a category with NNO $\mathbb{N}$. Then:*

1. *If $\mathscr{C}$ has binary coproducts or binary products, the diagram*

$$\top \xrightarrow{0} \mathbb{N} \xleftarrow{s} \mathbb{N}$$

   is a coporoduct diagram;

2. *The diagram*

$$\mathbb{N} \underset{s}{\overset{\mathrm{id}_{\mathbb{N}}}{\rightrightarrows}} \mathbb{N} \xrightarrow{!_{\mathbb{N}}} \top$$

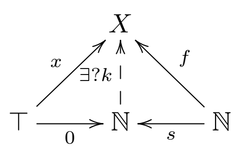
   is a coequalizer diagram.

Proof. We first prove (1). Assume that $\mathscr{C}$ has binary coproducts. We then have from Lemma 1.9 that $\mathbb{N}$ is an initial T-algebra for the endofunctor $T : \mathscr{C} \to \mathscr{C}$ given by $X \mapsto \top \coprod X$. Using Theorem A.3 we get that $\top \coprod \mathbb{N} = T\,\mathbb{N} \cong \mathbb{N}$. It then follows immediately that the diagram

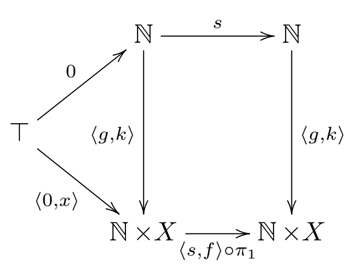
$$\top \xrightarrow{0} \mathbb{N} \xleftarrow{s} \mathbb{N}$$

is a coproduct diagram.

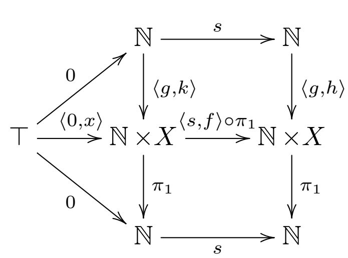
Assume now that $\mathscr{C}$ has binary products and that $X \in \mathscr{C}_0$ with morphisms $x : \top \to X$ and $f : \mathbb{N} \to X$. In order to show that the desired diagram is a coproduct, we must show that there exists a unique morphism $k : \mathbb{N} \to X$ making the diagram



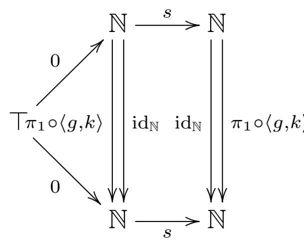
commute in $\mathscr{C}$. However, there is then a unique morphism $\langle g, k\rangle : \mathbb{N} \to \mathbb{N} \times X$ making the diagram



commute in $\mathscr{C}$. Note that $g \in \mathscr{C}(\mathbb{N}, \mathbb{N})$ and $k \in \mathscr{C}(\mathbb{N}, X)$. Now consider that the diagram



commutes, implying that



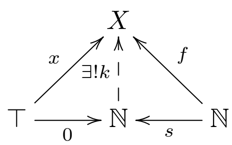
commutes. Thus it follows that $g = \mathrm{id}_{\mathbb{N}}$ by the universal property of the NNO. This in turn allows us to verify that $k : \mathbb{N} \to X$ is the unique morphism making the identities

$$h \circ 0 = x$$

and

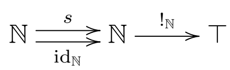
$$h \circ s = f \circ h$$

hold by a similar argument as the one above. It then follows that the diagram

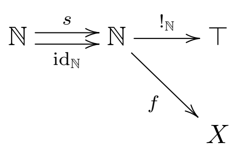


commutes, proving that $\top \xrightarrow{0} \mathbb{N} \xleftarrow{s} \mathbb{N}$ is a coproduct diagram.

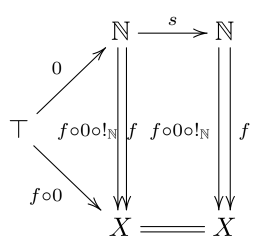
We now prove (2). To show that the diagram



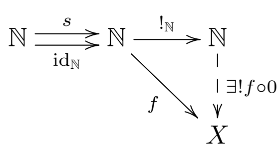
is a coequalizer, we consider the diagram



in $\mathscr{C}$. First, to factor f through $\top$, we note that the diagram



commutes in $\mathscr{C}$, showing that $f \circ 0 \circ !_{\mathbb{N}} = f$. Thus taking $f \circ 0 : \top \to \mathbb{N}$ makes the diagram



commute, where the uniqueness follows because $!_{\mathbb{N}}$ is a retract. $\square$

Remark 1.11. Freyd actually showed a stronger result (cf. [1]) than the proof presented above. In particular, Freyd showed that if $\mathscr{C}$ is a topos then an object $N \in \mathscr{C}_0$ for which there is a point $z : \top \to N$ and morphism $s : N \to N$ such that the cospan

$$\top \xrightarrow{z} N \xleftarrow{s} N$$

is a coproduct and the diagram

$$N \underset{s}{\overset{\mathrm{id}_N}{\rightrightarrows}} N \longrightarrow \top$$

is a coequalizer, then N is actually an NNO for $\mathscr{C}$. However, this proof requires the development of the internal logic of a topos, and so (the proof) is omitted from these notes. We will, however, instead present the theorem below.

Theorem 1.12. *In a topos $\mathcal{E}$, if an object N has morphisms $z : \top \to N$ and $s : N \to N$ such that the diagram*

$$\top \xrightarrow[z]{} N \xleftarrow[s]{} N$$

is a coproduct and such that

$$N \underset{\mathrm{id}_N}{\overset{s}{\rightrightarrows}} N \longrightarrow \top$$

is a coequalizer, then N is an NNO in $\mathcal{E}$.

We now would like to show that like any good model of the natural numbers, we have notions of addition and multiplication. At this point, **we momentarily assume that the category $\mathscr{C}$ is Cartesian closed**; this will be true in any topos and quasitopos, which are the situations in which we will generically care about NNOs, but is not true at the usual level of generality of this section. It is possible even from these definitions to show that the object $(\mathbb{N}, +, \cdot, 0, 1)$ forms a commutative rig object in $\mathscr{C}$. We will not do this in these notes; however, we will prove a result about how to do what is essentially parametrized primitive recursion in a Cartesian closed category before we can define/prove the existence of the addition and multiplication morphisms.

Proposition 1.13 (Proposition A2.5.2 of [2])**.** *Let $\mathscr{C}$ be a Cartesian closed category and let $\mathbb{N}$ be an NNO in $\mathscr{C}$. Then for any $A, B \in \mathscr{C}_0$ with $g \in \mathscr{C}(A, B)$ and $h \in \mathscr{C}(A \times \mathbb{N} \times B, B)$, there exists a unique morphism $f : A \times \mathbb{N} \to B$ which makes the diagrams*

$$\begin{array}{ccc} A \times \top & \xrightarrow{\mathrm{id}_A \times 0} & A \times \mathbb{N} \\ {\scriptstyle \cong}\uparrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[g]{} & B \end{array} \qquad \begin{array}{ccc} A \times \mathbb{N} & \xleftarrow{\mathrm{id}_A \times s} & A \times \mathbb{N} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \langle \mathrm{id}_{A \times \mathbb{N}}, f\rangle} \\ B & \xleftarrow[h]{} & A \times \mathbb{N} \times B \end{array}$$

commute in $\mathscr{C}$.

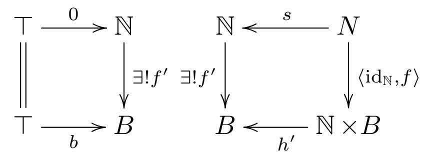
Proof. Our strategy is to first prove this for the case $A = \top$ and then use Cartesian closure to deduce the proposition for general A. With this in mind, assume that $A = \top$ and notice that since, for any $X \in \mathscr{C}_0$,

$$A \times X = \top \times X \cong X,$$

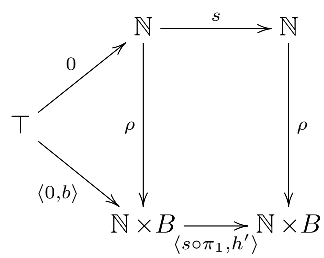
we have that for all morphisms $f : A \times X \to Y$ there is a correspondence:

$$\dfrac{\dfrac{A \times X \xrightarrow{f} Y}{\top \times X \xrightarrow{f} Y}}{X \xrightarrow{f'} Y}$$

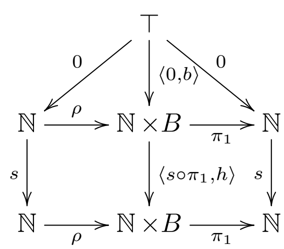
In particular, we note that proving the commutativity of the diagrams in the statement of the proposition reduces to, after being sufficiently careful with pre-and-post-composing with isomorphisms, showing the diagrams



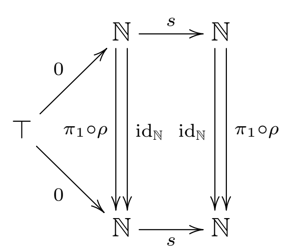
commute in the case $A = \top$. To this end, let $h' \in \mathscr{C}(\mathbb{N}\times B, B)$ be a morphism and let $b \in \mathscr{C}(\top, B)$ be a point. We can then find a unique morphism $\rho : \mathbb{N} \to \mathbb{N}\times B$ which makes the diagram



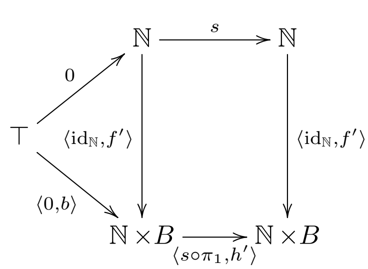
commute in $\mathscr{C}$. By stacking commuting diagrams, we derive that the diagram



commutes in $\mathscr{C}$, which in turn shows that



commutes. This allows us to deduce that $\pi_1 \circ \rho = \mathrm{id}_{\mathbb{N}}$ and hence that there exists a unique (by the uniqueness of ρ) morphism $f' = \pi_2 \circ \rho : \mathbb{N} \to B$ which makes the diagram



commute. However it then follows immediately from the diagram above that the two diagrams

$$\begin{array}{ccc} \top & \xrightarrow{0} & \mathbb{N} \\ \Big\| & & \Big\downarrow{\scriptstyle \exists! f'} \\ \top & \xrightarrow[b]{} & B \end{array} \qquad \begin{array}{ccc} \mathbb{N} & \xleftarrow{s} & \mathbb{N} \\ {\scriptstyle \exists! f'}\Big\downarrow & & \Big\downarrow{\scriptstyle \langle \mathrm{id}_{\mathbb{N}}, f' \rangle} \\ B & \xleftarrow[h']{} & \mathbb{N} \times B \end{array}$$

commute, which establishes the case $A = \top$.

Assume now that $A \in \mathscr{C}_0$ is arbitrary and that we have morphisms $g \in \mathscr{C}(A, B)$ and $h \in \mathscr{C}(A \times \mathbb{N} \times B, B)$. We first observe that since $A \cong \top \times A$ uniquely, there is a correspondence of morphisms

$$\frac{\dfrac{A \xrightarrow{g} B}{\top \times A \xrightarrow{g'} B}}{\top \xrightarrow{\overline{g}} [A, B]}$$

by using the Cartesian closed structure of $\mathscr{C}$. Similarly, if the morphism $\pi_{23} : \mathbb{N} \times [A, B] \times A \to [A, B] \times A$ is the morphism which projects away the first component of the triple product, then we derive a morphism

$$\mathbb{N} \times [A, B] \times A \xrightarrow{\pi_{23}} [A, B] \times A \xrightarrow{\mathrm{ev}} B.$$

Set $\overline{h} : \mathbb{N} \times [A, B] \to [A, B]$ to be the exponential transpose appearing in the last line:

$$\frac{\dfrac{\mathbb{N} \times [A, B] \times A \xrightarrow{\langle \pi_3, \pi_1, \mathrm{ev} \circ \pi_{23} \rangle} A \times \mathbb{N} \times B \xrightarrow{h} B}{\mathbb{N} \times [A, B] \times A \xrightarrow{h \circ \langle \pi_3, \pi_1, \mathrm{ev} \circ \pi_{23} \rangle} B}}{\mathbb{N} \times [A, B] \xrightarrow{\overline{h}} [A, B]}$$

Using the first part of the proof, we derive the existence a unique morphism $\overline{f} : \mathbb{N} \to [A, B]$ which makes the diagrams

$$\begin{array}{ccc} \top & \xrightarrow{0} & \mathbb{N} \\ \Big\| & & \Big\downarrow{\scriptstyle \exists! \overline{f}} \\ \top & \xrightarrow[\overline{g}]{} & [A, B] \end{array} \qquad \begin{array}{ccc} \mathbb{N} & \xleftarrow{s} & \mathbb{N} \\ {\scriptstyle \exists! \overline{f}}\Big\downarrow & & \Big\downarrow{\scriptstyle \langle \mathrm{id}_{\mathbb{N}}, \overline{f} \rangle} \\ [A, B] & \xleftarrow[\overline{h}]{} & \mathbb{N} \times [A, B] \end{array}$$

commute in $\mathscr{C}$. Taking the transpose of $\overline{f}$ as in the deduction

$$\frac{\dfrac{\mathbb{N} \xrightarrow{\overline{f}} [A, B]}{\mathbb{N} \times A \xrightarrow{f'} B}}{A \times \mathbb{N} \xrightarrow{f} B}$$

gives the commuting diagrams

$$\begin{array}{ccc} A \times \top & \xrightarrow{\mathrm{id}_A \times 0} & A \times \mathbb{N} \\ {\scriptstyle \cong}\Big\uparrow & & \Big\downarrow{\scriptstyle \exists! f} \\ A & \xrightarrow[g]{} & B \end{array}$$

as was to be shown. □

Remark 1.14. We can alternatively describe the morphism f constructed above as the unique morphism making the diagram

$$\begin{array}{ccccc} A\times\top & \xrightarrow{\mathrm{id}_A\times 0} & A\times\mathbb{N} & \xleftarrow{\mathrm{id}_A\times s} & A\times\mathbb{N} \\ {\scriptstyle\cong}\big\uparrow & & \big\downarrow{\scriptstyle\exists! f} & & \big\downarrow{\scriptstyle\langle \mathrm{id}_{A\times\mathbb{N}}, f\rangle} \\ A & \xrightarrow[g]{} & B & \xleftarrow[h]{} & A\times\mathbb{N}\times B \end{array}$$

commute in $\mathscr{C}$. We will use this form of Proposition 1.13 to construct the addition, multiplication, predecessor, and truncated subtraction morphisms.

In what follows we assume that $\mathscr{C}$ is Cartesian closed.

Example 1.15 (Addition of a natural numbers object)**.** We define the addition morphism $+ : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ on an NNO $\mathbb{N}$ recursively by letting $+ : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ be the unique morphism making

$$\begin{array}{ccccc} \mathbb{N}\times\top & \xrightarrow{\mathrm{id}_\mathbb{N}\times 0} & \mathbb{N}\times\mathbb{N} & \xleftarrow{\mathrm{id}_\mathbb{N}\times s} & \mathbb{N}\times\mathbb{N} \\ {\scriptstyle\cong}\big\uparrow & & \big\downarrow{\scriptstyle +} & & \big\downarrow{\scriptstyle\langle \mathrm{id}_{\mathbb{N}\times\mathbb{N}}, +\rangle} \\ \mathbb{N} & = & \mathbb{N} & \xleftarrow[s\circ\pi_3]{} & \mathbb{N}\times\mathbb{N}\times\mathbb{N} \end{array}$$

commute, which states that "$n+0=n$ and $n+(m+1)=(n+m)+1$ for all $n,m\in\mathbb{N}$."

Example 1.16 (Multiplication of a natural numbers object)**.** We define the multiplication morphism $\cdot$: $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ on an NNO $\mathbb{N}$ recursively by asserting that $\cdot$ is the unique morphism making

$$\begin{array}{ccccc} \mathbb{N}\times\top & \xrightarrow{\mathrm{id}_\mathbb{N}\times 0} & \mathbb{N}\times\mathbb{N} & \xleftarrow{\mathrm{id}_\mathbb{N}\times s} & \mathbb{N}\times\mathbb{N} \\ {\scriptstyle\cong}\big\uparrow & & {\scriptstyle\cdot}\big\downarrow & & {\scriptstyle\langle \mathrm{id}_{\mathbb{N}\times\mathbb{N}}, \cdot\rangle}\big\downarrow \\ \mathbb{N} & \xrightarrow[0\circ !_\mathbb{N}]{} & \mathbb{N} & \xleftarrow[s\circ\pi_3]{} & \mathbb{N}\times\mathbb{N}\times\mathbb{N} \end{array}$$

commute, which states that "$n\cdot 0=0$ and $n(m+1)=mn+m$ for all $m,n\in\mathbb{N}$."

Example 1.17 (Predecessor of a natural numbers object)**.** The predecessor map $p:\mathbb{N}\to\mathbb{N}$ is defined to be the unique retract of s such that the diagram

$$\begin{array}{ccccc} \top & \xrightarrow{0} & \mathbb{N} & \xleftarrow{s} & \mathbb{N} \\ \big\Vert & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle \mathrm{id}_\mathbb{N}, p} \\ \top & \xrightarrow[0]{} & \mathbb{N} & \xleftarrow[\pi_1]{} & \mathbb{N}\times\mathbb{N} \end{array}$$

commutes, which states that "$p(0)=0$ (or there is no proper predecessor to 0) and $p(n+1)=n$ for all $n\in\mathbb{N}$, or n comes before $n+1$."

Example 1.18. The truncated subtraction map is the unique morphism $\overset{\circ}{-} : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ which makes the diagram

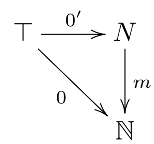
$$\begin{array}{ccccc} \mathbb{N}\times\top & \xrightarrow{\mathrm{id}_\mathbb{N}\times 0} & \mathbb{N}\times\mathbb{N} & \xleftarrow{\mathrm{id}_\mathbb{N}\times s} & \mathbb{N}\times\mathbb{N} \\ {\scriptstyle\cong}\big\uparrow & & \big\downarrow{\scriptstyle \overset{\circ}{-}} & & \big\downarrow{\scriptstyle\langle \mathrm{id}_{\mathbb{N}\times\mathbb{N}}, \overset{\circ}{-}\rangle} \\ \mathbb{N} & = & \mathbb{N} & \xleftarrow[p\circ\pi_3]{} & \mathbb{N}\times\mathbb{N}\times\mathbb{N} \end{array}$$

commute. Note that this says that "$n\overset{\circ}{-}0=n$ and $m\overset{\circ}{-}(n+1)=p(m\overset{\circ}{-}n)$ for all $m,n\in\mathbb{N}$."

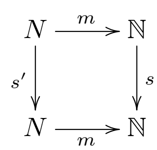
Remark 1.19. In case I don't prove it, the usual inductive proofs of the commutative rig status of $\mathbb{N}$, translated into their diagrammatic forms, are how you show that $\mathbb{N}$ is a crig in $\mathscr{C}$. The zero is $0 : \top \to \mathbb{N}$ and the multiplicative unit is the composite $s \circ 0 = 1_{\mathbb{N}} : \top \to \mathbb{N}$.

We close this section by showing that NNOs in a category satisfy Peano's fifth postulate. In particular, this amounts to showing that if there is any subobject N of $\mathbb{N}$ through which the successor and zero factor, then $N \cong \mathbb{N}$.

Proposition 1.20. *Let $m : N \to \mathbb{N}$ be a subobject of the NNO $\mathbb{N}$ such that there exist morphisms $0' : \top \to N$ and $s' : N \to N$ for which the diagrams*

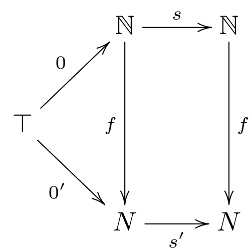


and

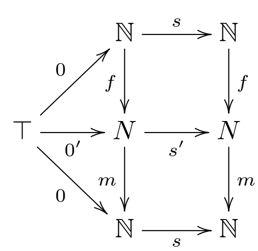


commute. Then $N \cong \mathbb{N}$.

Proof. Begin by observing that there is a unique morphism making the diagram



commute. From this, however, we derive that the diagram



commutes, which allows us to deduce that $m \circ f = \mathrm{id}_{\mathbb{N}}$. This shows that m is a monic retract and finally proves that $N \cong \mathbb{N}$ through m. ☐

2 Quasitoposes

The notion of a quasitopos is to a topos what separated presheaves are to sheaves on a site. In particular, we also have that quasitoposes arise by considering, instead of full subobject classification, what happens when we can only classify *strong* subobjects. We will first introduce the notion of strong morphisms and then present the strong subobject classifier, which will then lead to the study of quasitoposes.

Definition 2.1. Let $\mathcal{L}, \mathcal{R} \subseteq \mathscr{C}_1$ be classes of morphisms. We say that morphisms $\rho \in \mathcal{R}$ satisfy the *(orthogonal) right lifting property with* $\mathcal{L}$ if given any diagram

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\rho} \\ \bullet & \longrightarrow & \bullet \end{array}$$

with $\lambda \in \mathcal{L}$ is arbitrary, there exists a unique γ making the diagram

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ {\scriptstyle\lambda}\downarrow & \exists!\gamma \nearrow & \downarrow{\scriptstyle\rho} \\ \bullet & \longrightarrow & \bullet \end{array}$$

commute. Dually, we say that λ has the *(orthogonal) left lifting property with* $\mathcal{R}$.

Definition 2.2. Given a class of morphisms $A \subseteq \mathscr{C}_1$, we define

$$A^{\perp} := \{f \in \mathscr{C}_1 \mid f \text{ has the right lifting property with } A\}$$

and

$${}^{\perp}A := \{f \in \mathscr{C}_1 \mid f \text{ has the left lifting property with } A\}.$$

Definition 2.3. Let $\mathscr{C}$ be a category and let $E_{\mathscr{C}} = E$ be the class of epimorphisms of $\mathscr{C}$. We say that $s \in \mathscr{C}_1$ is a *strong morphism* if $s \in E^{\perp}$. Additionally, if s is a monomorphism as well, we say that s is a *strong monomorphism* or a strong monic.

Definition 2.4. An object Ω in a category $\mathscr{C}$ is a *strong subobject classifier* if there is a morphism true : $\top \to \Omega$ and given any strong monomorphism $m : A \to B$ there exists a unique morphism $\chi_m : B \to \Omega$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{!_A} & \top \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle \text{true}} \\ B & \xrightarrow[\chi_m]{} & \Omega \end{array}$$

a pullback.

One reason behind giving the definitions in this way is that we can prove first that regular monics are strong, and second that in a category with coequalizers, strong morphisms are monic. This will give us not only that in the class of regular monics is a subclass of the strong monics, but also that in a finitely cocomplete category, the class of strong monics coincides with morphisms that satisfy the right lifting property against epimorphisms.

We will now collect various facts and lemmas about strong morphisms (and in particular about strong monics) that will be useful later. In particular, we will show as an example that given a strong subobject classifier Ω, the monic true : $\top \to \Omega$ is strong.

Lemma 2.5. *Let $\mathscr{C}$ be a category and let $m : X \to Y$ be a regular monic with kernel pair $(f, g) : Y \to Z$. Then $m \in E^{\perp}$.*

Proof. Let $e : A \to B$ be an arbitrary epimorphism such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle m} \\ B & \xrightarrow[\beta]{} & Y \end{array}$$

commutes in $\mathscr{C}$. Note that it is immediate from the fact that m is monic that if a lift of the diagram exists, it is necessarily unique. As such, we only need to produce such a lift of the diagram. To this end, we first observe that

$$f \circ \beta \circ e = f \circ m \circ \alpha = g \circ m \circ \alpha = g \circ \beta \circ e$$

by the fact that m equalizes the kernel pair (f, g) and from the commutativity of the diagram. Using that e is epic now yields that

$$f \circ \beta = g \circ \beta$$

and hence provides a unique morphism $\gamma : B \to X$ making the diagram

$$\begin{array}{ccccc} X & \xrightarrow{m} & Y & \overset{f}{\underset{g}{\rightrightarrows}} & Z \\ {\scriptstyle \exists!\gamma}\uparrow & \nearrow{\scriptstyle \beta} & & & \\ B & & & & \end{array}$$

commute. Because γ factorizes β, we need only verify that $\gamma \circ e$ factorizes α; however, we observe that

$$m \circ \alpha = \beta \circ e = m \circ \gamma \circ e,$$

which gives that $\alpha = \gamma \circ e$ by the fact that m is monic. This shows that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle e}\downarrow & \overset{\exists!\gamma}{\nearrow} & \downarrow{\scriptstyle m} \\ B & \xrightarrow[\beta]{} & Y \end{array}$$

commutes and hence that $m \in E^{\perp}$. $\square$

Lemma 2.6. *Let $\mathscr{C}$ be a category with all coequalizers. Then every morphism $m \in E^{\perp}$ is monic.*

Proof. Assume that $m \in E^{\perp}$, write $m : X \to Y$, and assume that the diagram

$$Z \overset{h}{\underset{k}{\rightrightarrows}} X \xrightarrow{m} Y$$

commutes. Now coequalize h and k to produce the diagram

$$Z \overset{h}{\underset{k}{\rightrightarrows}} X \xrightarrow{\mathrm{coeq}(h,k)} C$$

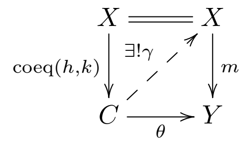
and recall that $\mathrm{coeq}(h, k)$ is epic. Since m coequalizes h and k, there is a unique morphism $\theta : C \to Y$ making the diagram

$$\begin{array}{ccccc} Z & \overset{h}{\underset{k}{\rightrightarrows}} & X & \xrightarrow{\mathrm{coeq}(h,k)} & C \\ & & & \searrow{\scriptstyle m} & \downarrow{\scriptstyle \exists!\theta} \\ & & & & Y \end{array}$$

commute. Place these maps into the commuting square

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle \mathrm{coeq}(h,k)}\downarrow & & \downarrow{\scriptstyle m} \\ C & \xrightarrow[\theta]{} & Y \end{array}$$

and note that since $m \in E^{\perp}$ there exists a unique $\gamma : C \to X$ making the diagram



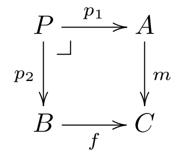
commute. However, from this we deduce that

$$h = \mathrm{id}_X \circ h = \gamma \circ \mathrm{coeq}(h,k) \circ h = \gamma \circ \mathrm{coeq}(h,k) \circ k = \mathrm{id}_X \circ k = k,$$

which shows that m is monic. □

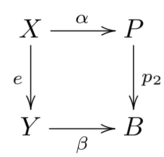
Corollary 2.7. *In a category $\mathscr{C}$ with coequalizers, the class $E^{\perp}$ coincides with the class M_{strong} of strong monomorphisms.*

Lemma 2.8. *Let $m : A \to C$ be a strong monomorphism and let*

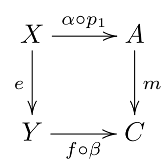


be a pullback in a category $\mathscr{C}$. Then p_2 is strong monic as well.

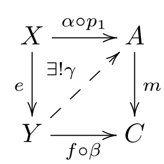
Proof. Begin by assuming that $e : X \to Y$ is an epimorphism such that the diagram



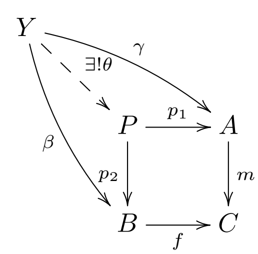
commutes in $\mathscr{C}$. Note that it is immediate that p_2 is monic by the fact that m is monic, so it suffices to simply construct a lift of the diagram. However, from this diagram we get a commuting square



and by the fact that m is strong, a unique lift γ making



commute. By the universal property of the pullback we can produce the commuting diagram:



Note that this gives the equation

$$p_2 \circ \theta = \beta.$$

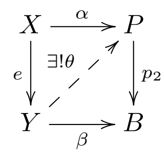
For the other equation desired of the lift, we note that

$$p_2 \circ \alpha = \beta \circ e = p_2 \circ \theta \circ e$$

so by the fact that p_2 is monic we have

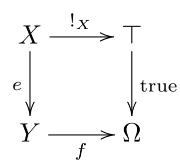
$$\alpha = \theta \circ e.$$

This implies that the diagram



commutes with θ unique. □

Example 2.9. If $\mathscr{C}$ has a strong subobject classifier, then the morphism true : $\top \to \Omega$ is strong monic. To see this, note that if $e : X \to Y$ is an epimorphism for which the diagram



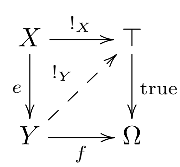
commutes, then by $\top$ being a terminal object there is a unique morphism $!_Y : Y \to \top$. This then implies immediately that

$$!_X =!_Y \circ e.$$

Finally the commutativity of the diagram gives

$$f \circ e = \text{true} \circ !_X = \text{true} \circ !_Y \circ e$$

and hence that $f = \text{true} \circ !_Y$ by the fact that e is epic. Thus the diagram



commutes with $!_Y$ evidently unique.

We will need one further lemma before proceeding that states simply that in a category $\mathscr{C}$ with coequalizers, if there are morphisms $m : A \to B$ and $g : A \to C$ with m a strong monic, and if the product $B \times C$ exists, then the morphism $\langle m, g \rangle : A \to B \times C$ is also strong monic. We will use this in a proposition below to conclude that in a quasitopos $\mathscr{C}$, strong monics are preserved by pushouts.

Lemma 2.10. *Let $\mathscr{C}$ be a category with coequalizers and assume that there are morphisms $m : A \to B$ and $g : A \to C$ with m strong. Then if $B \times C$ exits, the morphism $\langle m, g \rangle : A \to B \times C$ is strong monic.*

Proof. Assume that e is an epimorphism in $\mathscr{C}$ for which there is a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \langle m,g\rangle} \\ Y & \xrightarrow[\beta]{} & B \times C \end{array}$$

Since $\beta : Y \to B \times C$, there exist morphisms $\beta_1 : Y \to B$ and $\beta_2 : Y \to C$ which make $\beta = \langle \beta_1, \beta_2 \rangle$. We then have from the commutativity of the diagram that

$$\langle m \circ \alpha, g \circ \alpha \rangle = \langle m, g \rangle \circ \alpha = \beta \circ e = \langle \beta_1, \beta_2 \rangle \circ e = \langle \beta_1 \circ e, \beta_2 \circ e \rangle.$$

Thus, from m being strong, there exists a unique morphism $\gamma : Y \to B$ which makes the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ {\scriptstyle e}\downarrow & \overset{\exists!\gamma}{\nearrow} & \downarrow{\scriptstyle m} \\ Y & \xrightarrow[\beta_1]{} & B \end{array}$$

commute. We then see that, since $\alpha = \gamma \circ e$,

$$\langle m, g \rangle \circ \gamma \circ e = \langle m, g \rangle \circ \alpha = \beta \circ e.$$

Using that e is epic implies that

$$\langle m, g \rangle \circ \gamma = \beta$$

and hence implies that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & A \\ {\scriptstyle e}\downarrow & \overset{\exists!\gamma}{\nearrow} & \downarrow{\scriptstyle \langle m,g\rangle} \\ Y & \xrightarrow[\beta]{} & B \times C \end{array}$$

commutes. Note that the uniqueness follows because by Corollary 2.7, a strong morphism is monic and hence γ is unique by the fact that m must be monic. □

We now consider the definition of a quasitopos.

Definition 2.11. A category $\mathscr{C}$ is a *quasitopos* if $\mathscr{C}$ is finitely cocomplete, locally Cartesian closed, and has a strong subobject classifier.

Example 2.12. If $\mathcal{E}$ is a topos, then $\mathcal{E}$ is a quasitopos.

Example 2.13. Let $(\mathscr{C}, J)$ be a site in the sense of [4]. Then the category $\mathbf{Sep}(\mathscr{C}, J)$ of separated presheaves on $\mathscr{C}$ is a quasitopos.

Example 2.14. Let $\mathscr{C}$ be an Abelian category. Then $\mathscr{C}$ is not a quasitopos unless $\mathscr{C}$ is the category with one object and one morphism.

Remark 2.15. It is possible to define a power object functor $P : \mathscr{C} \to \mathscr{C}^{\text{op}}$, as in the case of a topos, for a quasitopos $\mathscr{C}$; however, the power object in a quasitopos is more poorly behaved and is not monadic. As such, the power objects in a quasitopos are called *weak power objects*. Let us construct this functor!

Let Ω be the strong subobject classifier in our quasitopos $\mathscr{C}$. We then define

$$P : \mathscr{C} \to \mathscr{C}^{\text{op}}$$

on objects $A \in \mathscr{C}_0$ by

$$A \mapsto [A, \Omega] =: PA.$$

For morphisms $f : A \to B$ in $\mathscr{C}$, we define $Pf : PB \to PA$ in $\mathscr{C}$ (so $Pf \in \mathscr{C}^{\mathrm{op}}(PA, PB)$) as follows. Recall that by Example 2.9 the unique morphism true : $\top \to \Omega$ is a strong monomorphism. Thus we get that for any $X \in \mathscr{C}_0$, the subobject $\in_X \rightarrowtail [X, \Omega] \times X$ is a strong subobject, as the diagram defining it is the pullback

$$\begin{array}{ccc} \in_X & \longrightarrow & \top \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{true}} \\ [X,\Omega] \times X & \xrightarrow[\mathrm{eval}]{} & \Omega \end{array}$$

and strong monomorphisms are preserved by pullbacks (cf. Lemma 2.8). With this we have that in any quasitopos we have "member of" relations. We now construct Pf by considering first the pullback:

$$\begin{array}{ccc} E_f & \longrightarrow & \in_B \\ {\scriptstyle m}\big\downarrow & \lrcorner & \big\downarrow \\ [B,\Omega] \times A & \xrightarrow[\mathrm{id}_{[B,\Omega]} \times f]{} & B \end{array}$$

Because $\in_B \rightarrowtail [B, \Omega] \times B$ is a strong monomorphism, so is $m : E_f \to [B, \Omega] \times A$. Thus we can classify m in $\mathscr{C}$ and produce a name to this relation. Explicitly:

$$\begin{array}{c} E_f \xrightarrow{m} [B,\Omega] \times A \\ \hline [B,\Omega] \times A \xrightarrow{\chi_m} \Omega \\ \hline [B,\Omega] \xrightarrow{\overline{\chi}_m} [A,\Omega] \\ \hline\hline PB \xrightarrow{Pf} PA \end{array}$$

Furthermore, we get a singleton morphism $\{\} : A \to [A, \Omega]$ in a quasitopos for any $A \in \mathscr{C}_0$ just as we did in a topos. Note that is because the diagonal morphism

$$\Delta : A \to A \times A$$

is a regular monic; in particular, it has kernel pair $(\pi_1, \pi_2) : A \times A \to A$. Thus Δ is a strong monomorphism by Lemma 2.5 and so there is a classifying map $\chi_\Delta : A \times A \to \Omega$ of the diagonal; taking this exponential transpose gives the desired signleton map. Explicitly:

$$\begin{array}{c} \Delta : A \to A \times A \\ \hline \chi_\Delta : A \times A \to \Omega \\ \hline \{\} : A \to [A, \Omega] \end{array}$$

This gives us much of the logical structure we have in a quasitopos that we have in a topos. However, we should be careful, as the Monadicity theorem is *false* in general in quasitoposes.

A Preliminary and Background Results

Here we discuss background material that is likely neither about quasitoposes or natural numbers objects explicitly. In particular, it is peripheral to our discussions, but is necessary as technical background or for some technique we use at some point.

The first main piece of background/peripheral information we need is that of T-algebras, where T is an endofunctor. These objects are in some sense less natural than algebras for a monad $\mathbb{T}$; however, the category of algebras for an endofunctor is related to the category of monad algebras (cf. Proposition A.6). For the next few propositions, $\mathscr{C}$ is some category and $T \in [\mathscr{C}, \mathscr{C}]_0$ is an endofunctor.

Definition A.1. An object $A \in \mathscr{C}_0$ is said to be a *T-algebra* if there is a morphism $\alpha \in \mathscr{C}(TA, A)$ and is denoted as a pair $(A, \alpha : TA \to A)$. A morphism of T-algebras $\rho : (A, \alpha) \to (B, \beta)$ is a morphism $\rho \in \mathscr{C}(A, B)$ such that the diagram

$$\begin{array}{ccc} TA & \xrightarrow{T\rho} & TB \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \beta} \\ A & \xrightarrow[\rho]{} & B \end{array}$$

commutes in $\mathscr{C}$. The category of all T-algebras will be denoted by $\mathscr{C}^T$.

Remark A.2. As usual for objects denoted by pairs, we will often refer to a T-algebra (A, α) simply by the object A if no confusion is likely to result.

The main technical result we need from the theory of T-algebras is Lambek's Theorem (cf. Theorem A.3 below). This is a theorem originally discovered by Lambek which allows one to characterize initial algebras in the category $\mathscr{C}^T$. In particular, it allows us to conclude that for any initial algebra A, TA is initial as well.

Theorem A.3 (Lambek's Theorem)**.** *Assume that $\mathscr{C}^T$ has an initial object A. The object TA is initial in $\mathscr{C}^T$.*

Proof. Begin by observing that the object TA is a T-algebra with structure map $T\alpha : T^2A \to TA$. Moreover, because the diagram since the diagram

$$\begin{array}{ccc} T^2A & \xrightarrow{T\alpha} & TA \\ {\scriptstyle T\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha} \\ TA & \xrightarrow[\alpha]{} & A \end{array}$$

commutes, there $\alpha : (TA, T\alpha) \to (A, \alpha)$ is a morphism in $\mathscr{C}^T$.

Because the object A is initial, there exists a morphism $\rho : (A, \alpha) \to (TA, T\alpha)$ which makes the diagram

$$\begin{array}{ccc} TA & \xrightarrow{T\rho} & T^2A \\ {\scriptstyle \alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle T\alpha} \\ A & \xrightarrow{\rho} & TA \end{array}$$

commute. Using that A is initial and that $\rho \circ \alpha$ is an endomorphism of A shows that the triangle

$$\begin{array}{ccc} A & \xrightarrow{\rho} & TA \\ & \diagdown\!\!\diagdown & \Big\downarrow{\scriptstyle \alpha} \\ & & A \end{array}$$

commutes in $\mathscr{C}^T$ and hence that $\rho \circ \alpha = \mathrm{id}_A$. From this we derive that

$$\mathrm{id}_{TA} = T(\mathrm{id}_A) = T(\rho \circ \alpha) = T\rho \circ T\alpha = \alpha \circ \rho,$$

where the last equality holds from using the commutativity of the diagram defining ρ. Thus ρ and α are mutual inverses and so $A \cong TA$. □

We now give the relation between the category of monad algebras and the category of endofunctor algebras, we first need to clarify what we mean by a "free monad generated by an endofunctor." This is the definition below.

Definition A.4. We say that a monad $\mathbb{T}$ on a category $\mathscr{C}$ is a *free monad generated by an endofunctor* T if $\mathbb{T} \cong \mathbb{F}(T)$, where $\mathbb{F}$ is a left adjoint to the forgetful functor $U : \mathbf{Monad}(\mathscr{C}) \to [\mathscr{C}, \mathscr{C}]$ and $T \in [\mathscr{C}, \mathscr{C}]_0$.

Remark A.5. If $\mathbb{T} = (T, \eta, \mu)$ is a monad with endofunctor T on a category $\mathscr{C}$, there is always forgetful functor $U : \mathscr{C}^{\mathbb{T}} \to \mathscr{C}^{T}$ which simply "forgets" the extra structure of the monad algebras.

Proposition A.6. *Let $\mathscr{C}$ be a category such that there is an adjunction:*

$$s$$

Then $\mathscr{C}^T \simeq \mathscr{C}^{\mathbb{F}(T)}$.

Corollary A.7. *If $\mathscr{C}^T$ has an inital object, so does $C^{\mathbb{F}(T)}$. Moreover, if A is the initial object in $\mathscr{C}^{\mathbb{F}(T)}$, then $\mathbb{F}(T)^n(A) \cong A$ for all $n \in \mathbb{N}$.*

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