CPSC 531:
System Modeling and Simulation

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Outline

- Probability and random variables
  - Random experiment and random variable
  - Probability mass/density functions
  - Expectation, variance, covariance, correlation
- Probability distributions
  - Discrete probability distributions
  - Continuous probability distributions
  - Empirical probability distributions

Random Experiment

- An experiment is called *random* if the outcome of the experiment is uncertain
- For a random experiment:
  - The set of all possible outcomes is known before the experiment
  - The outcome of the experiment is not known in advance
- *Sample space* Ω of an experiment is the set of all possible outcomes of the experiment
- Example: Consider random experiment of tossing a coin twice. Sample space is:

$$\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$$

Probability of Events

- An *event* is a subset of sample space

  Example 1: in tossing a coin twice, $E=\{(H,H)\}$ is the event of having two heads

  Example 2: in tossing a coin twice, $E=\{(H,H), (H,T)\}$ is the event of having a head in the first toss

- *Probability* of an event E is a numerical measure of the likelihood that event E will occur, expressed as a number between 0 and 1,

$$0 \leq \mathbb{P}(E) \leq 1$$

  – If all possible outcomes are equally likely: $\mathbb{P}(E) = |E|/|\Omega|$
  – Probability of the sample space is 1: $\mathbb{P}(\Omega) = 1$

Joint Probability

- Probability that two events A and B occur in a single experiment:

$$\mathbb{P}(A \textbf{ and } B) = \mathbb{P}(A \cap B)$$

- Example: drawing a single card at random from a regular deck of cards, probability of getting a red king
  - A: getting a red card
  - B: getting a king
  - $\mathbb{P}(A \cap B) = \frac{2}{52}$

Independent Events

- Two events A and B are independent if the occurrence of one does not affect the occurrence of the other:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

- Example: drawing a single card at random from a regular deck of cards, probability of getting a red king
  - A: getting a red card $\Rightarrow \mathbb{P}(A) = 26/52$
  - B: getting a king $\Rightarrow \mathbb{P}(B) = 4/52$
  - $\mathbb{P}(A \cap B) = \frac{2}{52} = \mathbb{P}(A)\mathbb{P}(B) \Rightarrow A$ and B are independent

Mutually Exclusive Events

- Events A and B are mutually exclusive if the occurrence of one implies the non-occurrence of the other, i.e., $A \cap B = \phi$:

$$\mathbb{P}(A \cap B) = 0$$

- Example: drawing a single card at random from a regular deck of cards, probability of getting a red club
  - A: getting a red card
  - B: getting a club
  - $\mathbb{P}(A \cap B) = 0$

- Complementary event of event A is event $[not\ A]$, i.e., the event that A does not occur, denoted by $\bar{A}$
  - Events A and $\bar{A}$ are mutually exclusive
  - $\mathbb{P}(\bar{A}) = 1 - \mathbb{P}(A)$

Union Probability

- Union of events A and B:

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

- If A and B are mutually exclusive:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$$

- Example: drawing a single card at random from a regular deck of cards, probability of getting a red card or a king
  - A: getting a red card $\Rightarrow \mathbb{P}(A) = 26/52$
  - B: getting a king $\Rightarrow \mathbb{P}(B) = 4/52$
  - $\mathbb{P}(A \cap B) = \frac{2}{52}$
  - $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$

Conditional Probability

- Probability of event A given the occurrence of some event B:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

- If events A and B are independent:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = P(A)$$

- Example: drawing a single card at random from a regular deck of cards, probability of getting a king given that the card is red
  - A: getting a red card $\Rightarrow \mathbb{P}(A) = 26/52$
  - B: getting a king $\Rightarrow \mathbb{P}(B) = 4/52$
  - $\mathbb{P}(A \cap B) = \frac{2}{52}$
  - $\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{2}{26} = \mathbb{P}(B)$

Types of Random Variables

- Discrete
  - Random variables whose set of possible values can be written as a finite or infinite sequence
  - Example: number of requests sent to a web server
- Continuous
  - Random variables that take a continuum of possible values
  - Example: time between requests sent to a web server

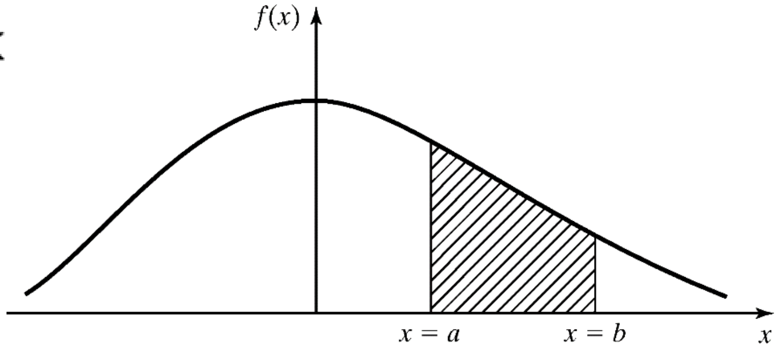
Probability Density Function (PDF)

- X: continuous random variable
- $f(x)$: probability density function of X

$$f(x) = \frac{d}{dx} F(x)$$

CDF of X

- Note:
  - $\mathbb{P}(X = x) = 0$!!
  - $f(x) \neq \mathbb{P}(X = x)$
  - $\mathbb{P}(x \leq X \leq x + \Delta x) \approx f(x)\Delta x$
- Properties:
  - $\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx$
  - $\int_{-\infty}^{+\infty} f(x)dx = 1$



Cumulative Distribution Function (CDF)

- X: discrete or continuous random variable
- $F(x)$: cumulative probability distribution function of X, or simply, probability distribution function of X

$$F(x) = \mathbb{P}(X \leq x)$$

  - If X is discrete, then $F(x) = \sum_{x_i \leq x} p(x_i)$
  - If X is continuous, then $F(x) = \int_{-\infty}^{x} f(t)dt$

- Properties
  - $F(x)$ is a non-decreasing function, i.e., if $a < b$, then $F(a) \leq F(b)$
  - $\lim_{x \to +\infty} F(x) = 1$, and $\lim_{x \to -\infty} F(x) = 0$

- All probability questions about X can be answered in terms of the CDF, e.g.:

$$\mathbb{P}(a < X \leq b) = F(b) - F(a), \text{ for all } a \leq b$$

Expectation of a Random Variable

- **Mean** or **Expected Value**:

$$\mu = E[X] = \begin{cases} \sum_{i=1}^{n} x_i p(x_i) & \text{discrete } X \\ \int_{-\infty}^{\infty} x f(x) dx & \text{continuous } X \end{cases}$$

- Example: number of heads in tossing three coins

$$E[X] = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3)$$
$$= 1 \cdot 3/8 + 2 \cdot 3/8 + 3 \cdot 1/8$$
$$= 12/8$$
$$= 1.5$$

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Properties of Expectation

- X, Y: two random variables
- a, b: two constants

$$E[aX] = aE[X]$$

$$E[X + b] = E[X] + b$$

$$E[X + Y] = E[X] + E[Y]$$

$$E[X - Y] = E[X] - E[Y]$$

Misuses of Expectations

- ***Multiplying means to get the mean of a product***

$$E[XY] \neq E[X]E[Y]$$

- Example: tossing three coins
  - X: number of heads
  - Y: number of tails
  - $E[X] = E[Y] = 3/2 \Rightarrow E[X]E[Y] = 9/4$
  - $E[XY] = 3/2$

$$\Rightarrow E[XY] \neq E[X]E[Y]$$

- ***Dividing means to get the mean of a ratio***

$$E\left[\frac{X}{Y}\right] \neq \frac{E[X]}{E[Y]}$$

Variance of a Random Variable

- The variance is a measure of the *spread* of a distribution around its mean value
- Variance is symbolized by $V[X]$ or $Var[X]$ or σ^2:
  - Mean is a way to describe the *location* of a distribution
  - Variance is a way to capture its *scale or degree* of being spread out
  - The unit of variance is the square of the unit of the original variable
- σ: standard deviation
  - Defined as the square root of variance $V[X]$
  - Expressed in the same units as the mean

Variance of a Random Variable

- **Variance**: The expected value of the square of distance between a random variable and its mean

$$\sigma^2 = V[X]$$

$$= E[(X-\mu)^2] = \begin{cases} \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i) & \text{discrete } X \\ \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx & \text{continuous } X \end{cases}$$

where, $\mu = E[X]$

- Equivalently:

$$\sigma^2 = E[X^2] - (E[X])^2$$

Properties of Variance

- X, Y: two random variables
- a, b: two constants

$$V[X] \geq 0$$

$$V[aX] = a^2 V[X]$$

$$V[X + b] = V[X]$$

- If X and Y are independent:

$$V[X + Y] = V[X] + V[Y]$$

Coefficient of Variation

- **Coefficient of Variation**:

$$\mathrm{CV} = \frac{\text{Standard Deviation}}{\text{Mean}} = \frac{\sigma}{\mu}$$

- Example: number of heads in tossing three coins

$$\mathrm{CV} = \frac{\sqrt{3/4}}{3/2} = \frac{1}{\sqrt{3}}$$

Covariance

- Covariance between random variables X and Y denoted by $Cov(X,Y)$ or $\sigma^2_{X,Y}$ is a measure of how much X and Y change together

$$\sigma^2_{X,Y} = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

- For independent variables, the covariance is zero:

$$E[XY] = E[X]E[Y]$$

- Note: Although independence always implies zero covariance, the reverse is **not** true

Covariance

- Example: tossing three coins
  - X: number of heads
  - Y: number of tails
  - $E[X] = E[Y] = 3/2$
- $E[XY]$?
  - X and Y depend on each other
  - $Y = 3 - X$
  - $E[XY] = 0 \times P(0) + 2 \times P(2)$
    $= 3/2$
- $\sigma^2_{X,Y} = E[XY] - E[X]E[Y]$
  $= 3/2 - 3/2 \times 3/2$
  $= -3/4$

x	y	xy	$p(x)$
0	3	0	1/8
1	2	2	3/8
2	1	2	3/8
3	0	0	1/8

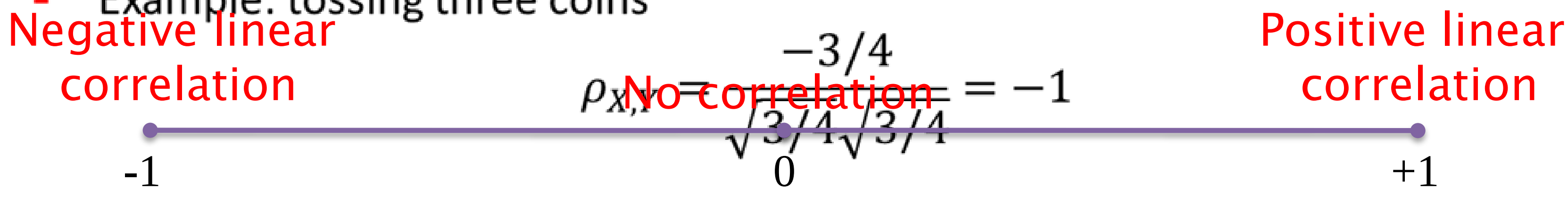
xy	$p(xy)$
0	2/8
2	6/8

Correlation

- Correlation Coefficient between random variables X and Y, denoted by $\rho_{X,Y}$, is the normalized value of their covariance:

$$\rho_{X,Y} = \frac{\sigma_{X,Y}^2}{\sigma_X \sigma_Y}$$

- Indicates the strength and direction of a linear relationship between two random variables
- The correlation always lies between -1 and +1



- Example: tossing three coins

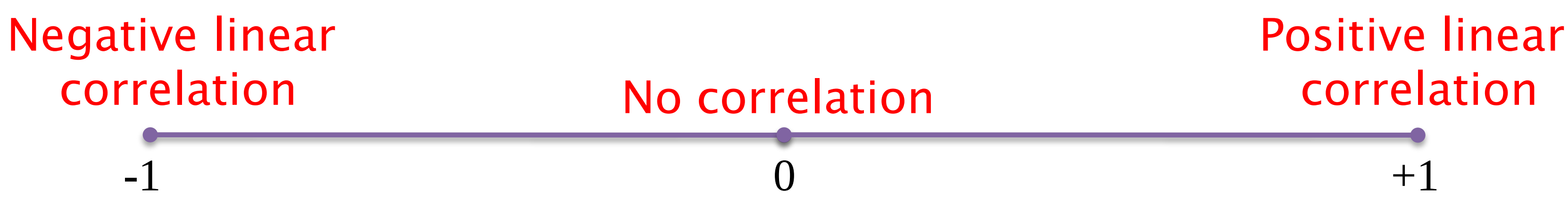
$$\rho_{X,Y} = \frac{-3/4}{\sqrt{3/4}\sqrt{3/4}} = -1$$

Autocorrelation

- Correlation Coefficient between a random variable and itself at different time lags within a sequence

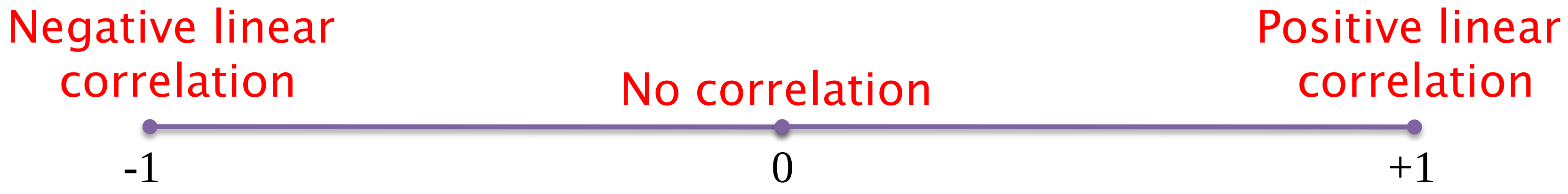
$$\rho_{X,X} = \frac{\sigma^2_{X,X}}{\sigma_X \sigma_X}$$

- Indicates the strength and direction of a linear relationship between two random variables
- The correlation always lies between -1 and +1
- It is always +1 at lag 0



Demo: Correlation and Autocorrelation

- Correlation (if desired) can be induced by sharing or re-using random numbers between two (or more) random variables
- Example: height and weight of medical patients
- Example: a coin that remembers some of its recent history



Geometric Distribution

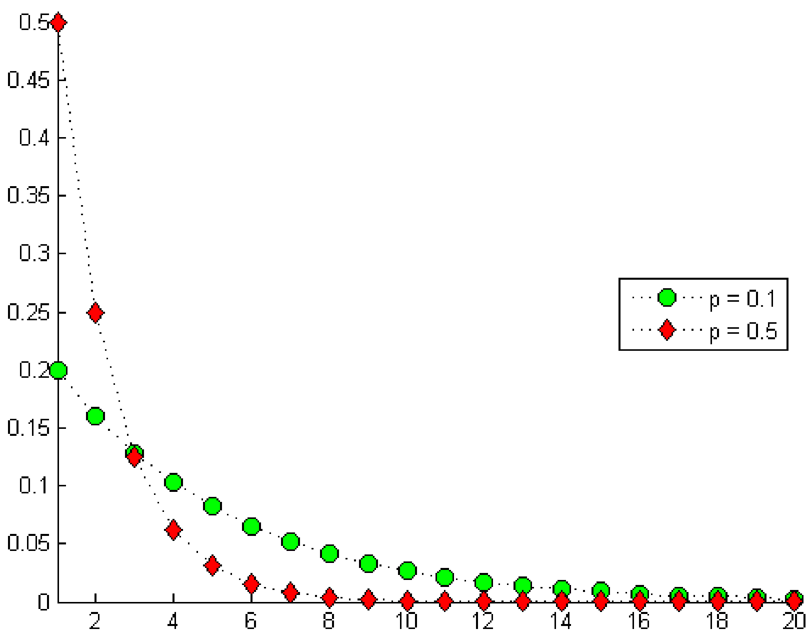
- X: number of Bernoulli trials until achieving the first success
- X is a geometric random variable with success probability p
- PMF: probability of k $(\mathrm{k} = 1,2,3,\ldots)$ trials until the first success

$$p(k) = p(1-p)^{k-1}$$

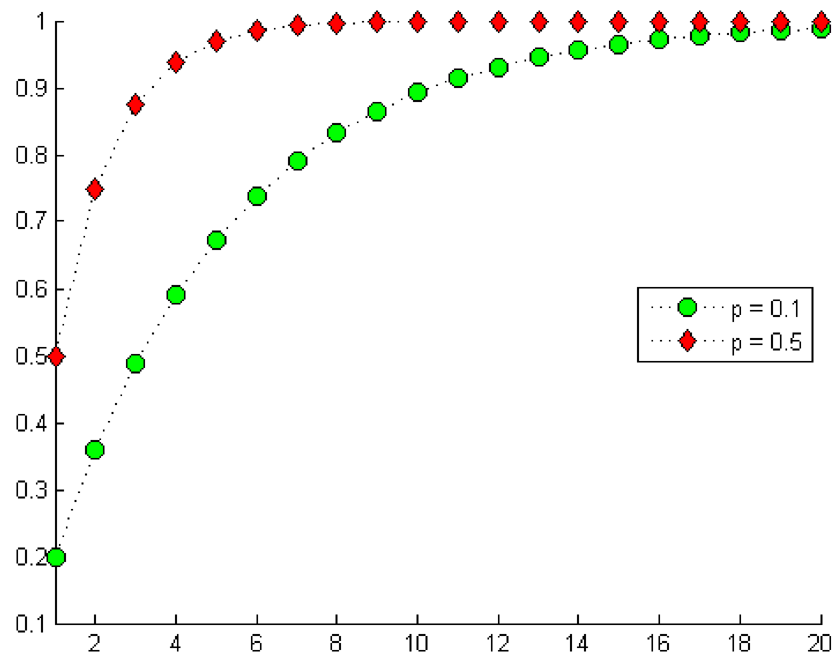
- CDF: $F(k) = 1 - (1-p)^k$
- Properties:

$$E[X] = \frac{1}{p}, \text{ and } V[X] = \frac{1-p}{p^2}$$

Example: Geometric Distribution



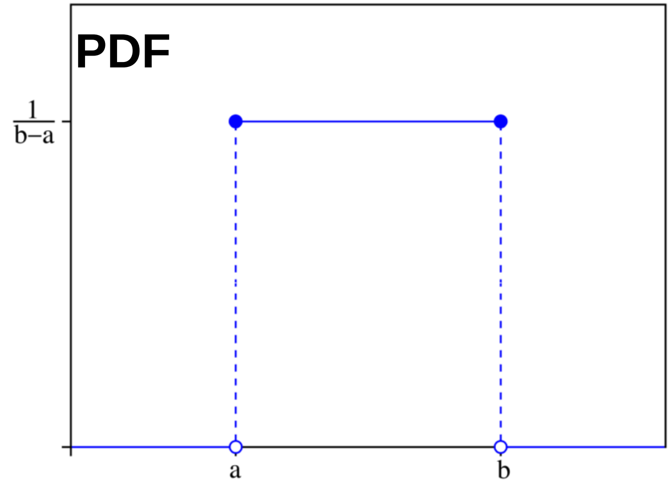
Geometric distribution PMF

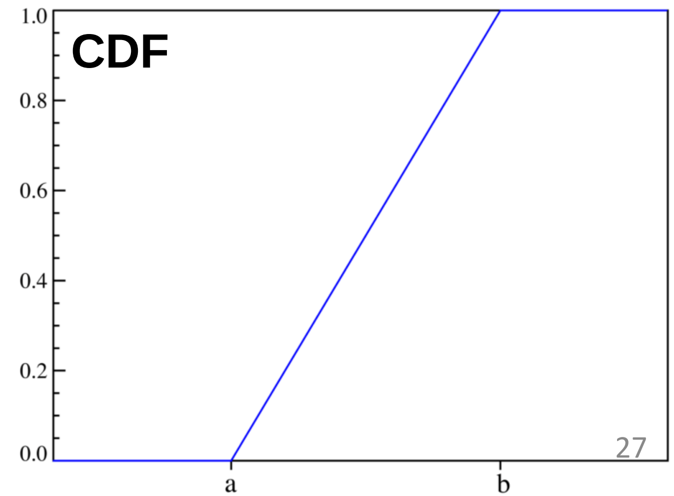


Geometric distribution CDF

Uniform Distribution

- A random variable X has **continuous uniform distribution** on the interval $[a,b]$, if its PDF and CDF are:
  - PDF: $f(x) = \frac{1}{b-a}$, for $a \leq x \leq b$
  - CDF: $F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq \mathrm{b} \\ 1 & x > b \end{cases}$





- Properties:

$$E[X] = \frac{a+b}{2}, \text{ and } V[X] = \frac{(a-b)^2}{12}$$

Uniform Distribution Properties

- $\mathbb{P}(x_1 < X < x_2)$ is proportional to the length of the interval $x_2 - x_1$

$$\mathbb{P}(x_1 < X < x_2) = \frac{x_2 - x_1}{b - a}$$

- Special case: standard uniform distribution denoted by $X \sim U(0,1)$

Example: Life of an inspection device is given by X, a continuous random variable with PDF:

$$f(x) = \frac{1}{2} e^{-\frac{x}{2}}, \quad \text{for } x \geq 0$$

- X has an exponential distribution with mean 2 years
- Probability that the device's life is between 2 and 3 years:

$$\mathbb{P}(2 \leq X \leq 3) = \frac{1}{2} \int_2^3 e^{-\frac{x}{2}} dx = 0.14$$

  - Very useful for random number generation in simulations

Exponential Distribution

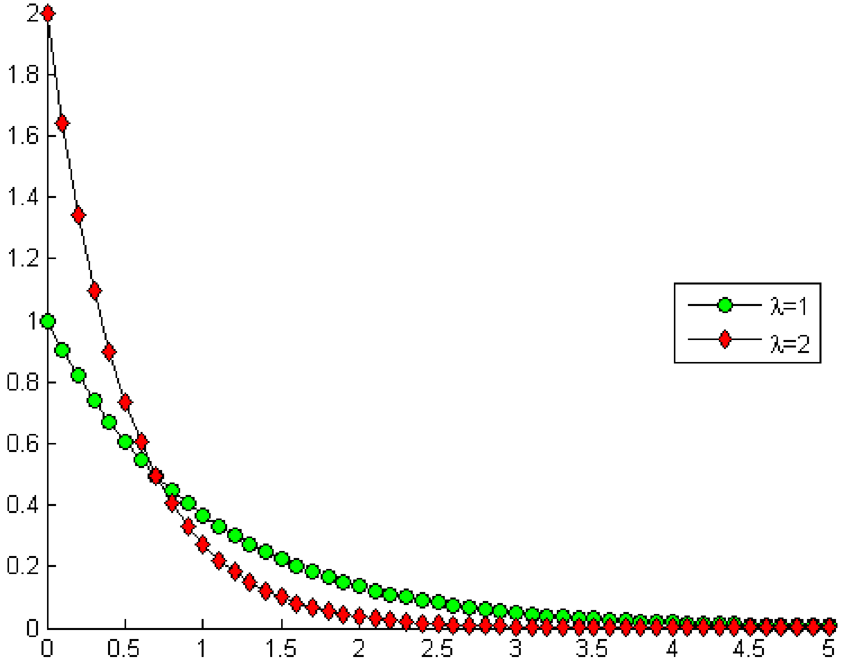
- A random variable X is exponentially distributed with parameter λ if its PDF and CDF are:
  - PDF: $f(x) = \lambda\, e^{-\lambda x}$, for $x \geq 0$
  - CDF: $F(x) = 1 - e^{-\lambda x}$, for $x \geq 0$

- Properties:

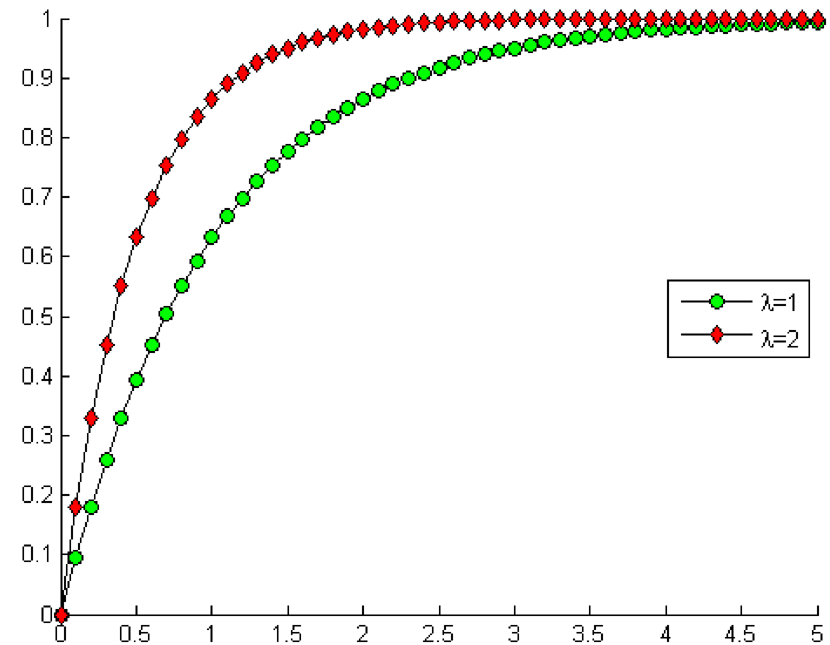
$E[X] = \frac{1}{\lambda}$, and $V[X] = \frac{1}{\lambda^2}$

- The exponential distribution describes the time between consecutive events in a Poisson process of rate λ

Example: Exponential Distribution



Exponential distribution PDF



Exponential distribution CDF

Light Bulb Testing (1 of 5)

- Scenario: Walmart has a giant bin of lightbulbs on sale. You buy one and bring it home for testing and observation.
- Assume: All light bulbs last exactly 100 hours.
- Observation: Your light bulb has worked for 70 hours.
- Question: How much longer is it expected to last?
- Answer: 30 hours

Light Bulb Testing (2 of 5)

- Scenario: Walmart has a giant bin of lightbulbs on sale. You buy one and bring it home for testing and observation.
- Assume: Half of the light bulbs last exactly 50 hours, while the other half last exactly 150 hours. The mean is 100 hours.
- Observation: Your light bulb has worked for 70 hours.
- Question: How much longer is it expected to last?
- Answer: 80 hours

Light Bulb Testing (3 of 5)

- Scenario: Walmart has a giant bin of lightbulbs on sale. You buy one and bring it home for testing and observation.
- Assume: Half of the light bulbs last exactly 50 hours, while the other half last exactly 150 hours. The mean is 100 hours.
- Observation: Your light bulb has worked for 40 hours.
- Question: How much longer is it expected to last?
- Answer: 60 hours

Light Bulb Testing (4 of 5)

- Scenario: Walmart has a giant bin of lightbulbs on sale. You buy one and bring it home for testing and observation.

- Assume: Light bulbs have a working duration that is uniformly distributed (continuous) between 50 hours and 150 hours. The mean is 100 hours.

- Observation: Your light bulb has worked for 70 hours.

- Question: How much longer is it expected to last?

- Answer: 40 hours

Light Bulb Testing (5 of 5)

- Scenario: Walmart has a giant bin of lightbulbs on sale. You buy one and bring it home for testing and observation.
- Assume: Light bulbs have a working duration that is exponentially distributed with a mean of 100 hours.
- Observation: Your light bulb has worked for 70 hours.
- Question: How much longer is it expected to last?
- Answer: 100 hours

Memoryless Property

- Memoryless is a property of certain probability distributions such as exponential distribution and geometric distribution
  - future events do not depend on the past events, but only on the present event

- Formally: random variable X has a memoryless distribution if

$$\mathbb{P}(X > t + s \mid X > s) = \mathbb{P}(X > t), \text{ for } s, t \geq 0$$

- Example: The probability that you will wait t more minutes given that you have already been waiting s minutes is the same as the probability that you wait for more than t minutes from the beginning!

Example: Exponential Distribution

- X: discrete or continuous random variable
- $F(x)$: cumulative probability distribution function of X, or simply, probability distribution function of X

$$F(x) = \mathbb{P}(X \leq x)$$

  - If X is discrete, then $F(x) = \sum_{x_i \leq x} p(x_i)$
  - If X is continuous, then $F(x) = \int_{-\infty}^{x} f(t)dt$

- Properties
  - $F(x)$ is a non-decreasing function, i.e., if $a < b$, then $F(a) \leq F(b)$
  - $\lim_{x \to +\infty} F(x) = 1$, and $\lim_{x \to -\infty} F(x) = 0$

- All probability questions about X can be answered in terms of the CDF, e.g.:

$$\mathbb{P}(a \leq X \leq b) = F(b) - F(a), \text{ for all } a \leq b$$