Reading #6

Asymptotic Notation and Standard Functions

Asymptotic Notation

Asymptotic Notation provides information about the ***relative rates of growth*** of a pair of functions (each functions of a single integer or real variable), but ignores or hides other details, including

- behaviour on *small* inputs — because results are generally most significant or meaningful when inputs are extremely *large* — and
- multiplicative constants and lower-order terms — which can be implementation- or platform-dependent anyway.

This provides the classification of algorithms into *classes* (such as linear-time, quadratic-time, polynomial-time or exponential-time, *etc.*), and is useful for stating bounds on the running times of algorithms.

Asymptotically Positive Functions

Consider either a total or partial function $f : \mathbb{N} \to \mathbb{N}$ or $f : \mathbb{R} \to \mathbb{R}$.

Definition 1. f is ***asymptotically positive*** (or "eventually positive") if there exists a constant c such that

- when $f : \mathbb{N} \to \mathbb{N}$, $f(n)$ is defined and $f(n) > 0$ for all $n \in \mathbb{N}$ such that $n \geq c$;
- when $f : \mathbb{R} \to \mathbb{R}$, $f(x)$ is defined and $f(x) > 0$ for all $x \in \mathbb{R}$ such that $x \geq c$.

Asymptotically positive functions are the only functions that will be considered in this course as (bounds for) the running times or storage requirements for algorithms.

Most of the time, functions $f : \mathbb{N} \to \mathbb{N}$ will be considered — but real-valued functions arise as bounds for the running times and storage requirements too.

Big-Oh Notation

Definition

Definition 2. Suppose that either $f, g : \mathbb{N} \to \mathbb{N}$ or $f : \mathbb{R} \to \mathbb{R}$, and that f and g are both asymptotically positive. Then $\boldsymbol{f \in O(g)}$ if there exist constants $c > 0$ and $N_0 \geq 0$ such that

$$f(n) \leq c \cdot g(n)$$

for all n (in the domain of f) such that $n \geq N_0$.

In essence, this means that the *rate of growth* of f is, to within a multiplicative constant, *at most* that of g. This is sometimes written "$f = O(g)$" instead of "$f \in O(g)$", even though the above definition effectively defines a ***set*** of functions f that are in the emphset $O(g)$.

One Way To Prove That $\boldsymbol{f \in O(g)}$

Consider, for example, the functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 4x^2 + 2$ and $g(x) = x^2$ for all $x \in \mathbb{R}$.

Claim 3. *$4x^2 + 2 \in O(x^2)$.*

Proof. By the definition of "$O(x^2)$", it suffices to show that *there exist* constants $c > 0$ and $N_0 \geq 0$ such $4^2 + 2 \leq cx^2$ *for all* $x \in \mathbb{R}$ such that $x \geq N_0$.

Let $c = 5$ and let $N_0 = 2$.

Now let x be arbitrarily chosen in $\mathbb{R}$ such that $x \geq N_0 = 2$. Then

$$4x^2 + 2 \leq 4x^2 + x^2 = 5x^2,$$

since $2 \leq 4 \leq x^2$ whenever $n \geq 2$.

Since x was arbitrarily chosen from $\mathbb{R}$ it now follows that $4x^2 + 2 \leq 5x^2 = cx^2$ for *all* $x \in \mathbb{R}$ such that $x \geq 2 = N_0$ and, since $c = 5$ and $N_0 = 2$ are constants such that $c > 0$ and $N_0 \geq 0$, this establishes the claim. $\square$

This can also be seen by an examination of the graph in Figure 1 on page 3.

How This Was Proved: The definition of "$O(x^2)$" was applied as follows:

1. This asks for a proof of a claim in which constants $c > 0$ and $N_0 \geq 0$ are ***existentially quantified*** — so one can start by *choosing* values for c and N_0 — and clearly stating the values for these constants that are chosen.

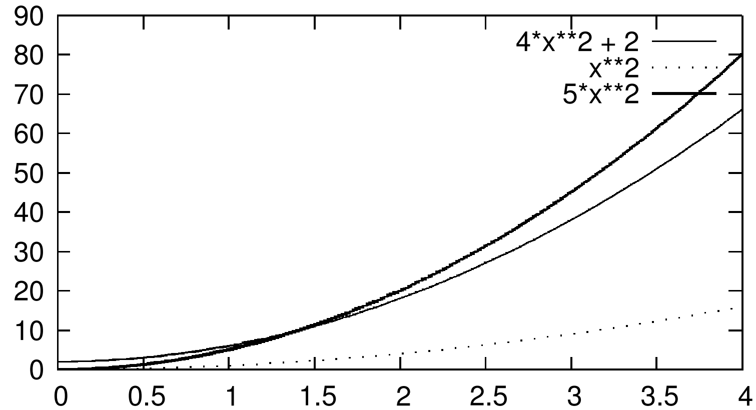

Figure 1: $4x^2 + 2 \in O(x^2)$

2. This leaves us with a claim in which a value x is ***universally quantified*** — it must hold *for all* x in the range of f such that $x \geq N_0$.

   One must therefore let x be an ***arbitrarily chosen*** element of the range of f such $x \geq N_0$ and prove that the claim holds for this choice of x — without assuming **anything** more about x, at all.

3. Once we have done this, we have proved the claim in which n is ***universally quantified***. In particular, we have proved a result for all x in the range of f such that $x \geq N_0$ (and it is a good idea to say so, in the proof).

4. This proves the claim in which the constants c and N_0 are ***existentially quantified*** — as needed to conclude that $f \in O(g)$.

Note: You **do not** need to choose c or N_0 to be as small as possible when you do this! Sometimes choosing constants that are a bit larger than necessary makes the proof easier to write down — and to read, as well.

Another Way To Prove That $f \in O(g)$ — Sometimes!

Theorem 4 (Limit Test for $O(g)$)**.** *Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ or that $f, g : \mathbb{N} \to \mathbb{N}$ and that f and g are asymptotically positive. If*

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)}$$

exists and is a real constant*, so that, in particular, it is* not *equal to $+\infty$, then $f \in O(g)$.*

The proof of this will not be given here — but it is a straightforward consequence of "the ϵ-δ definition of a limit", which you *might* have seen in a sufficiently advanced (or rigorous) calculus course.

Note: This gives a *sufficient* condition, but not a "necessary" one: Sometimes $f \in O(g)$ even though the above limit does not exist at all, and the above theorem cannot be applied.

So you should know how to prove that $f \in O(g)$ "by application of the definition" too, even if you want to use the limit test whenever you can.

Alternate Proof of Claim 3. Let $f(x) = 4x^2 + 2$ and let $g(x) = x^2$. Then, by the Limit Test for $O(g)$, it is sufficient for us to prove that

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)}$$

exists, and is a nonnegative *constant*, in order to prove the claim. Now

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{4x^2 + 2}{x^2} = \lim_{x \to +\infty} \left(4 + \frac{2}{x^2}\right) = 4$$

and, since 4 is a real constant, this suffices to establish the claim. □

L'Hôpital's Rule

Consider the functions $f(x) = \ln x$ and $g(x) = x$. These are both asymptotically positive, since their values are defined and positive when $x \geq 2$.

If we wished to show that $f \in O(g)$ using the Limit Test for $O(g)$ we would need to compute

$$\lim_{x \to +\infty} \frac{\ln x}{x}.$$

Unfortunately, $\lim_{x \to +\infty} \ln x = \lim_{x \to +\infty} x = +\infty$, and there is no clear way to cancel terms in order to see the value of the above limit.

L'Hôpital's Rule is a classical result from calculus (whose proof, again, will not be given in these notes) that can be often applied in order to overcome this type of problem.

Theorem 5 (l'Hôpital's Rule)**.** *Suppose that f and g are both asymptotically positive and **differentiable** at x, for $x \in \mathbb{R}$, when $x \geq c$ for some constant c. Suppose, as well, that*

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} g(x) = +\infty.$$

Then

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)}.$$

Note, now, that the functions $f(x) = \ln x$ and $g(x) = x$ are both asymptotically positive, and differentiable at x when $x \geq 2$. Furthermore, $f'(x) = \frac{1}{x}$ and $g'(x) = 1$. This allows us to prove the following.

Claim 6. $\ln x \in O(x)$.

Proof. Note that the functions $f(x) = \ln x$ and $g(x) = x$ are asymptotically positive, and differentiable at x when $x > 0$. Furthermore, $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = +\infty$. It therefore follows that

$$\begin{aligned}
\lim_{x\to+\infty} \frac{\ln x}{x} &= \lim_{x\to+\infty} \frac{f(x)}{g(x)} \\
&= \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} && \text{(by l'Hôpital's Rule)} \\
&= \lim_{x\to+\infty} \frac{(1/x)}{1} \\
&= 0.
\end{aligned}$$

It now follows by the Limit Test for $O(g)$ that $\ln x \in O(x)$. $\square$

Big-Omega Notation

Definition

Definition 7. Suppose that either $f, g : \mathbb{R} \to \mathbb{R}$ or $f, g : \mathbb{N} \to \mathbb{N}$, and that f and g are asymptotically positive. Then $\boldsymbol{f \in \Omega(g)}$ if there exist constants $c > 0$ and $N_0 \geq 0$ such that

$$f(n) \geq c \cdot g(n)$$

for all n (in the domain of f) such that $n \geq N_0$.

In essence this means that the rate of growth of f is, to within a positive multiplicative factor, *at least* that of g.

One Way To Prove That $\boldsymbol{f \in \Omega(g)}$

Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$ and $g(x) = 4x^2 + 2$.

Claim 8. $x^2 \in \Omega(4x^2 + 2)$.

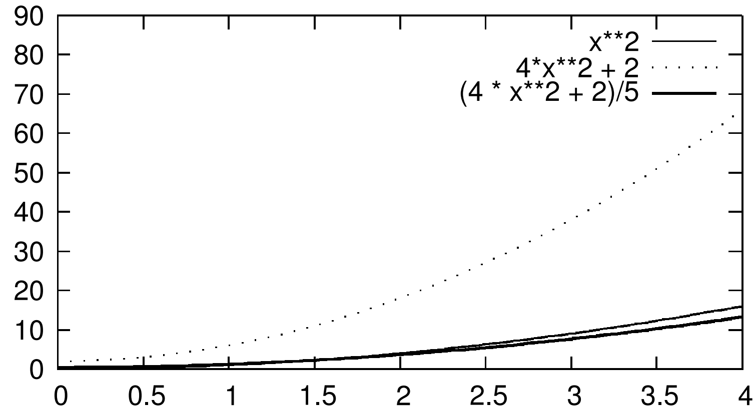

Figure 2: $x^2 \in \Omega(4x^2+2)$

Proof. By the definition of $\Omega(4x^2+2)$, it suffices to show that *there exist* constants $c > 0$ and $N_0 \geq 0$ such that $x^2 \geq c \cdot (4x^2+2)$ *for all* $x \in \mathbb{R}$ such that $x \geq N_0$.

Let $c = \frac{1}{5}$ and $N_0 = 2$.

Now let x be arbitrarily chosen in $\mathbb{R}$ such that $x \geq N_0 = 2$. Then

$$x^2 = \tfrac{1}{5} \cdot 5x^2 \geq \tfrac{1}{5} \cdot (4x^2+2) = c \cdot (4x^2+2),$$

since $x^2 \geq 4 \geq 2$ whenever $x \geq 2 = N_0$.

Since x was arbitrarily chosen from $\mathbb{R}$ such that $x \geq N_0$ it now follows that $x^2 \geq c \cdot (4x^2+2)$ for *all* $x \in \mathbb{R}$ such that $x \geq N_0$ and, since $c = \frac{1}{5}$ and $N_0 = 2$ are constants such that $c > 0$ and $N_0 \geq 0$, this establishes the claim. $\square$

This can also be seen by an examination of the graph in Figure 2.

A second example will be useful later.

Claim 9. $4x^2 + 2 \in \Omega(x^2)$.

Proof. By the definition of $\Omega(x^2)$, it suffices to show that *there exist* constants $c > 0$ and $N_0 \geq 0$ such that $4x^2+2 \geq c \cdot x^2$ *for all* $x \in \mathbb{R}$ such that $x \geq N_0$.

Let $c = 4$ and $N_0 = 0$.

Now let x be arbitrarily chosen in $\mathbb{R}$ such that $x \geq N_0 = 0$. Then

$$4x^2 + 2 \geq 4x^2 = c \cdot x^2.$$

Since x was arbitrarily chosen from $\mathbb{R}$ such that $x \geq N_0$ it now follows that $4x^2 + 2 \geq c \cdot x^2$ for *all* $x \in \mathbb{R}$ such that $x \geq N_0$ and, since $c = 4$ and $N_0 = 0$ are constants such that $c > 0$ and $N_0 \geq 0$, this establishes the claim. $\square$

How These Were Proved: The definition of "$\Omega(4x^2 + 2)$" was applied in the first proof, and the definition of "$\Omega(x^2)$" was applied in the second.

Note that the structure of the definition of $\Omega(g)$ is *identical* to that of the definition of $O(g)$ — only an inequality with "$\geq$" appears instead of an inequality with "$\leq$."

So the structure of a *proof* (by application of the definition) that $f \in \Omega(g)$ will generally be identical to the structure of a *proof* (by application of the definition) of $f \in O(g)$ — except that you will be working with inequalities including "$\geq$" instead of inequalities including "$\leq$".

Exercise: Compare the structure of the proof of Claims 3 and 8 in order to confirm that this is the case.

Another Way To Prove That $f \in \Omega(g)$ — Sometimes!

Theorem 10 (Limit Test for $\Omega(g)$)**.** *Suppose that either $f, g : \mathbb{R} \to \mathbb{R}$ or that $f, g : \mathbb{N} \to \mathbb{N}$, and that f and g are asymptotically positive. If*

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)}$$

exists and is greater than zero*, so that it is either a positive real constant or $+\infty$, then $f \in \Omega(g)$.*

Once again, the proof involves the "ϵ-δ definition of a limit" and is not included in these notes.

Note: Once again, this gives a *sufficient* condition, but not a "necessary" one: Sometimes $f \in \Omega(g)$ even though the above limit does not exist at all, and the above theorem cannot be applied.

Exercise: Use the above Limit Test to write *another* proof that $x^2 \in \Omega(4x^2 + 2)$.

Yet Another Way To Prove That $f \in \Omega(g)$

Theorem 11 (Transpose Symmetry for big-Oh and big-Omega)**.** *Suppose that either $f, g : \mathbb{R} \to \mathbb{R}$ or $f, g : \mathbb{N} \to \mathbb{N}$ and that f and g are asymptotically positive. Then $f \in O(g)$ if and only if $g \in \Omega(f)$.*

The ***proof*** of this is easy — it follows from the definitions of $O(g)$ and $\Omega(f)$ — and is left as an ***exercise***.

This gives us yet another way to prove that $f \in \Omega(g)$, namely, by proving that $g \in O(f)$ instead.

Exercise: Confirm that — now that you know Theorem 11 — Claim 8 can be seen to be a ***corollary*** (that is, “straightforward consequence”) of Claim 3.

Big-Theta Notation

Definition

Definition 12. Suppose that either $f, g : \mathbb{R} \to \mathbb{R}$ or $f, g : \mathbb{N} \to \mathbb{N}$, and that f and g are asymptotically positive. Then $\boldsymbol{f \in \Theta(g)}$ if there exist constants $c_L, c_U > 0$ and $N_0 \geq 0$ such that

$$c_L \cdot g(n) \leq f(n) \leq c_U \cdot g(n)$$

for all n in the domain of f such that $n \geq N_0$.

In essence this means that — up to positive multiplicative constants — the rate of growth of f is *the same as* the rate of growth of g.

One Way To Prove That $\boldsymbol{f \in \Theta(g)}$

Once again, consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 4x^2 + 2$ and $g(x) = x^2$.

Claim 13. $4x^2 + 2 \in \Theta(x^2)$.

Proof. By the definition of “$\Theta(x^2)$”, it suffices to show that *there exist* constants $c_L > 0$, $c_U > 0$ and $N_0 \geq 0$ such that $c_L \cdot x^2 \leq 4x^2 + 2 \leq c_U \cdot x^2$ for *all* $x \in \mathbb{R}$ such that $x \geq N_0$.

Let $c_L = 4$, $c_U = 5$, and $N_0 = 2$.

Now let x be arbitrarily chosen in $\mathbb{R}$ such that $x \geq N_0 = 2$. Then

$$c_L \cdot x^2 = 4x^2 \leq 4x^2 + 2$$

and

$$\begin{aligned} 4x^2 + 2 &\leq 4x^2 + x^2 && \text{(since } 2 \leq 4 = 2^2 \leq x^2\text{)} \\ &= 5x^2 = c_U \cdot x^2. \end{aligned}$$

Thus

$$c_L \cdot x^2 \leq 4x^2 + 2 \leq c_U \cdot x^2.$$

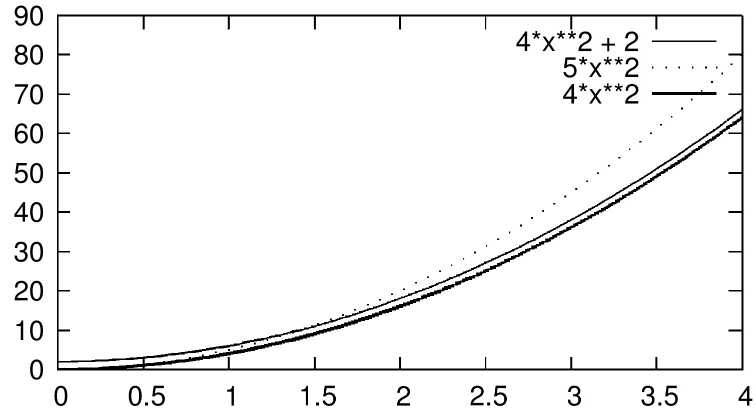

Figure 3: $4x^2 + 2 \in \Theta(x^2)$

Since x was arbitrarily chosen from $\mathbb{R}$ such that $x \geq N_0$ it now follows that

$$c_L \cdot x^2 \leq 4x^2 + 2 \leq c_U \cdot x^2$$

for *all* $x \in \mathbb{R}$ such that $x \geq N_0$ and, since $c_L = 4$, $c_U = 5$ and $N_0 = 2$ are constants such that $c_L, c_U > 0$ and $N_0 \geq 0$, this establishes the claim. $\square$

This can also be seen by an examination of the graph in Figure 3 on page 9.

Exercise: Modify the description of a process that can be followed to prove that $f \in O(g)$, using the definition of "$O(g)$", that is given above, to describe a process that can be followed to prove that $f \in \Theta(g)$ using the definition of "$\Theta(g)$". Ideally, you should then be able to confirm that the above proof is following the process you have described.

An Equivalent Definition — and Another Way To Prove that $f \in \Theta(g)$

Theorem 14. *Suppose that either $f, g : \mathbb{R} \to \mathbb{R}$ or $f, g : \mathbb{N} \to \mathbb{N}$ and that f and g are asymptotically positive. Then $f \in \Theta(g)$ if and only if $f \in O(g)$* and *$f \in \Omega(g)$.*

The proof of this follows (easily) from the definitions of $O(g)$, $\Omega(g)$ and $\Theta(g)$ — Definitions 2, 7 and 12, respectively — and is also left as an ***exercise***.

This provides another way to prove that $f \in \Theta(g)$:

1. Prove that $f \in O(g)$... either by
   (a) an application of the definition of $O(g)$, or
   (b) using the limit text for $O(g)$.
2. Prove that $f \in \Omega(g)$ — either
   (a) by an application of the definition of $\Omega(g)$,
   (b) using the limit test for $\Omega(g)$, or
   (c) by using transpose symmetry — that is, by proving that $g \in O(f)$, instead.

Note: One might apply by this, by observing that Claim 13 is now a corollary of Claims 3 and 9.

A Common Mistake

People sometimes write "f is $O(g)$" (which is yet another way to write "$f \in O(g)$" or "$f = O(g)$") when they actually mean "$f \in \Theta(g)$."

You may even notice this in some textbooks and mathematical papers.

Please note that if $f \in O(g)$ then it is ***not*** necessarily true that $f \in \Theta(g)$ as well

- For example, as functions of n, $n \in O(n^2)$ but $n \notin \Theta(n^2)$.

So: If you want people to understand that "$f \in \Theta(g)$" then this is what you should write!

Little-oh Notation

This, and the next, type of asymptotic notation, might *not* have been introduced before. However, they are both also extremely useful.

Definition

Definition 15. Suppose that either $f, g : \mathbb{R} \to \mathbb{R}$ or $f, g : \mathbb{N} \to \mathbb{N}$, and that f and g are asymptotically positive. Then $\boldsymbol{f \in o(g)}$ if it is true that ***for every*** constant $c > 0$ ***there exists a constant*** $N_0 \geq 0$ such that

$$f(n) \leq c \cdot g(n)$$

for all n in the domain of f such that $n \geq N_0$.

In essence, this means that the rate of growth of f is *strictly less* than that of g — regardless of any multiplicative constants.

One Way To Prove That $f \in o(g)$

Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x$ and $g(x) = x^2$.

Claim 16. $x \in o(x^2)$.

Proof. It follows by the definition of "$O(x^2)$" that it is sufficient to show that for *every* constant $c > 0$ *there exists* a constant N_0 (which may depend on c), such that

$$x \leq c \cdot x^2$$

for *all* $x \in \mathbb{R}$ such that $x \geq N_0$.

With that noted, let c be an arbitrary real constant such that $c > 0$.

Let N_0 be the constant $1/c$. (Since c now has a fixed positive value, $N_0 = 1/c$ is a constant such that $N_0 \geq 0$, as required here.)

Now let x be an arbitrarily chosen real number such that $x \geq N_0$. Then, since $c > 0$ and $x \geq N_0 = 1/c$, $cx \geq cN_0 = 1$. That is, $1 \leq cx$.

Thus (since $x \geq 0$ as well)

$$\begin{aligned} x &= 1 \cdot x \\ &\leq (cx) \cdot x && \text{(by the above)} \\ &= cx^2. \end{aligned}$$

Now, since x was an arbitrarily chosen real number such that $x \geq N_0$, it follows that $x \leq cx^2$ *for all* $x \in \mathbb{R}$ such that $x \geq N_0$.

It follows from *that* that *there exists* a constant $N_0 \geq 0$ such that $x \leq cx^2$ for all $x \leq N_0$.

Finally, since c was an arbitrarily chosen real constant such that $c > 0$, it follows that *for all* $c > 0$, *there exists* a constant $N_0 \geq 0$ such that $x \leq cx^2$ *for all* $x \in \mathbb{R}$ such that $x \geq N_0$.

It therefore follows by the definition of $o(x^2)$ that $x \in o(x^2)$. $\square$

This can also be seen by an examination of the graph in Figure 4 on page 12.

How Was This Proved?

The definition of "$O(x^2)$" as applied:

1. The value c was set to be an *arbitrarily chosen* real constant such that $c > 0$.

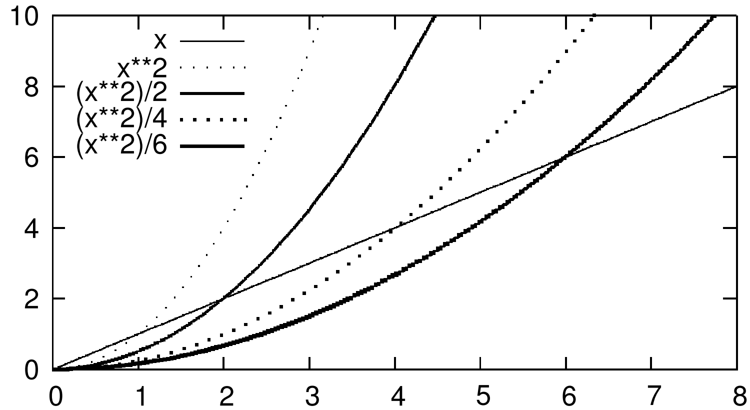

Figure 4: $x \in o(x^2)$

2. A value was then given for a constant N_0 such that $N_0 \geq 0$ and $f(x) \leq cg(x)$ for all n in the domain of f such that $x \geq N_0$.

   Note that — as above, the value of N_0 can (and virtually always *must*) depend on the value of c (now that c is "fixed" but unknown).

3. One should then set x to be an *arbitrarily chosen* element of the domain of f such that $x \geq N_0$.

4. One should then *prove* that $f(x) \leq cg(x)$ – without assuming *anything* more about x, N_0, or c.

5. One can then conclude that — since x was arbitrarily chosen — $f(x) \leq cg(x)$ *for all* x in the domain of f such that $x \geq N_0$.

6. One can conclude, from this, that *there exists* a constant $N_0 \geq 0$ such that $f(x) \leq cg(x)$ *for all* x in the domain of f such that $x \geq N_0$.

7. Finally, since c was an arbitrarily chosen real constant such that $c > 0$ one can conclude, from *that*, that *for all* $c > 0$ *there exists* a constant N_0 such that $f(x) \leq cg(x)$ *for all* x in the domain of f such that $x \geq N_0$.

8. It now follows by the definition of "$o(g)$" that $f \in o(g)$.

Another Way To Prove That $f \in o(g)$ — *Always!*

Theorem 17 (Limit Test for $o(g)$)**.** *Suppose that either $f, g : \mathbb{R} \to \mathbb{R}$ or that $f, g : \mathbb{N} \to \mathbb{N}$, and that f and g are asymptotically positive. Then $f \in o(g)$* ***if and only if***

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 0.$$

Once again, the proof of this depends on the "ϵ-δ definition of a limit", and is not included in these notes.

Note that this limit test *can always* be used if $f \in o(g)$ — provided that you are able to compute the limit in the theorem!

Alternate proof for Claim 16. Note that

$$\lim_{x \to +\infty} \frac{x}{x^2} = \lim_{x \to +\infty} \frac{1}{x} = 0.$$

If follows by the limit test for $o(x^2)$ that $x \in o(x^2)$ $\square$

An Occasional Misunderstanding

Students sometimes believe that you can convert a proof that $f \in O(g)$ into a proof that $f \in o(g)$ by changing the inequality in the argument from "$\leq$" to "$<$."

This does not work.

This simply produces another proof that $f \in O(g)$. It is the ***quantification*** of the constant c ("$\forall$" instead of "$\exists$") that is important here!

Little omega Notation

Definition

Definition 18. Suppose that either $f, g : \mathbb{R} \to \mathbb{R}$ or $f, g : \mathbb{N} \to \mathbb{N}$, and that f and g are asymptotically positive. Then $\boldsymbol{f \in \omega(g)}$ if it is true that ***for every*** constant $c > 0$ ***there exists a constant*** $N_0 \geq 0$ such that

$$f(n) \geq c \cdot g(n)$$

for all n in the domain of f such that $n \geq N_0$.

In essence, this means that the rate of growth of f is *strictly greater than* that of g — regardless of any multiplicative constants.

One Way To Prove That $f \in \omega(g)$

Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$ and $g(x) = x$.

Claim 19. $x^2 \in \omega(x)$.

Proof. It follows by the definition of "$\omega(x)$" that it is sufficient to show that for *every* constant $c > 0$ *there exists* a constant N_0 (which may depend on c), such that

$$x^2 \geq c \cdot x$$

for *all* $x \in \mathbb{R}$ such that $x \geq N_0$.

With that noted, let c be an arbitrary real constant such that $c > 0$.

Let N_0 be the constant c. (Since c now has a fixed positive value, $N_0 = c$ is a constant such that $N_0 \geq 0$, as required here.)

Now let x be an arbitrarily chosen real number such that $x \geq N_0$. Then, since $x \geq N_0 = c > 0$,

$$\begin{aligned} x^2 &= x \cdot x \\ &\geq N_0 \cdot x && \text{(since } x \geq N_0 \text{ and } x > 0\text{)} \\ &= c \cdot x && \text{(since } N_0 = c\text{).} \end{aligned}$$

Now, since x was an arbitrarily chosen real number such that $x \geq N_0$, it follows that $x^2 \geq c \cdot x$ *for all* $x \in \mathbb{R}$ such that $x \geq N_0$.

It follows from *that* that *there exists* a constant $N_0 \geq 0$ such that $x^2 \geq c \cdot x$ for all $x \geq N_0$.

Finally, since c was an arbitrarily chosen real constant such that $c > 0$, it follows that *for all* $c > 0$, *there exists* a constant $N_0 \geq 0$ such that $x^2 \geq c \cdot x$ *for all* $x \in \mathbb{R}$ such that $x^2 \geq c \cdot x$.

It therefore follows by the definition of "$\omega(x)$" that $x^2 \in \omega(x)$. □

This can also be seen by an examination of the graph in Figure 5 on page 15.

How Was This Proved?

The structure of the definition of $\omega(g)$ is *identical* to that of the definition of $o(g)$ — only an inequality with "$\geq$" appears instead of an inequality with "$\leq$.

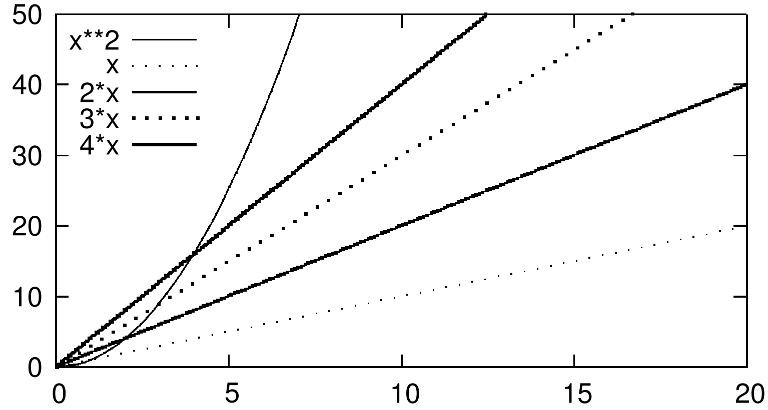

Figure 5: $x^2 \in \omega(x)$

So the structure of a *proof* (by application of the definition) that $f \in \omega(g)$ will be identical to the structure of a *proof* (by application of the definition) of $f \in o(g)$ — except that you will be working with inequalities including "$\geq$" instead of inequalities including "$\leq$".

Exercise: Compare the proofs of Claims 16 and 19 in order to confirm that the structures of these proofs really *are* the same.

Another Way To Prove That $f \in \omega(g)$ — *Always!*

Theorem 20 (Limit Test for $\omega(g)$)**.** *Suppose that either $f, g : \mathbb{R} \to \mathbb{R}$ or $f, g : \mathbb{N} \to \mathbb{N}$ and that f and g are asymptotically positive. Then $f \in \omega(g)$* if and only if

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = +\infty.$$

Once again, a proof of this depends on the "ϵ-δ definition of a limit" and is not included in these notes.

Note: Like the limit test for $o(g)$, this limit test *can always* be used if $f \in \omega(g)$ — provided that you are able to compute the limit in the theorem!

Exercise: Use Theorem 20 to provide a different proof for Claim 19.

Yet Another Way To Prove That $f \in \omega(g)$

Theorem 21 (Transpose Symmetry for little-oh and little-omega)**.** *Suppose that either $f, g : \mathbb{R} \to \mathbb{R}$ or $f, g : \mathbb{N} \to \mathbb{N}$ and that f and g are asymptotically positive. Then $f \in o(g)$ if and only if $g \in \omega(f)$.*

Once again, a proof of this is easily obtained using the definitions of little-oh and little-omega (Definitions 15 and 18, respectively) and is left as an ***exercise***.

Exercise: Use this to confirm that Claim 19 is just a corollary (or “straightforward consequence”) of Claim 16.

Standard Functions

Polynomial Functions

Definition 22. Let $d \in \mathbb{N}$. A ***polynomial function*** with ***degree*** d is a function

$$p(n) = a_d n^d + a_{d-1} n^{d-1} + \cdots + a_1 n + a_0$$

where $a_d, a_{d-1}, \ldots, a_1, a_0 \in \mathbb{R}$ and $a_d \neq 0$.

- If $a_d > 0$ then $p(n) \in \Theta(n^d)$.
- If $a_d > 0$ then $p(n) \in o(n^e)$ for all $e \in \mathbb{R}$ such that $e > d$, and $p(n) \in \omega(n^f)$ for all $f \in \mathbb{R}$ such that $f < d$.

You should take advantage of the above, when using asymptotic notation to write down information about the worst case running times of algorithms, when these functions are polynomials functions (with some degree d) of the input size n.

Exponential Functions

Definition 23. An ***exponential function*** of n is a function $e(n) = a^n$ for some value $a \in \mathbb{R}$ such that $a > 0$.

- If $a > 1$ then $e(n) \in \omega(p(n))$ for *every* polynomial function p of n — no matter how high the degree of p is!
- If $a = 1$ then $a(n) = 1$, so $a(n) \in \Theta(1)$.

- If $a < 1$ then $a(n) \in o(1)$.

Furthermore, if $a, b \in R$, $a, b > 0$, $e_a(n) = a^n$ and $e_b(n) = b^n$, then

- $e_a(n) \in o(e_b(n))$ if $a < b$,
- $e_a(n) \in \Theta(e_b(n))$ if $a = b$, and
- $e_a(n) \in \omega(e_b(n))$ if $a > b$.

Exponential functions also arise as the (worst case) running times of algorithms.

Logarithmic Functions

Definition 24. A ***logarithmic function*** is a function $\ell(n) = \log_a n$ for $a \in R$ such that $a > 1$.

- If $a, b \in \mathbb{R}$ such that $a > 1$ and $b > 1$, $\ell_a(n) = \log_a n$ and $\ell_b(n) = \log_b n$, then $\ell_a(n) \in \Theta(\ell_b(n))$.
- If $a \in \mathbb{R}$ such that $a > 1$, $\ell_a(n) = \log_a n$, and p is a polynomial function with degree $d \geq 1$ whose leading coefficient is positive, then $\ell_a(n) \in o(p(n))$.

Logarithmic functions can arise as the worst case running times of algorithms (or as factors of these running times) too.

For Further Reading

Most of the material on asymptotic notation and standard functions, given above, can be found (with different examples) in Chapter 3 of the text *Introduction to Algorithms* [1]. This also includes information about other kinds of functions that can arise as running times for algorithms.

References

[1] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms*. The MIT Press, third edition, 2009.