Topos Theory Notes: Cartesian Closed Categories and Subobject Classifiers

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February 24, 2020

1 Cartesian Closed Categories

Throughout these two sections, when referring to a *Cartesian category* we mean a category which has all finite limits. Of course, such a category will have all pullbacks, and thus any morphism $f : A \to B$ in a Cartesian category $\mathbf{C}$ induces a functor between slice categories

$$f^* : \mathbf{C}/B \to \mathbf{C}/A$$

so that for any object $\varphi : X \to B$ in $\mathbf{C}/B$, its image under f^* is given by the morphism $\varphi \wedge f :\to C$ given by the pullback diagram

$$\begin{array}{ccc} \varphi \wedge f & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \varphi} \\ A & \xrightarrow{f} & B \end{array}$$

Any Cartesian category has a terminal object as it is the limit of the empty diagram. Thus, in the case where the considered morphism is the unique map into the terminal object $B \to 1$, for an object B in $\mathbf{C}$, then according functor

$$B^* : \mathbf{C} \to \mathbf{C}/B$$

will send an object A in $\mathbf{C}$ to the map $B^*(A) \to B$ given by the pullback diagram

$$\begin{array}{ccc} B^*(A) \cong B \times A & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & 1 \end{array}$$

Note that this construction is a special case of our previous construction by realizing the obvious isomorphism of categories $\mathbf{C}/1 \cong \mathbf{C}$.

This functor has an important left adjoint, likewise induced by the morphism $f : A \to B$, which is denoted

$$\Sigma_f : \mathbf{C}/A \to \mathbf{C}/B$$

and sends any object $g : X \to A$ in $\mathbf{C}/A$ to the object in $\mathbf{C}/B$ given by postcomposition with f, i.e. $gf : X \to B$. As was the case above, if the morphism inducing the functor is given by the unique morphism into the terminal object, say $B \to 1$, then we denote the functor as

$$\Sigma_B : \mathbf{C}/B \to \mathbf{C}$$

appealing once again to the isomorphism $\mathbf{C}/1 \cong \mathbf{C}$.

Definition 1.1. Let $\mathbf{C}$ be a category with finite products. We say an object A in $\mathbf{C}$ is *exponentiable* if its associated (left) product functor

$$(-) \times A : \mathbf{C} \to \mathbf{C}$$

exists and has a right adjoint, usually denoted $(-)^A$. We say $\mathbf{C}$ is *Cartesian closed* if every object in the category is exponentiable.

It does seem odd that we do not insist that a Cartesian closed category be a Cartesian category, but [1] insists that there are important examples of categories that are Cartesian closed but do not have all finite limits (e.g. the relationship between Cartesian closure and the lambda calculus, which we shall see later in the course). However, if a category is both Cartesian and Cartesian closed, we say that it is *properly Cartesian closed*, and note that it suffices for a Cartesian closed category to have all equalizers.

To understand the adjunction providing Cartesian closure, it is useful to consider the equivalent construction of the couniversal property, where the counit of the adjunction is (suggestively) denoted as the *evaluation map*

$$ev : B^A \times A \to B$$

such that it satisfies the usual couniversal property being that for any map $h : C \times A \to B$ in $\mathbf{C}$, there exists a unique $\overline{h} : C \to B^A$ so that the diagram

$$\begin{array}{ccc} C \times A & \xrightarrow{\overline{h} \times 1_A} & B^A \times A \\ & \searrow^{h} & \downarrow \text{ev} \\ & & B \end{array}$$

commutes. Here, the map $\overline{h}$ is commonly referred to as the *exponential transpose* of h and simply corresponds to the usual transpose provided by natural isomorphism

$$\mathbf{C}(C \times A, B) \cong \mathbf{C}(C, B^A)$$

corresponding to the adjunction.

Example 1.1. The category **Set** obviously has finite products, and is, in fact, Cartesian closed where the exponential functor is given by the covariant hom-functor i.e. the structure is given by the well known adjunction

$$(-) \times A \dashv \mathbf{Set}(A, -) : \mathbf{Set} \to \mathbf{Set}$$

for any set A. As mentioned before, one can check that in this case, the counit of the adjunction is exactly the evaluation map

$$\text{ev} : \mathbf{Set}(A, B) \times A \to B; \;\; (f, a) \mapsto f(a)$$

Example 1.2. [2] Let $\mathbf{C}$ be any small category. Then, we can see that the category of presheaves $[\mathbf{C}^{\text{op}}, \mathbf{Set}]$ is Cartesian closed. Given the precedent given by the Cartesian closure of $\mathbf{Set}$, one would think that the proper adjoint to the product functor would take the form

$$Q^P(C) = \mathbf{Set}(P(C), Q(C))$$

for presheaves Q and P. However, it is a simple exercise to check that this construction is not functorial. Instead, we suppose what would result should such a functor exist and work backwards. If this were the case, then we would have the natural bijection

$$[\mathbf{C}^{\text{op}}, \mathbf{Set}](R \times P, Q) \cong [\mathbf{C}^{\text{op}}, \mathbf{Set}](R, Q^P)$$

for any presheaves P, Q, and R. Then, considering the case where R is representable, say $R = \mathbf{C}(-, C)$ for some object C in $\mathbf{C}$, then we would have

$$\begin{aligned} Q^P(C) &\cong [\mathbf{C}^{\text{op}}, \mathbf{Set}](\mathbf{C}(-, C), Q^P) \\ &\cong [\mathbf{C}^{\text{op}}, \mathbf{Set}](\mathbf{C}(-, C) \times P, Q) \end{aligned}$$

And thus the proper adjoint turns out to be the construction

$$Q^P(C) := [\mathbf{C}^{\text{op}}, \mathbf{Set}](\mathbf{C}(-, C) \times P, Q)$$

being the set of all natural transformations from $\mathbf{C}(-, C) \times P \to Q$.

Recall that characterizing a category as Cartesian is a local property i.e. for any Cartesian category $\mathbf{C}$, the slice category $\mathbf{C}/B$ is a Cartesian category for any object B in $\mathbf{C}$.

Lemma 1.1. Let $\mathbf{C}$ be a Cartesian category. Then, the following properties hold:

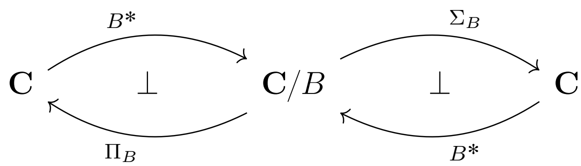
1. An object B of $\mathbf{C}$ is exponentiable if and only if the functor $B^* : \mathbf{C} \to \mathbf{C}/B$ has a right adjoint
$$\Pi_B : \mathbf{C}/B, \to \mathbf{C}.$$

2. If A is exponentiable in $\mathbf{C}$, then $B^*(A)$ is exponentiable in $\mathbf{C}/B$ for any B. Moreover, B^* preserves any exponentiabls which exist in $\mathbf{C}$.

Proof. (i): $(\Leftarrow)$: Suppose that $(B^* \dashv \Pi_B)$ form an adjoint pair. It is easy to check that $(-) \times B$ is equal to the composite

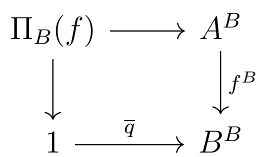
$$\mathbf{C} \xrightarrow{B^*} \mathbf{C}/B \xrightarrow{\Sigma_B} \mathbf{C}$$

So, if Π_B exists, define $(-)^B := \Pi_B \circ B^*$. Then, we have the diagram

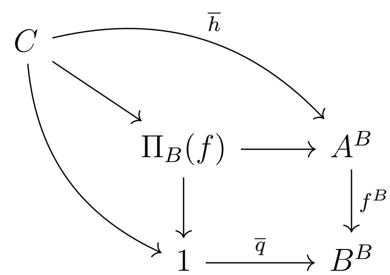


and thus $(-) \times B \dashv (-)^B$, since adjoints compose.

$(\Rightarrow)$: Conversely suppose B is exponentiable in $\mathbf{C}$. Then, for every object $f : A \to B$ in $\mathbf{C}/B$, define $\Pi_B(f)$ to be the pullback given by the diagram



where $\overline{q}$ is the exponential transpose of $q : 1 \times B \to B$. Now, for any object C of $\mathbf{C}$, the universal property of pullbacks says that any morphism $C \to \Pi_B(f)$ corresponds to a morphism $\overline{h} : C \to A^B$, as in the diagram



By the adjoint, these morphism then correspond to morphisms $h : C \times B \to A$ so that $hf : C \times B \to B$ is the product projection. There are then isomorsphism

$$C \times B \cong B \times C \cong B^*(C)$$

thus providing the final correspondence with morphsisms $B^*(C) \to f$ in $\mathbf{C}/B$. After checking for naturality, we can infer the natural isomorphism

$$\mathbf{C}(C, \Pi_B(f)) \cong \mathbf{C}/B(B^*(C), f)$$

(ii) : Given an object $f : C \to B$ in $\mathbf{C}/B$, define $f^{B^*(A)}$ to be the left vertical map in the pullback square

$$\begin{array}{ccc} f^A \wedge \overline{p} & \longrightarrow & C^A \\ {\scriptstyle f^{B^*(A)}}\big\downarrow & & \big\downarrow{\scriptstyle f^A} \\ B & \xrightarrow{\overline{p}} & B^A \end{array}$$

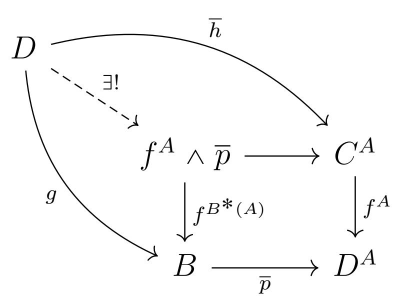
where $\overline{p}$ is the exponential transpose of the projection $p : B \times A \to B$. Then, if $g : D \to B$ is another object of $\mathbf{C}/B$, the product $g \times B^*(A)$ is the diagonal map (in $\mathbf{C}$) of the diagram

$$\begin{array}{ccc} D \times A & \xrightarrow{\langle g, 1_A \rangle} & B \times A \\ {\scriptstyle \pi_0}\big\downarrow & \searrow & \big\downarrow{\scriptstyle p} \\ D & \xrightarrow{g} & B \end{array}$$

which is easily verified to be a pullback square. Then, morphisms $h : g \times B^*(A) \to f$ in $\mathbf{C}/B$ correspond to morphisms $\overline{h} : D \to C^A$ in $\mathbf{C}$ so that $\overline{h} f^A = g\overline{p}$. This is given explicitly in the following two diagrams

$$\begin{array}{ccc} D \times B^*(A) & \xrightarrow{h} & C \\ {\scriptstyle \langle g, \pi_0 \rangle}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & = & B \end{array} \qquad\qquad \begin{array}{ccc} D & \xrightarrow{\overline{h}} & C^A \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f^A} \\ B & \xrightarrow{\overline{p}} & D^A \end{array}$$

where the correspondence is given using the acknowledgement of the isomorphism $B^*(A) \cong B \times A$. This must then correspond to morphisms $g \to f^{B^*(A)}$ in $\mathbf{C}/B$, using the universal property of pullbacks to achieve the diagram



Finally, if f is itself of the form $B^*(E)$ for some object E in $\mathbf{C}$, then the pullbacks square we considered above is reduced to

$$\begin{array}{ccc} E^A \times B & \xrightarrow{\langle 1, \overline{p} \rangle} & E^A \times B^A \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow{\overline{p}} & B^A \end{array}$$

since the functor $(-)^A$ preserves products as it is a right adjoint. This shows that exponentials are preserved. $\square$

Corollary 1.1. Let $\mathbf{C}$ be a Cartesian category. Then $\mathbf{C}$ is locally Cartesian closed, i.e. $\mathbf{C}/B$ is Cartesian closed for every object B of $\mathbf{C}$, if and only if, for every morphism $f : A \to B$ in $\mathbf{C}$, the functor $f^* : \mathbf{C}/B \to \mathbf{C}/A$ has a right adjoint $\Pi_f : \mathbf{C}/A \to \mathbf{C}/B$.

Proof. This result follows immediately from the previous lemma and the well known equivalence $(\mathbf{C}/B)/f \cong \mathbf{C}/A$, for a morphism $f : A \to B$. $\square$

Definition 1.2. Let $\mathbf{C}$ and $\mathbf{D}$ be Cartesian closed categories and let $F : \mathbf{C} \to \mathbf{D}$ be a functor that preserves finite products. Then F is called *Cartesian closed* if the map $\theta_{A,B} : F(B^A) \to F(B)^{F(A)}$ is an isomorphism for all objects A, B in $\mathbf{C}$, where $\theta_{A,B}$ is the transpose of the counit

$$F(B^A) \times F(A) \cong F(B^A \times A) \xrightarrow{F(\mathrm{ev})} F(B)$$

under the natural isomorphism given by the adjunction,

$$\mathbf{D}(F(B^A) \times F(A), F(B)) \cong \mathbf{C}(F(B^A), F(B)^{F(A)}).$$

Lemma 1.2. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between categories that are Cartesian closed with a left adjoint L. Then F is Cartesian closed if and only if the cannonical morphism

$$(L(\pi_0), L(\pi_1)\varepsilon_A) : L(B \times F(A)) \to L(B) \times A$$

is an isomorphism for all object A and B of $\mathbf{C}$ and $\mathbf{D}$ respectively, where ε is the counit of the adjunction $(L \dashv F)$.

In practice, this lemma has obvious utility for testing for the Cartesian closure of functors. However we omit the proof as it necessitates the verification that quite an intricate pair of compositions provide the required inverse morphisms, though an exposition of these inverses can be found in [1], pp. 51,52.

To conclude this section, we state without proof one final result connecting Cartesian closed categories with regular categories studied in the previous seminar. The proof of this proposition can be found in [1], pp. 55,56.

Proposition 1.1. For any properly Cartesian closed category $\mathbf{C}$, the category $\mathbf{Reg}(\mathbf{C})$ is Cartesian closed.

2 Subobject Classifiers

Before we begin, it is worth noting that many authours introduce subobject classifiers through a construction defining its domain to be the terminal object in the category. Johnstone, on the other hand, introduces this concept showing that this property is, in fact, a consequence of the subobject classifier we wish to see in a subobject classifier. We follow this construction:

Definition 2.1. Let **C** be a category with pullbacks. A *generic subobject* in **C** is a monomorphism $\top : B' \rightarrowtail B$ so that for any other monomorphism $m : A' \rightarrowtail A$ in **C**, there exists a unique morphism $f : A \to B$ such that the following diagram forms a pullback square:

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \top} \\ A & \underset{\exists! f}{\longrightarrow} & B \end{array}$$

In this case, f is call the *classifying map* or *characterstic morphism* associated to m.

Lemma 2.1. Let **C** be a category with pullbacks and a generic subobject. Then, the domain of the generic subobject is necessarily the terminal object in **C**.

Proof. Let $\top : B' \rightarrowtail B$ be a generic subobject in a category with pullbacks **C**. The identiy on any object A in **C** is monic, and so there exists a map $\psi : A \to B'$ which is given by the pullback

$$\begin{array}{ccc} A & \overset{\psi}{\longrightarrow} & B' \\ \| & & \downarrow{\scriptstyle \top} \\ A & \underset{\exists! \varphi}{\longrightarrow} & B \end{array}$$

Moreover, if $\omega : A \to B'$ is any other morphism, then

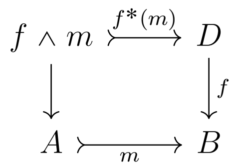
$$\begin{array}{ccc} A & \overset{\omega}{\longrightarrow} & B' \\ \| & & \downarrow{\scriptstyle \top} \\ A & \underset{\omega\top}{\longrightarrow} & B \end{array}$$

But φ is the unique charictaristic map for 1_A, and so $\omega\top = \varphi = \psi\top$ and hence $\omega = \psi$ since $\top$ is monic. □

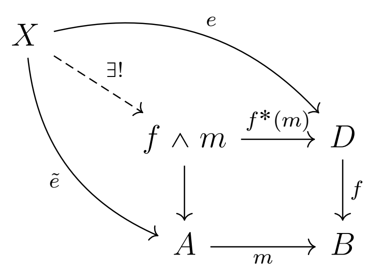
Definition 2.2. Let **C** be a category with pullbacks and a generic subobject 1Ω. Then the codomain of the generic subobject Ω is called the *subobject classifier* of **C**.

Lemma 2.2. In any category, the pullback of a regular monomorphism, should it exist, is itself regular.

Proof. In a category $\mathbf{C}$, suppose that $m : A \rightarrowtail B$ is a regular monomorphism so that it equalizes $g, h : B \to C$. Furthermore, suppose that $f : D \to B$ is some morphism so that the pullback



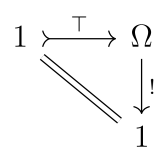
exists. Then, it is easy to see that $f^*(m)$ equalises the pair fg, fh. Indeed, if $e : XD$ is any other arrow so that $efg = efh$, then ef equalizes the pair (g, h), and so the universal property of equalizers gives a map $\tilde{e} : X \to m$ so that $\tilde{e}m = ef$. In this case, there is then a unique arrow $X \to f \wedge m$ so that the diagram



by the universal property of pullbacks. By the uniqueness of this arrow and the uniqueness of $\tilde{e}$, we can see that $f^*(m)$ is an equalizer and thus a regular monomorphism. $\square$

Corollary 2.1. In any category with pullbacks and a subobject classifier, every monomorphism is regular. In particular, any such category is balanced (every morphism that is both epic and monic is an isomorphism).

Proof. Let $\mathbf{C}$ be a category with pullbacks and a subobject classifier $1 \rightarrowtail \Omega$. Notice that the triangle



commutes, and so $\top$ equalizes the pair $(!\top, 1_\Omega)$. So, $\top$ is a regular monomorphim and any other monomorphism in $\mathbf{C}$ is the pullback along $\top$ along its characteristic map, and thus must also be regular by the previous lemma. Furthermore, it is a well known fact that any epic equalizer is an isomorphism, and thus $\mathbf{C}$ is balanced. $\square$

Example 2.1. In the category **Set**, the subobject classifier is given by

$$\top : \{*\} \rightarrowtail \Omega := \{0,1\};\ * \mapsto 1$$

Then, if $m : A \rightarrowtail B$ is any other monomorphism of sets, that is an injective function, define the set function $f : B \to \Omega$ so that for any $b \in B$,

$$f(b) = \begin{cases} 1 \text{ if } b = m(a) \text{ for some } a \in A; \\ 0 \text{ otherwise.} \end{cases}$$

Then, from a brief diagram chase, one can check that

$$\begin{array}{ccc} A & \xrightarrow{!} & \{*\} \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \top} \\ B & \xrightarrow[f]{} & \Omega \end{array}$$

is a pullback square, and that f is the unique map allowing this property.

Example 2.2. [2] For any small category $\mathbf{C}$, the category of copresheaves $[\mathbf{C}, \mathbf{Set}]$ has a subobject classifier, given by the following construction: Let F be a subfunctor of $\mathbf{C}(A, -)$ for A an object of $\mathbf{C}$. Then, this functor is completely characterized by the set

$$R = \{f \in \mathbf{C}_1 \mid \text{dom } f = A \text{ and } f \in F(\text{codom } f)\}.$$

Then R is called a *cosieve* on A. Thus, the subobject classifer Ω in $[\mathbf{C}, \mathbf{Set}]$ is defined to behave on objects so that

$$\Omega(A) = \{\text{cosieves on A}\}$$

and on morphisms so that, for any $R \in \Omega(A)$ and any morphism $f : A \to B$,

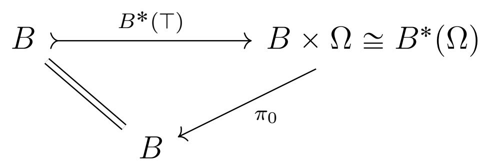
$$\Omega(f)(R) = \{g \in \mathbf{C}_1 \mid \ \text{dom } g = B \text{ and } gf \in R\}$$

The case for the presheaf category $[\mathbf{C}^{\text{op}}, \mathbf{Set}]$ is of course given by dualizing this notion. Importantly, in the case where $\mathbf{C} = \mathbf{Open}(X)$, the open lattice of a topological space X, then, for any open subset $U \subseteq X$, we have

$$\Omega(U) = \{V \mid V \text{ is open and } V \subset U\}.$$

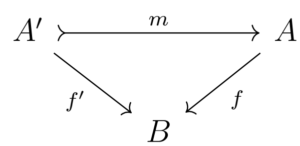
In fact, one can check that, in this case as well as in the more general case above, the presheaf Ω is, in fact, a sheaf.

Lemma 2.3. If Ω is a subobject classifier in a category $\mathbf{C}$ (with pullbacks), then the triangle

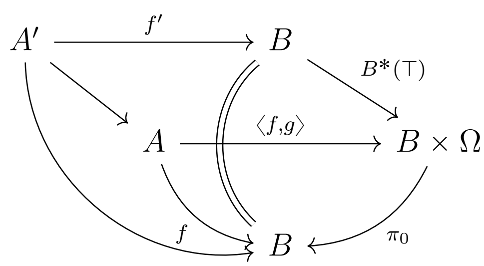


is a generic subobject, and thus defines a subobject classifier, in the slice category $\mathbf{C}/B$, for any object B of $\mathbf{C}$.

Proof. Let the triangle



denote a monomorphism in the category $\mathbf{C}$. Then $m : A' \rightarrowtail A$ is a monomorphism in $\mathbf{C}$, so let $g : A \to \Omega$ be its classifying map in this category. Define the component map $\langle f, g\rangle : A \to B \times \Omega$. Then, it is easy to check that the prism



forms a pullback $\mathbf{C}/B$, and the uniqueness of $\langle f, g\rangle$ is given by the uniqueness of g in $\mathbf{C}$ (since the forgetful functor $U : \mathbf{C}/B \to \mathbf{C}$ creates limits). □

We end with a fun little lemma, describing the behaviour of monic endomorphism on a subobject classifier, as the proof provides a good explication of the behaviour of the subobject classifier itself.

Lemma 2.4. Let $f : \Omega \to \Omega$ be a monomorphism between subobject classifiers. Then, $ff = 1_\Omega$.

Proof. First, form the pullback diagram given by the subobject classifier

Since f is monic, this implies that there are monomorphisms $V \xrightarrow{h} U \to 1$, which we can use to form the diagram

$$\begin{array}{ccccccc}
V & = & V & \overset{h}{\rightarrowtail} & U & \overset{!}{\longrightarrow} & 1 \\
{\scriptstyle h}\downarrow & & \downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \top} \\
U & \overset{!}{\longrightarrow} & 1 & \overset{\top}{\rightarrowtail} & \Omega & \overset{f}{\rightarrowtail} & \Omega
\end{array}$$

from which it is easy to verify that each square, and hence the outer rectangle, is a pullback square. Therefore, the composition of the three bottom arrows is a classifying map, and we have $!_U \top f = g$. Furthermore, this gives $gff = g$ since $!_U \top = gf$, and so the diagram

$$\begin{array}{ccc}
U & = & U \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\
\Omega & \overset{ff}{\longrightarrow} & \Omega
\end{array}$$

which is a pullback, which is seen using the fact that ff and g are both monic. Putting this diagram together with what we started with gives that the outer rectangle

$$\begin{array}{ccccc}
U & = & U & \overset{!}{\longrightarrow} & 1 \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \top} \\
\Omega & \overset{ff}{\longrightarrow} & \Omega & \overset{f}{\longrightarrow} & \Omega
\end{array}$$

is a pullback square, and hence $fff = f$ by the uniques of the classifying map. Furthermore, since f is monic, and monomorphisms compose, we have $ff = 1_\Omega$, as required. □

References

[1] Peter T. Johstone *Sketches of an Elephant*, Oxford University Press, 2002;

[2] Saunders Mac Lane, Ieke Moerdijk, *Sheaves in Geometry and Logic*, Springer, 1992.