

CPSC 313 — Supplemental Material for Lecture #2

Extended Transition Functions: Equivalence of Definitions

1 Introduction

During the first lecture on deterministic finite automata, *two* definitions were given for an extended transition function

$$\delta^* : Q \times \Sigma^* \rightarrow Q$$

for a deterministic finite automaton

$$M = (Q, \Sigma, \delta, q_0, F).$$

This document includes a **proof** that these definitions really are equivalent. CPSC 313 students are asked to read this because

- the equivalence of these definitions is quite important, and
- this document includes proofs of several properties, concerning strings, using **mathematical induction**. Students in this course will be expected to read and understand proofs like this throughout the course — and may also be asked to *write* proofs like these on assignments.

2 Extended Transition Function — First Definition and Some Properties

As defined in the lecture notes, the **extended transition function** is a total function

$$\delta^* : Q \times \Sigma^* \rightarrow Q$$

such that if the automaton M is in state $q \in Q$, and the (sequence of symbols in) the string $\omega \in \Sigma^*$ is received, then M is in state $\delta^*(q, \omega)$ after that.

Thus

$$\delta^*(q, \lambda) = q \quad (1)$$

for every state $q \in Q$.

Suppose, more generally, that

$$\omega = \sigma_1\sigma_2\dots\sigma_n \in \Sigma^*$$

is a string with length $n \geq 0$ in Σ^* — so that, for $1 \leq i \leq n$, the i^{th} symbol in ω is σ_i .

Let $q \in Q$. Then a *sequence* of states r_0, r_1, \dots, r_n , with length $n + 1$, can be defined as follows:

- $r_0 = q$, and
- $r_{i+1} = \delta(r_i, \sigma_{i+1})$ for every integer i such that $0 \leq i < n$.

In this case,

$$\delta^*(q, \omega) = r_n, \quad (2)$$

the *last* state in the above sequence.

Note that the definitions given at lines (1) and (2) agree if $\omega = \lambda$ — for $n = 0$ in this case, so that $r_n = r_0 = q$.

The following *lemma* presents a property of sequences of states like the above that will be useful when properties of the extended transition function are considered.

Lemma 2.1. *Let $q \in Q$, let*

$$\mu = \sigma_1\sigma_2\dots\sigma_m \in \Sigma^* \quad (3)$$

and

$$\nu = \tau_1\tau_2\dots\tau_n \in \Sigma^* \quad (4)$$

be strings with lengths m and n respectively, and let

$$\omega = \mu \cdot \nu = \sigma_1\sigma_2\dots\sigma_m\tau_1\tau_2\dots\tau_n \in \Sigma^* \quad (5)$$

be the concatenation of μ and ν , so that ω has length $n + m$.

Consider the sequences r_0, r_1, \dots, r_m and s_0, s_1, \dots, s_{m+n} of states, with lengths $m + 1$ and $m + n + 1$, respectively, that are defined using the following rules.

- (a) $r_0 = q$.
- (b) $r_{i+1} = \delta(r_i, \sigma_{i+1})$ for every integer i such that $0 \leq i \leq m - 1$.

- (c) $s_0 = q$.
- (d) $s_{i+1} = \delta(s_i, \sigma_{i+1})$ for every integer i such that $0 \leq i \leq m - 1$.
- (e) $s_{m+j+1} = \delta(s_{m+j}, \tau_{j+1})$ for every integer j such that $0 \leq j \leq n - 1$.

Then $r_i = s_i$ for every integer i such that $0 \leq i \leq m$.

Proof. It will be shown that $r_i = s_i$, for every integer i such that $0 \leq i \leq m$, by induction on i . The standard form of mathematical induction will be used.

Basis: if $i = 0$ then

$$\begin{aligned}
 r_i &= r_0 \\
 &= q && \text{(by the equation in part (a) of the claim, above)} \\
 &= s_0 && \text{(by the equation in part (c) of the claim, above)} \\
 &= s_i
 \end{aligned}$$

as required to establish the claim in this case.

Inductive Step: Let k be an integer such that $k \geq 0$. It is necessary and sufficient to use the following

Inductive Hypothesis: If $k \leq m$ then $r_k = s_k$.

to prove the following

Inductive Claim: If $k + 1 \leq m$ then $r_{k+1} = s_{k+1}$.

There is nothing to prove if $k + 1 > m$. Suppose, therefore, that $k + 1 \leq m$. Since $k \geq 0$ it follows that $0 \leq k \leq m - 1 \leq m$, and it follows by the inductive hypothesis that $r_k = s_k$. Thus

$$\begin{aligned}
 r_{k+1} &= \delta(r_k, \sigma_{k+1}) && \text{(by part (b) of the above claim, since } 0 \leq k \leq m - 1) \\
 &= \delta(s_k, \sigma_{k+1}) && \text{(by the inductive hypothesis, as noted above)} \\
 &= s_{k+1} && \text{(by part (d) of the above claim, since } 0 \leq k \leq m - 1)
 \end{aligned}$$

as needed to complete the inductive step and to prove the claim. \square

Note: The above claim is (arguably) so obvious that it probably does not need a proof. A proof has been given, anyway, to give an example of a proof of a property of a string using mathematical induction.

Lemma 2.2. Let $q \in Q$, let μ be a string in Σ^* and let τ be a symbol in Σ . Then

$$\delta^*(q, \mu \cdot \tau) = \delta(\delta^*(q, \mu), \tau).$$

Proof. This is a straightforward consequence of Lemma 2.1, for μ as in the claim (so that $m = |\mu|$), and when setting ν to be the string with length one whose first (and only) symbol is τ (so that $\tau_1 = \tau$ and $n = 1$). Then it follows by the definition of the extended transition function, given above, that $\delta^*(q, \mu) = r_m$ — and, since

$$\mu \cdot \tau = \sigma_1 \sigma_2 \dots \sigma_m \tau,$$

it also follows by the definition of the extended transition function that $\delta^*(q, \mu \cdot \tau) = s_{m+n} = s_{m+1}$. Thus

$$\begin{aligned} \delta^*(q, \mu \cdot \tau) &= s_{m+1} && \text{(as noted above)} \\ &= \delta(s_m, \tau) && \text{(by part (e) of the definition in Lemma 2.1, with } j = 0\text{)} \\ &= \delta(r_m, \tau) && \text{(since } s_m = r_m, \text{ by Lemma 2.1)} \\ &= \delta(\delta^*(q, \mu), \tau), && \text{(since } \delta^*(q, \mu) = r_m, \text{ as noted above)} \end{aligned}$$

as claimed. □

3 Extended Transition Function — Second Definition

Now consider the (possibly different) function

$$\delta^\times : Q \times \Sigma^* \rightarrow Q$$

that corresponds to the *second* definition. That is, for every state $q \in Q$ and for every string $\omega \in \Sigma^*$, let

$$\delta^\times(q, \omega) = \begin{cases} q & \text{if } \omega = \lambda, \\ \delta(\delta^\times(q, \mu), \tau) & \text{if } \omega = \mu \cdot \tau \text{ for } \mu \in \Sigma^* \text{ and } \tau \in \Sigma. \end{cases} \quad (6)$$

4 Equivalence of Definitions

Theorem 4.1. *Let $\delta^* : Q \times \Sigma^* \rightarrow Q$ and $\delta^\times : Q \times \Sigma^* \rightarrow Q$ be the functions defined in Sections 2 and 3, respectively. Then $\delta^* = \delta^\times$, that is, $\delta^*(q, \omega) = \delta^\times(q, \omega)$ for every state $q \in Q$ and for every string $\omega \in \Sigma^*$.*

Proof. The result will be proved by induction on the length of the string ω mentioned in the claim. The standard form of mathematical induction will be used.

Basis: If $|\omega| = 0$ then $\omega = \lambda$, the empty string. It is therefore necessary and sufficient to prove that $\delta^*(q, \lambda) = \delta^\times(q, \lambda)$ for every state $q \in Q$.

Suppose, therefore, that $q \in Q$. Then

$$\begin{aligned}\delta^*(q, \lambda) &= q && \text{(by the equation at line (1))} \\ &= \delta^\times(q, \lambda), && \text{(by the equation at line (6))}\end{aligned}$$

as required.

Inductive Step: Let k be an integer such that $k \geq 0$. It is necessary and sufficient to use the following

Inductive Hypothesis: $\delta^*(q, \omega) = \delta^\times(q, \omega)$ for every state $q \in Q$ and for every string $\omega \in \Sigma^*$ whose length is k .

to prove the following

Inductive Claim: $\delta^*(q, \omega) = \delta^\times(q, \omega)$ for every state $q \in Q$ and for every string $\omega \in \Sigma^*$ whose length is $k + 1$.

With that noted, let q be a state in Q and let ω be a string in Σ^* with length $k + 1$. Then, since $k + 1 \geq 1$, there exists a string $\mu \in \Sigma^*$ and a symbol $\tau \in \Sigma$ such that $\omega = \mu \cdot \tau$. Now $k + 1 = |\omega| = |\mu| + 1$, so that the length of μ is k , and it follows by the inductive hypothesis that

$$\delta^*(q, \mu) = \delta^\times(q, \mu).$$

Thus

$$\begin{aligned}\delta^*(q, \omega) &= \delta^*(q, \mu \cdot \tau) && \text{(since } \omega = \mu \cdot \tau) \\ &= \delta(\delta^*(q, \mu), \tau) && \text{(by Lemma 2.2)} \\ &= \delta(\delta^\times(q, \mu), \tau) && \text{(by the inductive hypothesis, as noted above)} \\ &= \delta^\times(q, \mu \cdot \tau) && \text{(by the equation at line (6))} \\ &= \delta^\times(q, \omega) && \text{(since } \omega = \mu \cdot \tau).\end{aligned}$$

Since q and ω were arbitrarily chosen it follows that $\delta^*(q, \omega) = \delta^\times(q, \omega)$ for every state $q \in Q$ and for every string $\omega \in \Sigma^*$ whose length is $k + 1$ — as required to complete the inductive step, and the proof of the claim. \square

Thus the two definitions of an “extended transition function,” given in the lecture notes, really are (provably) equivalent.