UNIVERSITY OF CALGARY

An investigation of some theoretical aspects of reversible computing

by

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A DISSERTATION

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF COMPUTER SCIENCE

CALGARY, ALBERTA

October, 2014

Abstract

The categorical semantics of reversible computing must be a category which combines the concepts of partiality and the ability to reverse any map in the category. Inverse categories, restriction categories in which each map is a partial isomorphism, provide exactly this structure. This thesis explores inverse categories and relates them to both quantum computing and standard non-reversible computing. The former is achieved by showing that commutative Frobenius algebras form an inverse category. The latter is by establishing the equivalence of the category of discrete inverse categories to the category of discrete Cartesian restriction categories — this is the main result of this thesis. This allows one to transfer the formulation of computability given by Turing categories onto discrete inverse categories.

Acknowledgements

First and foremost, many thanks to my supervisor of the past many, many years, Dr. Robin Cockett. His indefatigable thirst for mathematical exploration of computer science has been an inspiration for both my master's degree and my doctoral studies.

Secondly, thank you to Dr. Peter Selinger, both for his paper on the semantics of quantum computing which inspired me to write LQPL and the opportunity to work for him at MSRI in Berkeley for the fall of 2012. His clarity of exposition in research writing remains a target to which I aspire.

Thank you to Dr. Kristine Bauer and Dr. Peter Høyer for serving on my supervisory committee, partaking in my qualifying exam and serving on my thesis defence committee. I always felt your doors were open to me.

Thank you to Dr. Gilad Gour for being my internal examiner.

Thank you to Dr. Bob Coecke for agreeing to be my external examiner.

Many thanks to Dr. Sudarshan Sehgal for encouraging me to return to academia and get my doctorate.

Thanks to my fellow graduate students (and post-docs) over the years for great conversations, people to hang with while at conferences and critiques of research. Thanks to Craig Pastro, Xiuzhan Guo, Dana Harrington, Ross Duncan, Jon Grattage, Sean Nichols, Ning Tang, Brian Redmond, Geoff Cruttwell, Alexander Green, Neil Ross, Jonathan Gallagher, Chad Nester and Prashant Kumar.

Finally, and most of all, thank you so much to all of my family. My wife, for being with me for all these years and proof-reading both my master's and this thesis — learning more about math than she ever thought she would; my kids, who still remember me; and my grandkids — who likely don't remember a time when Grandpa was not in school.

Dedication

To my wife, Marie Gelinas Giles.

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List of Symbols, Abbreviations and Nomenclature

Symbol	Definition
$\overline{f}$	$\overline{f}$ is the restriction of f, see Definition 3.1.1.
$\cup$	Set union.
$\cap$	Set intersection.
$X \times Y$	The Cartesian product of X, Y.
$X \subseteq Y$	X is a subset/subcategory of Y.
$\emptyset$	The empty set.
$\partial_0(f)$	The domain object of the map f, see Definition 2.1.1.
$\partial_1(f)$	The codomain object of the map f, see Definition 2.1.1.
$\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{D}, \mathbb{R}, \ldots$	Categories, see Definition 2.1.1.
$\mathbb{X}_o$, $\mathbb{X}_m$	The objects and maps of the category $\mathbb{X}$.
$\mathbb{X}^{op}$	The dual of the category $\mathbb{X}$.
$\textsc{Sets}, \textsc{stabLat}, \textsc{Pinj}, \ldots$	Specific categories are set in small caps type.
$\in$	Element of, in sets.
$\to$	Separate domain and codomain of a map.
$\mathbb{X}(A, B)$	Hom-set in $\mathbb{X}$.
$\{x \mid \text{condition on } x\}$	Define a set via a condition.
$\exists x.$	This means "there exists an x such that".
$\mathbb{N}$	The set of natural numbers, i.e., $\{0, 1, 2, \ldots\}$.
$[a_{ij}]$	The matrix whose i, j element is a_{ij}.
$f^{\diamond}$	A right inverse of f.
$f_{\diamond}$	A left inverse of f.
$\mathrm{K}_E(\mathbb{B})$	The Karoubi envelope. See Definition 2.2.2.
$\langle f, g\rangle$	The product map of f, g. See Definition 2.7.3.

$A \times B$	The product of A, B. See Definition 2.7.3.
$[f, g]$	The coproduct map of f, g. See Definition 2.7.4.
$A + B$	The coproduct of A, B. See Definition 2.7.4.
$A \boxplus B$	The biproduct of A, B. See Definition 2.7.6.
$\alpha : F \Rightarrow G$	α is a natural transformation.
$F \cong G$	The natural transformation $F \Rightarrow G$ is an isomorphism.
$F \vdash G$	F is the left adjoint of G.
$\equiv$	Used informally to say "is the same as".
$\mathcal{P}(S)$	The power set of the set S.
$\mathcal{O}(A)$	Open sets of A or restriction idempotents of A.
$\oplus, \otimes, \odot$	Tensors of categories.
$\uparrow$	Signifies that a function is undefined at some value.
$\leq$	An ordering relation.
$f \smile g$	f is compatible with g. See Definition 3.3.1.
$f \vee g$	The join of f, g. See Definition 3.3.3.
$\setminus$	Minus operation on sets.
$\cap$	Meets of maps, see Definition 3.4.1.
$f^{(-1)}$	Partial inverse of f, see Definition 3.5.1.
$\hat{f}$	$\hat{f}$ is the range of f, see Definition 3.6.1.
$A \triangleleft B$	A is a retract of B.
$A \triangleleft_m^r B$	$m : A \to B$, $r : B \to A$, witness maps to $A \triangleleft B$.
$\top$	Restriction terminal object. See Definition 3.9.2.
$!$	The unique map to the restriction terminal object.
$\mathcal{M}$	A collection of monic maps.
$\forall X$	For all X.
$f^\dagger$	Apply the functor $\dagger$ to f.

$\implies$	Implies.
$\mathbb{C}$	The field of complex numbers.
$f \xrightarrow{\cong} g$	f is isomorphic to g.
$\widetilde{\mathbb{X}}$	The completion of the discrete inverse category $\mathbb{X}$.
$f \overset{h}{\simeq} g$	f is equivalent to g. See Definition 5.1.2.
(f, C)	Equivalence class of maps $f : A \to B \otimes C$.
$f : A \to B_{\mid C}$	Alternate way to write $(f, C) : A \to B$.
$:=$	Defines the left hand side as the right hand side.
$?$	The unique map from the initial object.
$\perp$	Disjointness relationship. See Definition 6.2.1.
$\underline{\perp}$	Open disjointness. See Definition 6.2.4.
$\sqcup$	Disjoint join. See Definition 6.3.1.
$\uplus$	The disjoint union of sets.
$f \triangledown g$	Partial operation on f, g. See Definition 7.2.6.
$f \vartriangle g$	Partial operation on f, g. See Definition 7.2.6.
$\iff$	If and only if.

Chapter 1

Introduction

1.1 Summary

A "quantum" setting has a duality given by the "dagger" of dagger categories [1, 64]. On the other hand, classical computation is fundamentally asymmetric and has no duality. In passing from a quantum setting to a more classical setting, one may want to keep this duality for as long as possible and, thus, consider the intermediate step of passing to "reversible" computation — which has an obvious self-duality given by the ability to reverse the computation. It is reasonable to wonder whether one can then pass from a reversible setting to a classical setting quite independently from the underlying quantum setting. Such an abstract passage would allow a direct translation into the reversible world of the classical notions of computation, as an example.

Of course, from a quantum setting, it is already possible to pass directly to a classical setting by taking the homomorphisms between special coalgebras, where "special" means the coalgebra must be the algebra part of a separable Frobenius algebra. That the coalgebra should be special in this manner may be justified by regarding this as a two step process through reversible computation. However, this leaves some gaps: How does one pass, in general, between a quantum setting to a reversible setting and how does one obtain a classical setting from a reversible setting? This thesis answers these questions.

1.2 Background of reversible computation

In 1961, Landauer [44] examined logically irreversible computing and showed that it must dissipate energy, i.e., produce heat at a specified minimal level. This is due to applying a physically irreversible operation to non-random data, leading to an entropy increase in the computer. While there are various objections to the connection between *logical* irreversibility and heat generation, summarized by Bennett [7], this led to an interest in exploring reversible computation, because of its potential energy advantage.

Bennett, in 1973, [6] showed how one can simulate an ordinary Turing machine using a reversible Turing machine, based on reversible transitions. Since approximately 2000, there has been an increased interest in reversible computing. Active areas include research in database theory, specifically the view-update problem [10, 30, 40] and quantum computing. Recall that quantum computation may be modelled by unitary transforms [56], each of which is reversible, followed by an irreversible measurement.

An important aspect of the treatment of reversible computing is the consideration of the partiality inherit in programs, as it is possible for programs to never provide a result for certain inputs. Some of the reversibility research referenced in this section considers partiality to a greater or lesser degree, but none of them treat it as a central consideration.

Partiality was shown to have a purely algebraic description by Cockett and Lack in [21–23]. They introduce a restriction operator on maps, which associated to a map a partial identity on its domain. In [21], they recalled the concept of *inverse category*, a category equipped with a restriction operator in which all the maps have partial inverses, i.e., are reversible. Categories with restriction operators are presented in Chapter 3, while inverse categories are explored in Chapter 4.

The semantics of reversible computing has been explored in a variety of ways, including by developing various reversible programming languages. An early example of this is Janus [48], an imperative language written as an experiment in producing a language that did not erase information. However, it does not appear that any semantic underpinnings were developed for this language. Additionally, there are special purpose reversible languages, such as biXid [42], a language developed explicitly to transform XML [11] from one data schema to another. The main novelty of biXid is that a single program targets two schemas and will transform in either direction.

Zuliani [68] provides a reversible language with an operational semantics. He examines logical reversibility via comparing the probabilistic Guarded Command Language ($pGCL$) [53] to the quantum Guarded Command Language ($qGCL$) [61]. Zuliani provides a method for transforming an irreversible $pGCL$ program into a reversible one. This is accomplished via an application of expectation transform semantics to the $pGCL$ program. Interestingly, in this work, partial programs are specifically excluded from the definition of reversible programs. The initial definition of a reversible program is strict, i.e., the program is equivalent to *skip* which does nothing. To alleviate this and allow us to extract the output, Zuliani follows the example of [6] and modifies the result so that the output is copied before reversing the rest of the program.

In addition, a number of reversible calculi have been developed. Danos and Krivine [28] extend CCS (Calculus for Communicating Systems) [51, 52] to produce RCCS, which adds reversible transitions to CCS. This is done by adding a syntax for backtracking, together with a labelling which guides the backtracking. The interesting aspect of this work is the applicability to concurrent programs.

Phillips and Uladowski [59] take a different approach to creating a reversible CCS from that of Danos and Krivine. Rather, their stated goal is to use a structural approach, inspired by [2]. The paper is only an initial step in this process, primarily explaining how to turn dynamic rules (such as choice operators) into a series of static rules that keep all the information of the input. For example (from the paper), in standard CCS, we have the rule

$$\frac{X \to X'}{X + Y \to X'}.$$

To preserve information and allow reversibility, this is replaced with

$$\frac{X \to X'}{X + Y \to X' + Y}.$$

In [2], Abramsky considers linear logic as his computational model. This is done by producing a Linear Combinatory Algebra [4] from the involutive reversible maps over a term algebra and showing these are bi-orthogonal automata. (An automata is considered orthogonal if it is non-ambiguous and left-linear. It is bi-orthogonal when both the automata and its converse are both orthogonal). While the paper does use reversible term rewritings as the basis for computation, its emphasis is on how one can derive a linear combinatory algebra. Linear combinatory algebras are themselves not reversible systems.

In [54] and [55], Mu, Hu and Takeichi introduce the language **Inv**, a language that is composed only of partial injective functions. The language has an operational semantics based on determinate relations and converses. They provide a variety of examples of the language, including translations from XML to HTML and simple functions such as *wrap*, which wraps its argument into a list. They continue by describing how non-injective functions may be converted to injective ones in **Inv** via the addition of logging. In fact, they use this logging to argue the language is equivalent in terms of power to the reversible Turing machine of [6].

Some of the ideas of [54, 55] are closely related to the theory developed in this thesis. For example, given two functions f, g of **Inv**, constructing their union, $f \cup g$ requires that both $\operatorname{dom} f \cap \operatorname{dom} g = \phi$ and $\operatorname{range} f \cap \operatorname{range} g = \phi$. This is an example of a disjoint join as introduced in Section 6.3.

In [60], Di Pierro, Hankin and Wiklicky consider groupoids as their mathematical model for reversible computations. This leads them to develop **rCL**, reversible combinatory logic. This logic consists of a pair of terms, $\langle M|H\rangle$, where M is a term of standard combinatory logic and H is a history, with a specified syntax. In standard combinatory logic, the k term is irreversible in that it erases its second argument. In **rCL**, application of the k term copies the second argument into the history, preserving reversibility. Groupoids are a specific example of inverse categories, which are total.

A recent reversible language is Theseus, [39], by Sabry and James. Theseus is a functional language which compiles to a graphical language [37, 38] for reversible computation, based on isomorphisms of finite sums and products of types. Their chosen isomorphisms include commutativity and associativity for sums and products, units for product and distributivity of product over sums. The basic graphical language is extended with recursive types and looping operators and therefore introduces partiality due to the possibility of non-terminating loops. Their abstract model for this language is a dagger symmetric traced bimonoidal category [64].

Additionally, there are a number of quantum programming languages which, as noted, included reversible operations. Our primary example is LQPL [31], a compiled language based on the semantics of [62]. The language includes a variety of reversible operations (unitary transforms) as primitives, a linear type system and an operational semantics. More recently Quipper [32,33], which focuses on methods to handle very large circuits, is a quantum language embedded in Haskell [58]. Quipper uses quantum *and* classical circuits as an underlying model. An interesting aspect of reversibility in Quipper is the inclusion of an operator to compute the reverse of a given circuit.

In much of the research on reversibility, specific conditions are placed on some aspect of the computational model or reversible language to ensure "programs" in this model are reversible. The variety of models and languages obscures the fundamental commonality of reversibility. By basing the theory of this thesis on inverse categories, our treatment clarifies the relationship between these various approaches.

1.3 Objectives

This thesis proposes a categorical semantics for reversible computing. Based upon the review of current research as noted in Section 1.2, reversibility still lacks a unifying semantic model. Standard computability has Cartesian closed categories [5] and Turing categories [18], while quantum computing has had much success with dagger compact closed categories [1,63,64].

We present inverse categories as an abstract semantics for partial reversible computation. Inverse categories admit product-like and coproduct-like structures, respectively called inverse product and disjoint sum. Inverse categories with an inverse product are called discrete inverse categories. The name discrete is derived from topological spaces, where Δ has an inverse only when the topological space has the discrete topology. Similarly, when Δ in a Cartesian restriction category has an inverse, it will be called a discrete Cartesian restriction category. Section 5.1 shows how the "Cartesian Completion" of a discrete inverse category can be constructed. This enables us to create a discrete Cartesian restriction category from a discrete inverse category. It is then shown that we have an equivalence between the category of discrete inverse categories and the category of discrete Cartesian restriction categories.

The next step is to show how to add a disjoint sum to an inverse category and how the Cartesian Completion results in a distributive restriction category when one starts with a distributive inverse category.

An example of a discrete inverse category with disjoint sums is provided by the commutative Frobenius algebras in any additive symmetric monoidal category. As Frobenius algebras are related to bases in finite dimensional Hilbert spaces [27], this provides a connection between inverse categories and quantum computing.

Finally, we develop the structure of inverse Turing categories and inverse partial combinatory algebras, directly based on Turing category and partial combinatory algebras from [18,20], using the main result of this thesis. This places the connection between reversible and irreversible computing on a more abstract footing.

While the thesis does cover many important aspects of reversible computing, there are interesting areas related to reversible computing that are not within the scope of this thesis.

The thesis does not, in general, consider resource usage or complexity classes. For example, although we do mention Turing machines in the introduction, we do not develop this further. In particular the thesis does not consider whether the Cartesian Completion preserves a given complexity of programs when applied to an inverse category whose maps are programs. The reader wishing to relate resource theory and the work in this thesis could use the recent work by Coecke, Fritz and Spekkens [25], which defines a resource theory as a symmetric monoidal category, as a starting point. We do look at this briefly in SubSection 4.4.1.

Additionally, the thesis does not address the creation or invention of specific algorithms. For example, in the quantum world, finding the "right" set of invertible transforms to produce the desired answer is a significant problem [56].

1.4 Outline

We assume a knowledge of basic algebra including definitions and properties of groups, rings, fields, vector spaces and matrices. The reader may consult [45] if further details are needed.

Chapter 2 introduces the various categorical concepts that will be used throughout this thesis.

Chapter 3 describes restriction categories, an algebraic formulation of partiality in categories. We discuss joins, meets and ranges in restriction categories and their relation to partial map categories. We describe products in restriction categories, and define discrete Cartesian restriction categories, which will be important to the thesis. Various examples of restriction categories are given.

Chapter 4 introduces inverse categories and provides examples of them. We show that inverse categories with a restriction product collapse to a restriction preorder, that is, a restriction category in which all parallel maps agree wherever they are both defined. Then, Section 4.3 introduces the concept of inverse products and explores the properties of the inverse product. Inverse categories with inverse products are called *discrete inverse categories*.

Chapter 5 then presents the "Cartesian Completion" — a construction of a discrete

Cartesian restriction category from a discrete inverse category. Subsection 5.1.1 presents the details of the equivalence relation on maps of a discrete inverse category needed in the construction, while Section 5.1 contains the proof that the construction gives a Cartesian restriction category. Section 5.2 culminates in Theorem 5.2.6 giving an equivalence between the category of discrete inverse categories and the category of discrete Cartesian restriction categories. Note this is not a 2-equivalence of categories. We provide some simple examples.

Chapter 6 begins the exploration of how to add a coproduct-like construction to inverse categories. Paralleling the previous chapter, we show the existence of a restriction coproduct implies that an inverse category must be a preorder, i.e., that all parallel maps are equal. Section 6.2 defines a disjointness relation in an inverse category. We show that disjointness may be defined on all maps or equivalently only on the restriction idempotents of the inverse category. This allows us to define the disjoint join in Section 6.3.

Chapter 7 introduces the disjoint sum, an object in an inverse category with a disjoint join, which behaves like a coproduct. The disjoint sum has injection maps which are subject to certain conditions. When an inverse category has all disjoint sums, it is possible to define a symmetric monoidal tensor based on the disjoint sum. The remainder of the chapter explores what constraints on a tensor will allow the creation of a disjoint sum. We define a disjoint sum tensor, a symmetric monoidal tensor in the inverse category with specific additional constraints. A disjoint sum tensor allows us to define both a disjointness relation and a disjoint join based on the tensor. Disjoint sum tensors do produce disjoint sums and conversely, the tensor defined by disjoint sums is a disjoint sum tensor.

Chapter 8 introduces a matrix construction on inverse categories with disjoint joins in order to add disjoint sums. The functor from $\mathbb{X}$ to $\text{iM}\textsc{at}(\mathbb{X})$ gives us an adjunction between the category of inverse categories with disjoint joins and the category of inverse categories with disjoint sums.

In Chapter 9 a distributive inverse category is defined as an inverse category where the inverse product distributes over the disjoint sum. The Cartesian Completion of a distributive inverse category turns the disjoint sum into a coproduct and, in fact, will create a distributive restriction category.

Chapter 10 discusses commutative Frobenius algebras. The chapter starts with providing a background on dagger categories and Frobenius algebras, showing how the latter are equivalent to bases in a finite dimensional Hilbert space. The category $\text{CF}\textsc{rob}(\mathbb{X})$, the category of commutative Frobenius algebras in a symmetric monoidal category $\mathbb{X}$, is introduced. $\text{CF}\textsc{rob}(\mathbb{X})$ is shown to be a discrete inverse category. Furthermore, when $\mathbb{X}$ is an additive tensor category with zero maps, $\text{CF}\textsc{rob}(\mathbb{X})$ has disjoint sums.

In Chapter 11, Turing categories and partial combinatory algebras are introduced as a way to formulate computability. The corresponding structures in inverse categories, inverse Turing categories and inverse partial combinatory algebras, are then investigated. We show the equivalence of these structures to the ones in discrete Cartesian restriction categories.

Chapter 12 starts with a summary of the contributions of this thesis and concludes with a short section on potential areas of further exploration.

Chapter 2

Introduction to categories

This chapter introduces categories and fixes notation for them. More details for category theory can be found from, e.g., [5], [19], [49] and [66].

2.1 Definition of a category

A category may be defined in a variety of equivalent ways. As much of our work will involve the exploration of partial and reversible maps, we choose a definition that highlights the algebraic nature of these.

Definition 2.1.1. A *category* $\mathbb{A}$ is a directed graph consisting of objects $\mathbb{A}_o$ and maps $\mathbb{A}_m$. Each $f \in \mathbb{A}_m$ has two associated objects in $\mathbb{A}_o$, called the domain, $\partial_0(f)$, and codomain, $\partial_1(f)$. When $\partial_0(f)$ is the object X and $\partial_1(f)$ is the object Y, we will write $f : X \to Y$. For $f, g \in \mathbb{A}_m$, if $f : X \to Y$ and $g : Y \to Z$, there is a map called the *composite* of f and g, written fg,[1] such that $fg : X \to Z$. For any $W \in \mathbb{A}_o$ there is an *identity* map $1_W : W \to W$. Additionally, these two axioms must hold:

[**C.1**] for $f : X \to Y,\ 1_X f = f = f 1_Y$; (Unit laws)

[**C.2**] given $f : X \to Y,\ g : Y \to Z$ and $h : Z \to W$, then $f(gh) = (fg)h$. (Associativity)

For a category $\mathbb{A}$, given objects $X, Y \in A_o$, the set of maps between X and Y is referred to as the *hom-set* of $\mathbb{A}$ between X and Y and written as $\mathbb{A}(X, Y)$.

Throughout this thesis, we will be working with *small* categories, that is, those categories whose collection of maps and collection of objects is, in fact, a set. We will give categories

[1] Note that composition is written in diagrammatic order throughout this thesis.

of "all" sets as an example, and the reader can take that to mean all our sets are very small and all belong to a some sufficiently large set, $\mathcal{U}$.

We give a few examples of categories:

Example 2.1.2. We set $\mathbf{1}$ to be the category consisting of a single object A and the identity arrow $1 : A \to A$. This is obviously a category, with $\mathbf{1}_o = A$ and $\partial_0(1) = \partial_1(1) = A$.

There is also the category $\mathbf{2}$:

$$
\begin{tikzcd}
A \arrow[loop left, "1_A"] \arrow[r, "f"] & B \arrow[loop right, "1_B"]
\end{tikzcd}
$$

where f is the only non-identity arrow.

Example 2.1.3 (Preorders are categories)**.** Take any partially ordered set $(P, \leq)$ and define $f : a \to b$ for $a, b \in P$ if and only if $a \leq b$. This is a category as we always have:

(i) $a \leq a$ (Identity);

(ii) $a \leq b$ and $b \leq c$ implies $a \leq c$ (Composition).

Note that we have at most one map between any two objects in P, hence [**C.1**] and [**C.2**] are immediately satisfied.

Example 2.1.4 (Dual Category)**.** Given a category $\mathbb{B}$, we may form the *dual* of $\mathbb{B}$, written $\mathbb{B}^{op}$ as the following category:

Objects: The objects of $\mathbb{B}$;

Maps: $f^{op} : B \to A$ in $\mathbb{B}^{op}$ when $f : A \to B$ in $\mathbb{B}$;

Identity: The identity maps of $\mathbb{B}$;

Composition: If $fg = h$ in $\mathbb{B}$, $g^{op} f^{op} = h^{op}$.

Note the format of the previous example, where we list the four basic requirements of a category. This is typically how we will present categories in this thesis. Depending upon the complexity of the definition, we may add further proof that it meets [**C.1**] and [**C.2**].

The previous example is an important one, as we will often speak of *dualizing* a notion, or that concept "x" is the *dual* of concept "y". This means that when "y" holds in a category $\mathbb{B}$, the "x" holds in $\mathbb{B}^{op}$.

2.2 Properties of maps

Many useful properties of maps are generalizations of notions used for sets and functions. We present a few of these in Table 2.1, together with their categorical definition. Throughout Table 2.1, e, f, g are maps in a category C with $e : A \to A$ and $f, g : A \to B$.

Sets	**Categorical Property**	**Definition**
Injective	Monic	f is monic whenever $hf = kf$ means that $h = k$.
Surjective	Epic	The dual notion to monic, g is epic whenever $gh = gk$ means that $h = k$. A map that is both monic and epic is called *bijic*.
Left Inverse	Section	f is a section when there is a map $f^\diamond$ such that $ff^\diamond = 1_A$. f is also referred to as the *left inverse* of $f^\diamond$.
Right Inverse	Retraction	f is a retraction when there is a map $f_\diamond$ such that $f_\diamond f = 1_B$. f is also referred to as the *right inverse* of $f_\diamond$. A map that is both a section and a retraction is called an *isomorphism*.
Idempotent	Idempotent	An endomorphism e is idempotent whenever $ee = e$.

Table 2.1: Properties of maps in categories

There are number of basic properties of maps enumerated in Table 2.1.

Lemma 2.2.1. *In a category $\mathbb{B}$,*

(i) If f, g are monic, then fg is monic.

(ii) If fg is monic, then f is monic.

(iii) f being a section means it is monic.

(iv) f, g sections implies that fg is a section.

(v) fg a section means f is a section.

(vi) If $f : A \to B$ is both a section and a retraction, then $f^{\diamond} = f_{\diamond}$, where $f^{\diamond}$ and $f_{\diamond}$ are as defined in Table 2.1.

(vii) f is an isomorphism if and only if it is an epic section.

Proof.

(i) Suppose $hfg = kfg$. As g is monic, $hf = kf$. As f is monic, this gives us $h = k$ and therefore fg is monic.

(ii) See [5], chapter 2.

(iii) Suppose $hf = kf$. Then $hff^{\diamond} = kff^{\diamond}$ giving us $h1 = k1$ and therefore $h = k$ and f is monic.

(iv) We are given $ff^{\diamond} = 1$ and $gg^{\diamond} = 1$. But then $fgg^{\diamond}f^{\diamond} = ff^{\diamond} = 1$ and fg is a section.

(v) We are given there is an h such that $(fg)h = 1$. This means $f(gh) = 1$ and f is a section.

(vi) See [19], Lemma 1.2.2.

(vii) See [19], Lemma 1.2.3.

□

Note there are corresponding properties for epics and retractions, obtained by dualizing the statements of Lemma 2.2.1.

Suppose $f : A \to B$ is a retraction with left inverse $f_\diamond : B \to A$. Note that $ff_\diamond$ is idempotent as $ff_\diamond ff_\diamond = f1_B f_\diamond = ff_\diamond$. If we are given an idempotent e, we say e is *split* if there is a retraction f with $e = ff_\diamond$.

In general, not all idempotents in a category will split. The following construction allows us to create a category based on the original one, in which all idempotents do split.

Definition 2.2.2. Given a category $\mathbb{B}$ and a set of idempotents E of $\mathbb{B}$, we may create the category of $\mathbb{B}$ split over the idempotents E. This is normally written as $\mathrm{K}_E(\mathbb{B})$, and defined as:

Objects: (A, e), where A is an object of $\mathbb{B}$, $e : A \to A$ and $e \in E$.

Maps: $f_{d,e} : (A, d) \to (B, e)$ is given by $f : A \to B$ in $\mathbb{B}$, where $f = dfe$.

Identity: The map $e_{e,e}$ for (A, e).

Composition: Inherited from $\mathbb{B}$.

When E is the set of all idempotents in $\mathbb{B}$, we write $\mathrm{K}(\mathbb{B})$.

This is the standard idempotent splitting construction, variously known as the Karoubi envelope (whence the notation) or Cauchy completion.

2.3 Functors and natural transformations

Definition 2.3.1. A map $F : \mathbb{X} \to \mathbb{Y}$ between categories, as in Definition 2.1.1, is called a *functor*, provided it satisfies the following:

[**F.1**] $F(\partial_0(f)) = \partial_0(F(f))$ and $F(\partial_1(f)) = \partial_1(F(f))$;

[**F.2**] $F(fg) = F(f)F(g)$.

Lemma 2.3.2. *Categories and functors form a category* $\textsc{Cat}$.

Proof.

Objects: Categories.

Maps: Functors.

Identity: The identity functor which takes an object to the same object and a map to the same map.

Composition: Given $F : \mathbb{A} \to \mathbb{B}$, $G : \mathbb{B} \to \mathbb{D}$, define the functor $FG : \mathbb{A} \to \mathbb{D}$ such that $FG(x) = G(F(x))$. This is clearly associative.

Note Cat is often regarded as a *large* category, as its collection of objects is not a set. $\square$

A functor $F : \mathbb{B} \to \mathbb{D}$ induces a map between hom-sets in $\mathbb{B}$ and hom-sets in $\mathbb{D}$. For each object A, B in $\mathbb{B}$ we have the map:

$$F_{AB} : \mathbb{B}(A, B) \to \mathbb{D}(F(A), F(B)).$$

Definition 2.3.3. A functor $F : \mathbb{X} \to \mathbb{Y}$ is *full* when for each pair of objects A, B in $\mathbb{X}$, and map $g : FA \to FB$ in $\mathbb{Y}$, there is a map $f : A \to B$ in $\mathbb{X}$ such that $Ff = g$.

Definition 2.3.4. A functor $F : \mathbb{X} \to \mathbb{Y}$ is *faithful* when for parallel maps f, f', if $Ff = Ff'$ then $f = f'$.

We may also consider the notion of containment between categories:

Definition 2.3.5. Given the categories $\mathbb{B}$ and $\mathbb{D}$, we say that $\mathbb{B}$ is a *subcategory* of $\mathbb{D}$ when each object of $\mathbb{B}$ is an object of $\mathbb{D}$ and when each map of $\mathbb{B}$ is a map of $\mathbb{D}$.

When $\mathbb{B}$ is a subcategory of $\mathbb{D}$, the functor $J : \mathbb{B} \to \mathbb{D}$ which takes each object to itself in $\mathbb{D}$ and each map to itself in $\mathbb{D}$ is called the inclusion functor. When J is a full functor, we say $\mathbb{B}$ is a full subcategory of $\mathbb{D}$.

We now have the machinery to discuss the relationship between a category $\mathbb{B}$ and its splitting $\mathrm{K}(\mathbb{B})$:

Lemma 2.3.6. *Given a category $\mathbb{B}$, then it is a full subcategory of $K(\mathbb{B})$ and all idempotents split in $K(\mathbb{B})$.*

Proof. First, recall $\mathrm{K}(\mathbb{B})$ means we are splitting over all idempotents in $\mathbb{B}$, including the identity maps. We identify each object A in $\mathbb{B}$ with the object $(A, 1)$ in $\mathrm{K}(\mathbb{B})$. The only maps between $(A, 1)$ and $(B, 1)$ in $\mathrm{K}(\mathbb{B})$ are the maps between A and B in $\mathbb{B}$, hence we have a full subcategory.

Suppose we have the map $d_{e,e} : (A, e) \to (A, e)$ with $dd = d$, i.e., it is idempotent in $\mathbb{B}$ and $\mathrm{K}(\mathbb{B})$. In $\mathrm{K}(\mathbb{B})$, we have the maps $d_{e,d} : (A, e) \to (A, d)$ as $edd = e(ede)d = eded = dd = d$ and $d_{d,e} : (A, d) \to (A, e)$ as $dde = de = (ede)e = ede = d$. The compositions are $d_{d,e}d_{e,d} = d_{d,d} = 1_{(A,d)}$ and $d_{e,d}d_{d,e} = d_{e,e}$. Hence, it is a splitting of the map $d_{e,e}$. $\square$

Functors with two arguments, e.g., $F : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ which satisfy [**F.1**] and [**F.2**] for each argument independently are called *bi-functors*.

We will often restrict ourselves to specific classes of functors which either *preserve* or *reflect* certain characteristics of the domain category or codomain category.

Definition 2.3.7. Given a category $\mathbb{S}$, a *diagram* in a category $\mathbb{B}$ of *shape* $\mathbb{S}$ is a functor $D : \mathbb{S} \to \mathbb{B}$.

Definition 2.3.8. A *property* of a diagram D, written $P(D)$, is a logical relation expressed using the objects and maps of the diagram D.

For example, $P(f : A \to B) := \exists h : B \to A. hf = 1_A$ expresses that f is a retraction.

Definition 2.3.9. A functor F *preserves* the property P over maps f_i and objects A_j when:

$$P(f_1, \ldots, f_n, A_1, \ldots, A_m) \implies P(F(f_1), \ldots, F(f_n), F(A_1), \ldots, F(A_m)).$$

A functor F *reflects* the property P over maps f_i and objects A_j when:

$$P(F(f_1), \ldots, F(f_n), F(A_1), \ldots, F(A_m)) \implies P(f_1, \ldots, f_n, A_1, \ldots, A_m).$$

For example, all functors preserve the properties of being an idempotent or a retraction or section, but in general, not the property of being monic.

Definition 2.3.10. Given functors $F, G : \mathbb{X} \to \mathbb{Y}$, a *natural transformation* $\alpha : F \Rightarrow G$ is a collection of maps in $\mathbb{Y}$, $\alpha_X : F(X) \to G(X)$, indexed by the objects of $\mathbb{X}$ such that for all $f : X_1 \to X_2$ in $\mathbb{X}$ the following diagram in $\mathbb{Y}$ commutes:

$$\begin{tikzcd} F(X_1) \arrow[r, "F(f)"] \arrow[d, "\alpha_{X_1}"'] & F(X_2) \arrow[d, "\alpha_{X_2}"] \\ G(X_1) \arrow[r, "G(f)"'] & G(X_2). \end{tikzcd}$$

In the case where a natural transformation is a collection of isomorphisms, we write $\alpha : F \cong G$ or simply $F \cong G$.

2.4 Adjoint functors and equivalences

Referring to Section 2.3, we consider relationships between categories. While two categories may be isomorphic (i.e. there is an invertible functor between them), category theory normally considers two alternate relations between categories: Equivalence and adjointness.

Adjointness refers to an *adjoint pair* of functors. There are various equivalent ways of defining adjoints and we will approach them using universality.

Definition 2.4.1. Given $G : \mathbb{B} \to \mathbb{A}$ is a functor and $A \in \mathbb{A}$, then an object U in $\mathbb{B}$ together with a map $\eta_A : A \to G(U)$ is a *universal pair* for the G at A if whenever there is a map $f : A \to G(Y)$, there is a unique $f^\# : U \to Y$ such that

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & G(U) \arrow[d, dotted, "G(f^\#)"] \\ & G(Y) \end{tikzcd}$$

is a commutative diagram.

Consider what happens when we have the situation above, but the U in $\mathbb{B}$ is definable based on the $A \in \mathbb{A}$.

Lemma 2.4.2. *Suppose $G : \mathbb{B} \to \mathbb{A}$ is a functor such that for each $A \in \mathbb{A}_o$, there is a map on objects $F : \mathbb{A}_o \to \mathbb{B}_o$ such that $(F(A), \eta_A)$ is a universal pair. Then:*

- *F is a functor with $F(g) = (g\eta)^{\#}$;*
- *$\eta_A : A \to G(F(A))$ is a natural transformation;*
- *$\epsilon_B = 1^{\#}_{G(B)} : F(G(B)) \to B$ is a natural transformation;*
- *The triangle equalities, $\eta_{G(B)}G(\epsilon_B) = 1_{G(B)}$ and $F(\eta_A)\epsilon_{F(A)} = 1_{F(A)}$ hold.*

Conversely, if we have functors F, G with transformation η and ϵ which satisfy the triangle identities, then each $(F(A), \eta_A)$ is universal for G at A.

Proof. See Proposition 2.2.2 [19]. □

When we have this occurring we say F is *left adjoint* to G (and G is the *right adjoint* of F). The transformation η is referred to as the *unit* and the transformation ϵ is referred to as the *counit*. This is written as:

$$(\eta, \epsilon) : F \vdash G : A \to B.$$

Additionally, adjoints are often defined as a one to one correspondence between hom-sets in the following manner:

Let F, G be functors such that $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{A}$ such that there is a bijection ϕ between the hom-sets:

$$\phi : \mathbb{B}(F(X), Y) \to \mathbb{A}(X, G(Y)).$$

This is the path followed by Mac Lane in [49], where he proves this definition is equivalent to Definition 2.4.1 above. Rather than explicitly defining the bijection, this is often written in the form of an inference:

$$\frac{\mathbb{B}(F(X), Y)}{\mathbb{A}(X, G(Y)).}$$

Example 2.4.3 (Simple slice adjoint)**.** Consider the *simple slice* category [9], $\mathbb{B}[A]$ defined as:

Objects: Objects of $\mathbb{B}$.

Maps: a map $f_A : C \to D$ in $\mathbb{B}[A]$ is given by the map $f : C \times A \to D$ in $\mathbb{B}$.

Identity: projection – π_0.

Composition: $C \xrightarrow{f_A} D \xrightarrow{g_A} E$ in $\mathbb{B}[A]$ is given by $C \times A \xrightarrow{1 \times \Delta} C \times A \times A \xrightarrow{f \times 1} D \times A \xrightarrow{g} E$ in $\mathbb{B}$.

Note the definition of simple slice category relies on products in categories, to be introduced below in Section 2.7.

There is an adjunction between $\mathbb{B}[A]$ and $\mathbb{B}$.

Define $F : \mathbb{B}[A] \to \mathbb{B}$ by $F : C \mapsto C \times A$, $F : (f_A : C \to D) \mapsto (1 \times \Delta)(f \times 1)$. Define $G : \mathbb{B} \to \mathbb{B}[A]$ by $G : C \mapsto C$, $G : (f : C \to D) \mapsto (\pi_0 f)_A$. We see immediately there is a correspondence ϕ of hom-sets

$$\frac{\mathbb{B}(F(C), D)}{\mathbb{B}[A](C, G(D))} \qquad \equiv \frac{\mathbb{B}(C \times A, D)}{\mathbb{B}[A](C, D)}$$

where $\phi : \mathbb{B}(C \times A, D) \to \mathbb{B}[A](C, D)$ is given by $\phi(f) = f_A$. Thus $F \vdash G$.

Now, we may consider equivalence of categories.

Definition 2.4.4. Given two categories $\mathbb{A}$ and $\mathbb{B}$, we say they are *equivalent* when there exists two functors, $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{A}$ and natural isomorphisms such that $FG \cong I : \mathbb{A} \to \mathbb{A}$ and $GF \cong I : \mathbb{B} \to \mathbb{B}$.

The equivalence functors will also be an adjoint pair, typically they are chosen such that $F \vdash G$.

2.5 Enrichment of categories

If $\mathbb{X}$ is a category, then the maps from A to B in $\mathbb{X}$ are denoted $\mathbb{X}(A, B)$. If $\mathbb{X}(A, B)$ is a set for all objects in $\mathbb{X}$, we say $\mathbb{X}$ is *enriched* in sets. More generally, categories may be enriched in any monoidal category. For example a category may be enriched in abelian groups, vector spaces, posets, categories or commutative monoids.

Specific types of enrichment may force a structure on a category. Examples:

1. If $\mathbb{X}$ is enriched in sets of cardinality of 0 or 1, then $\mathbb{X}$ is a preorder.
2. If $\mathbb{X}$ is enriched in pointed sets with the monoid of smash product, then $\mathbb{X}$ has zero morphisms.

2.6 Examples of categories

In this section, we will offer a few examples of categories.

Example 2.6.1. A group G may be considered as a one-object category $\mathbb{G}$, with object $\{*\}$. The elements of the group are the maps between $\{*\}$. As G is a group, it has an identity e and multiplication is associative. In $\mathbb{G}$, the identity map is e, composition is given by the group multiplication and additionally, each map has an inverse. As $\mathbb{G}(\{*\}, \{*\}) = G$, this category is enriched in groups.

Four categories whose objects are sets are Rel, Sets, Par, and Pinj:

Example 2.6.2 (Rel)**.** Rel is often of interest in quantum programming language semantics:

Objects: All sets;

Maps: $R : X \to Y$ is a relation: $R \subseteq X \times Y$;

Identity: $1_X = \{(x, x) | x \in X\}$;

Composition: $RS = \{(x, z) | \exists y.(x, y) \in R \text{ and } (y, z) \in S\}$.

Note that Rel is enriched in posets (see Example 2.1.3), via set inclusion. Sets, Par and Pinj can be viewed as subcategories of Rel, with the same objects, but restricting the maps allowed — see each example below. Par is also enriched in posets, via the same inclusion ordering as in Rel.

Example 2.6.3 (Sets)**.** This is the subcategory of Rel which has the same objects (sets) but only allowing maps which are total functions, i.e., deterministic relations f where for all $x \in X$, there is a y such that $(x, y) \in f$ and if $(x, y), (x, y') \in f$, then $y = y'$.

Example 2.6.4 (Par)**.** This is the subcategory of Rel which has the same objects (sets) but only allowing maps which are partial functions, i.e., deterministic relations f where if $(x, y), (x, y') \in f$, then $y = y'$.

Example 2.6.5 (Pinj)**.** Our final example based on sets is one that will be used throughout this thesis. The category Pinj consists of the partial injective functions over sets. Similarly to Sets and Par, it is a subcategory of Rel. The maps f, g (relations in Rel) in Pinj are subject to two conditions:

$$(x, y) \in f \text{ and } (x, y') \in f \text{ implies } y = y', \tag{2.1}$$

$$(x, y) \in f \text{ and } (x', y) \in f \text{ implies } x = x'. \tag{2.2}$$

Example 2.6.6 (Top)**.** For a set S, $\mathcal{P}(S)$ is the power set of S, that is, the set of all subjects of S, including the empty set and S itself.

We define $\mathcal{O}(T)$, the *open sets* of T, as follows:

(i) $\mathcal{O}(T) \subseteq \mathcal{P}(T)$;

(ii) $T \in \mathcal{O}(T)$ and $\emptyset \in \mathcal{O}(T)$;

(iii) The union of any sets in $\mathcal{O}(T)$ is in $\mathcal{O}(T)$;

(iv) The intersection of finitely many sets in $\mathcal{O}(T)$ is in $\mathcal{O}(T)$.

A topological space is defined as a pair $(T, \mathcal{O}(T))$ where T is a set and $\mathcal{O}(T)$ are the open sets of T. Note that we may choose different open sets for T, resulting in different topological spaces.

Maps between two topological spaces $(T, \mathcal{O}(T))$ and $(S, \mathcal{O}(S))$ consist of a set map $f : T \to S$ such that the inverse image of any open set in S is an open set in T. Such maps are referred to as *continuous functions*.

The identity map is always continuous and composition of continuous functions yields another continuous function, hence setting $\textsc{Top}_o$ to be the set of all topological spaces and $\textsc{Top}_m$ to be the continuous maps between them gives us a category.

Our next example shows maps in categories need not always be something normally thought of as a function or relation.

Example 2.6.7 (Matrix category)**.** Given a rig R (i.e., a ri**n**g without **n**egatives, e.g., the natural numbers), one may form the category $\textsc{Mat}(R)$. For example, the category of matrices over natural numbers is:

Objects: $\mathbb{N}$;

Maps: $[r_{ij}] : n \to m$ where $[r_{ij}]$ is an $n \times m$ matrix over $\mathbb{N}$;

Identity: I_n;

Composition: Matrix multiplication.

2.7 Limits and colimits in categories

We shall review only a few basic limits/colimits in categories, in order to set up notation and terminology. First we discuss initial and terminal objects.

Definition 2.7.1. An *initial object* in a category $\mathbb{B}$ is an object which has exactly one map to each other object in the category. The dual notion is *terminal object*. Every object in the category has exactly one map to the terminal object.

Lemma 2.7.2. *Suppose I, J are initial objects in $\mathbb{B}$. Then there is a unique isomorphism $i : I \to J$.*

Proof. First, note that by definition there is only one map from I to I — which must be the identity map. As I is initial there is a map $i : I \to J$. As J is initial there is a map $j : J \to I$. But this means $ij : I \to I = 1$ and $ji : J \to J = 1$ and hence i is the unique isomorphism from I to J. □

Dually, we have the corresponding result to Lemma 2.7.2 for terminal objects — they are also unique up to a unique isomorphism.

In categories, following the terminology in sets, we normally designate the initial object by 0 and the terminal object by 1. A map *from* the terminal object to another object in the category is often referred to as an *element* or a *point*.

We now turn to products and coproducts.

Definition 2.7.3. Let A, B be objects of the category $\mathbb{B}$. Then the object $A \times B$ is a *product* of A and B when:

- There exist maps π_0, π_1 with $\pi_0 : A \times B \to A$, $\pi_1 : A \times B \to B$;
- Given an object C with maps $f : C \to A$ and $g : C \to B$, there is a unique map $\langle f, g\rangle$ such that the following diagram commutes:

$$\begin{tikzcd} & & A \\ C \arrow[urr, "f"] \arrow[r, dotted, "\langle f,g\rangle"] \arrow[drr, "g"'] & A \times B \arrow[ur, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ & & B. \end{tikzcd}$$

A coproduct is the dual of a product.

Definition 2.7.4. Let A, B be objects of the category $\mathbb{B}$. Then the object $A + B$ is a *coproduct* of A and B when:

- There exist maps $\amalg_1, \amalg_2$ with $\amalg_1 : A \to A + B$, $\amalg_2 : B \to A + B$;
- Given an object C with maps $h : A \to C$ and $k : B \to C$, there is a unique map $[h, k]$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[dr, "\amalg_1"'] \arrow[drr, "h"] & & \\
& A + B \arrow[r, dotted, "{[h,k]}"] & C \\
B. \arrow[ur, "\amalg_2"] \arrow[urr, "k"'] & &
\end{tikzcd}$$

It is possible for an object to be both a limit and a colimit at the same time:

Definition 2.7.5. Given a category $\mathbb{B}$, any object that is both a terminal and initial object is called a *zero object*. This object is labelled $\mathbf{0}$.

Note that any category with a zero object has a special map, $\mathbf{0}_{A,B}$ between any two objects A, B of the category given by: $\mathbf{0}_{A,B} : A \to \mathbf{0} \to B$.

Definition 2.7.6. In a category $\mathbb{B}$, with products and coproducts, where:

$$\amalg_i \pi_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and for any two objects, A, B, $A \times B$ is the same as $A + B$, then $A \times B$ is referred to as the *biproduct* and designated as $A \boxplus B$. A category $\mathbb{D}$ is said to have *finite biproducts* when it has a zero object $\mathbf{0}$ and when each pair of objects A, B have a biproduct $A \boxplus B$.

The biproduct is often written as $\oplus$. This thesis frequently uses $\oplus$ where it is not a biproduct, and thus uses $\boxplus$ for biproduct instead.

A category with finite biproducts is enriched in commutative monoids. If $f, g : A \to B$, define $f + g : A \to B$ as $\langle 1_A, 1_A \rangle (f \boxplus g) [1_B, 1_B]$. The unit for $+$ is $\mathbf{0}_{A,B}$. In this thesis, when discussing a category with finite biproducts, $\langle 1, 1 \rangle$ will be designated by Δ and $[1, 1]$ by ∇.

2.8 Symmetric monoidal categories

Definition 2.8.1. A *symmetric monoidal category* [5, 49] $\mathbb{D}$ is a category equipped with a monoid $\otimes$ (a bi-functor $\otimes : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$) together with four families of natural isomorphisms:

$$u^l_A : I \otimes A \to A \qquad\qquad u^r_A : A \otimes I \to A$$

$$a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C \qquad\qquad c_{A,B} : A \otimes B \to B \otimes A.$$

When the objects are identified in other ways, we often write the maps with the tensor as a subscript:

$$u^l_\otimes : I \otimes A \to A \qquad\qquad u^r_\otimes : A \otimes I \to A$$

$$a_\otimes : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \qquad\qquad c_\otimes : A \otimes B \to B \otimes A.$$

These maps must satisfy the coherence diagrams and equations shown in Figures 2.1, 2.2, 2.3, 2.4 and 2.5. The essence of the coherence diagrams is that any diagram composed solely of the structure isomorphisms will commute. The isomorphisms are referred to as the *structure isomorphisms*. I is the unit of the monoid. A symmetric monoidal category is called *strict* when each of $a_{A,B,C}$, u^r_A, u^l_A and $c_{A,B}$ are identity maps.

A tensor with these properties is also referred to as a *symmetric tensor*.

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r, "a_{A,B,(C\otimes D)}"] \arrow[d, "1\otimes a_{B,C,D}"'] & (A \otimes B) \otimes (C \otimes D) \arrow[r, "a_{(A\otimes B),C,D}"] & ((A \otimes B) \otimes C) \otimes D \arrow[d, "a_{A,B,C}\otimes 1"] \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "a_{A,(B\otimes C),D}"'] & & (A \otimes (B \otimes C)) \otimes D
\end{tikzcd}$$

Figure 2.1: Pentagon diagram for associativity in an SMC.

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "a_{A,I,B}"] \arrow[dr, "1\otimes u^l_B"'] & & (A \otimes I) \otimes B \arrow[dl, "u^r_A \otimes 1"] \\
& A \otimes B &
\end{tikzcd}$$

$$\text{and } u^r_I = u^l_I : I \otimes I \to I$$

Figure 2.2: Unit diagram and equation in an SMC.

$$\begin{tikzcd}
A \otimes B \arrow[r, "c_{A,B}"] \arrow[dr, equal] & B \otimes A \arrow[d, "c_{B,A}"] \\
& A \otimes B
\end{tikzcd}$$

Figure 2.3: Symmetry in an SMC.

$$\begin{tikzcd}
A \otimes I \arrow[rr, "c_{A,I}"] \arrow[dr, "u^r_A"'] & & I \otimes A \arrow[dl, "u^l_A"] \\
& A &
\end{tikzcd}$$

Figure 2.4: Unit symmetry in an SMC.

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "c_{(A\otimes B),C}"] \arrow[d, "a^{-1}_{A,B,C}"'] & C \otimes (A \otimes B) \arrow[d, "a_{C,A,B}"] \\
A \otimes (B \otimes C) \arrow[d, "1\otimes c_{B,C}"'] & (C \otimes A) \otimes B \arrow[d, "c_{C,A}\otimes 1"] \\
A \otimes (C \otimes B) \arrow[r, "a_{A,C,B}"] & C \otimes (A \otimes B).
\end{tikzcd}$$

Figure 2.5: Associativity symmetry in an SMC.

Example 2.8.2. Any category $\mathbb{D}$ with a product, $\times$, and a terminal object, $\top$, is a symmetric monoidal category with $\otimes := \times$ and $I := \top$.

Example 2.8.3 (PINJ has a symmetric tensor)**.** A symmetric tensor in PINJ is given by the Cartesian product of sets. In detail, this means:

$$\begin{aligned} A \otimes B &= \{(a,b) | a \in A, b \in B\} \\ f \otimes g &= \{((a,c),(b,d)) | (a,b) \in f, (c,d) \in g\} \\ 1 &= \{*\}, \text{ a single element set.} \end{aligned}$$

The symmetric monoid isomorphisms are:

$$\begin{aligned} u^l_\otimes &: \{(*,a)\} \mapsto \{a\} & u^r_\otimes &: \{(a,*)\} \mapsto \{a\} \\ a_\otimes &: \{((a,b),c)\} \mapsto \{(a,(b,c))\} & c_\otimes &: \{(a,b)\} \mapsto \{(b,a)\}. \end{aligned}$$

Chapter 3

Restriction categories

Restriction categories were introduced in [21–23] as a way to give an algebraic treatment of partiality. We will introduce restriction categories and provide a variety of results.

From this point forward in the thesis, we will use the symbol $\uparrow$ to mean a function is undefined on some value or values. For example, if $f : \{1, 2\} \to \{a, b, c\}$ where $f(1) = a$ and $f(2)$ is not defined, then we write $f(2)$ is $\uparrow$.

3.1 Definitions

Definition 3.1.1. A *restriction category* is a category $\mathbb{X}$ together with a *restriction operator* on maps,

$$\frac{f : A \to B}{\overline{f} : A \to A,}$$

where f is a map of $\mathbb{X}$ and A, B are objects of $\mathbb{X}$, such that the following four *restriction identities* hold, whenever the compositions are defined:

$$[\mathbf{R.1}]\ \overline{f}f = f \qquad\qquad [\mathbf{R.2}]\ \overline{g}\,\overline{f} = \overline{f}\,\overline{g}$$

$$[\mathbf{R.3}]\ \overline{\overline{f}g} = \overline{f}\,\overline{g} \qquad\qquad [\mathbf{R.4}]\ f\overline{h} = \overline{fh}f.$$

Definition 3.1.2. A *restriction functor* is a functor which preserves the restriction. That is, given a functor $F : \mathbb{X} \to \mathbb{Y}$ with $\mathbb{X}$ and $\mathbb{Y}$ restriction categories, F is a restriction functor if:

$$F(\overline{f}) = \overline{F(f)}.$$

Note that any map such that $r = \overline{r}$ is an idempotent, as $rr = \overline{r}r = r$. Such a map is called a *restriction idempotent*.

Here are some basic facts (see e.g., [21] and [24]) for restriction categories.

Lemma 3.1.3. *In a restriction category $\mathbb{X}$,*

(i) $\overline{f}$ *is idempotent;*

(ii) $\overline{fg} = \overline{fg}\,\overline{f}$*;*

(iii) $\overline{fg} = \overline{f\overline{g}}$ *;*

(iv) $\overline{\overline{f}} = \overline{f}$*;*

(v) $\overline{f}\,\overline{g} = \overline{\overline{f}\,\overline{g}}$*;*

(vi) f *monic implies* $\overline{f} = 1$*;*

(vii) $f = \overline{g}f \implies \overline{g}\,\overline{f} = \overline{f}$*.*

Proof.

(i) Using [**R.3**] and then [**R.1**], we see $\overline{f}\,\overline{f} = \overline{\overline{f}f} = \overline{f}$.

(ii) Using [**R.1**], [**R.3**] and then [**R.2**], $\overline{fg} = \overline{\overline{f}fg} = \overline{f}\,\overline{fg} = \overline{fg}\,\overline{f}$.

(iii) Using (ii), [**R.3**] and then [**R.4**], $\overline{fg} = \overline{fg}\,\overline{f} = \overline{\overline{fg}f} = \overline{f\overline{g}}$.

(iv) By (iii), $\overline{f} = \overline{1f} = \overline{1\overline{f}} = \overline{\overline{f}}$.

(v) Using [**R.3**], $\overline{\overline{f}\,\overline{g}} = \overline{f}\,\overline{\overline{g}} = \overline{f}\,\overline{g}$.

(vi) By [**R.1**] $\overline{f}f = 1f$, hence when f is monic, $\overline{f} = 1$.

(vii) $\overline{g}\,\overline{f} = \overline{\overline{g}f} = \overline{f}$.

$\square$

Note that by Lemma 3.1.3, all maps $\overline{f}$ are restriction idempotents as $\overline{f} = \overline{\overline{f}}$.

Definition 3.1.4. A map $f : A \to B$ in a restriction category is said to be *total* when $\overline{f} = 1_A$.

Lemma 3.1.5. *The total maps in a restriction category form a subcategory $Total(\mathbb{X}) \subseteq \mathbb{X}$.*

Proof. First, as the identity map 1 is monic, by Lemma 3.1.3, we have $\overline{1} = 1$ and therefore the identity map is in $Total(\mathbb{X})$. If f, g are composable maps in $Total(\mathbb{X})$, then $\overline{fg} = \overline{f\overline{g}} = \overline{f} = 1$ and hence fg is in $Total(\mathbb{X})$. Therefore, $Total(\mathbb{X})$ is a subcategory of $\mathbb{X}$. $\square$

Example 3.1.6 (Par)**.** Continuing from Example 2.6.4, Par is a restriction category. The restriction of $f : A \to B$ is:

$$\overline{f}(x) = \begin{cases} x & \text{if } f(x) \text{ is defined,} \\ \uparrow & \text{if } f(x) \text{ is } \uparrow. \end{cases}$$

In Par, the total maps correspond precisely to the functions that are defined on all elements of the domain.

Example 3.1.7 (Rel)**.** The category Rel from Example 2.6.2 is *not* a restriction category with the candidate restriction of $R = \{(a,b)\}$ being $\overline{R} = \{(a,a)|\exists b.(a,b) \in R\}$. The axiom that fails is [**R.4**], as can be seen by setting $R = \{(1,1),(1,2)\}, S = \{(2,3)\}$. Then we have $RS = \{(1,3)\}$, $\overline{RS} = \{(1,1)\}$ and therefore $\overline{RS}R = R$. However, $R\overline{S} = R\{(2,2)\} = \{(1,2)\}$.

Example 3.1.8 (PInj)**.** From Example 2.6.5, we see PInj is a restriction category and in fact is a sub-restriction category of Par. We will show the four restriction axioms:

[**R.1**] $\overline{f}f = \{(x,z)|\exists x.(x,x) \in \overline{f} \text{ and } (x,z) \in f\} = \{(x,z)|(x,z) \in f\} = f$,

[**R.2**] $\overline{f}\overline{g} = \{(x,z)|\exists y.(x,y) \in \overline{f} \text{ and } (y,z) \in \overline{g}\} = \{(x,x)|(x,x) \in \overline{f} \text{ and } (x,x) \in \overline{g}\} = \overline{g}\overline{f}$,

[**R.3**] $\overline{\overline{f}g} = \overline{\{(x,y)|(x,x) \in \overline{f},(x,y) \in g\}} = \{(x,x)|(x,x) in \overline{f},(x,x) \in \overline{g}\} = \overline{f}\overline{g}$,

[**R.4**] $f\overline{g} = \{(x,y)|(x,y) \in f,(y,y) \in \overline{g}\} = \{(x,y)|(x,y) \in f, \exists z.(y,z) \in g\}$

$$= \{(x,y)|(x,y) \in f, \exists z.(x,z) \in fg\} = \{(x,y)|(x,y) \in f,(x,x) \in \overline{fg}\} = \overline{fg}f.$$

Example 3.1.9 (Top_p)**.** This is the category of topological spaces with partial functions.

Objects: Topological spaces;

Maps: Any partial function f, where f is defined on some open subset of $\partial_0(f)$;

Identity: The identity function;

Composition: Function composition;

Restriction: The restriction of $f : A \to B$ is:

$$\overline{f}(x) = \begin{cases} x & \text{if } f(x) \text{ is defined,} \\ \uparrow & \text{if } f(x) \text{ is } \uparrow . \end{cases}$$

Example 3.1.10. Given $\mathbb{S}$ is a symmetric monoidal category, define $\textsc{Copy}(\mathbb{S})$ as the category whose objects are the objects of $\mathbb{S}$ which are commutative comonoids and maps are semigroup homomorphisms, i.e., maps which do not preserve the unit. As noted in [21], when there is a counit, $! : B \to I$ where I is the unit of the tensor, this is a restriction category, where the restriction is given by

$$\frac{A \xrightarrow{f} B}{A \xrightarrow{\overline{f}} A := A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes 1} B \otimes A \xrightarrow{! \otimes 1} I \otimes A \xrightarrow{u^l_\otimes} A.}$$

We may also consider commutative monoids with semigroup morphisms, $\textsc{Copy}(\mathbb{S})^{op}$. These will have a corestriction operator, i.e., an operator $\overline{(\_)}$ on $f : A \to B$ with $\overline{f} : B \to B$ fulfilling the duals of the restriction axioms.

For example, consider the category of commutative monoids in $\textsc{Ab}$, the category of abelian categories. This is the same as $(\textsc{Copy}(\textsc{Ab}^{op}))^{op}$ and thus is a corestriction category. At the same time, this is the category of commutative rings with homomorphisms which do not preserve the unit.

As another example, consider Frobenius algebras in $\mathbb{S}$ (defined below in 10.1.10) with coalgebra homomorphisms that do not preserve the unit. Again, this category will have a corestriction.

3.2 Partial order enrichment

We may use the restriction to define a partial order on the hom-sets of a restriction category. Intuitively, we would think of a map f being less than a map g if f is defined on fewer elements than g *and* they agree where they are defined. This can be expressed as:

Definition 3.2.1. In a restriction category, for any two parallel maps $f, g : A \to B$, define $f \leq g$ if and only if $\overline{f}g = f$.

Lemma 3.2.2. *Any restriction category $\mathbb{X}$ is enriched in a category of partial orders under the ordering $\leq$ from Definition 3.2.1 and the following hold:*

(i) $f \leq g \implies \overline{f} \leq \overline{g}$;

(ii) $\overline{fg} \leq \overline{f}$;

(iii) $f \leq g \implies hf \leq hg$;

(iv) $f \leq g \implies fh \leq gh$;

(v) $f \leq g$ *and* $\overline{f} = \overline{g}$ *implies* $f = g$;

(vi) $f \leq 1 \iff f = \overline{f}$;

(vii) $f \leq g,\ h \leq k \implies fh \leq gk$;

(viii) $\overline{g}f = f$ *implies* $\overline{f} \leq \overline{g}$.

Proof. First, we show the enrichment by showing $\leq$ is a partial order on $\mathbb{X}(A, B)$. With $f, g, h : A \to B$ parallel maps in $\mathbb{X}$, each of the requirements for a partial order is verified below:

Reflexivity: $\overline{f}f = f$ and therefore, $f \leq f$.

Anti-Symmetry: Given $\overline{f}g = f$ and $\overline{g}f = g$, it follows:

$$f = \overline{f}f = \overline{\overline{f}g}f = \overline{f}\,\overline{g}f = \overline{g}\,\overline{f}f = \overline{g}f = g.$$

Transitivity: Given $f \leq g$ and $g \leq h$,

$$\overline{f}h = \overline{\overline{f}g}h = \overline{f}\,\overline{g}h = \overline{f}g = f$$

showing that $f \leq h$.

We now show the rest of the claims.

(i) The premise is that $\overline{f}g = f$. From this, $\overline{f}\,\overline{g} = \overline{\overline{f}g} = \overline{f}$, showing $\overline{f} \leq \overline{g}$.

(ii) Computing, $\overline{\overline{fg}}\,\overline{f} = \overline{fg}\,\overline{f} = \overline{fg}$ where the last step is by Lemma 3.1.3(ii).

(iii) $\overline{hf}hg = h\overline{f}g = hf$ and therefore $hf \leq hg$.

(iv) $\overline{f}g = f$, this shows $\overline{fh}gh = \overline{\overline{f}gh}gh = \overline{f}\,\overline{gh}gh = \overline{f}gh = fh$ and therefore $fh \leq gh$.

(v) $g = \overline{g}g = \overline{f}g = f$.

(vi) As $f \leq 1$ means precisely $\overline{f}1 = f$.

(vii) $\overline{fh}gk = \overline{fh}\,\overline{f}gk = \overline{fh}fk = f\overline{h}k = fh$.

(viii) Assuming $\overline{g}f = f$, we need to show $\overline{f}\,\overline{g} = \overline{f}$. Using [**R.2**] and then [**R.3**] we have $\overline{f}\,\overline{g} = \overline{g}\overline{f} = \overline{\overline{g}f} = \overline{f}$. Hence, $\overline{f} \leq \overline{g}$.

$\square$

In a restriction category $\mathbb{X}$, we will use the notation $\mathcal{O}(A)$ for the restriction idempotents of A, an object of $\mathbb{X}$, that is, $\mathcal{O}(A) = \{x : A \to A | x = \overline{x}\}$. The notation $\mathcal{O}(A)$ was chosen to be suggestive of open sets, as in TOP_p, see Example 3.1.9.

Lemma 3.2.3. *In a restriction category $\mathbb{X}$, $\mathcal{O}(A)$ is a meet semi-lattice — a poset with a top element and binary meets.*

Proof. The top of the meet semi-lattice is 1_A, under the ordering from Definition 3.2.1. The meet of any two idempotents is given by their composition. $\square$

Let STABLAT be the category whose objects are meet semi-lattices and maps are stable homomorphisms, that is, they preserve the meets but not necessarily the top. From Example 13 in [21], this is a corestriction category, i.e. STABLAT^{op} is a restriction category. We see that the operation $\mathcal{O}$ is a functor, $\mathcal{O} : \mathbb{X} \to \text{STABLAT}^{op}$.

3.3 Joins

The restriction operator allows one to algebraically axiomatize the concept of "domain of definition" for a function. With that axiomatization, we may then consider other questions about the maps. In this section, we consider when maps are identical on their common domain of definition. Two maps having this property are called compatible.

Definition 3.3.1. Two parallel maps $f, g : A \to B$ in a restriction category are *compatible*, written as $f \smile g$, when $\overline{f}g = \overline{g}f$. A restriction category $\mathbb{X}$ is a *restriction preorder* when all parallel pairs of maps are compatible.

Example 3.3.2 (Compatibility in PAR)**.** In the restriction category PAR, two maps, f, g are compatible when $(x, y) \in f$ and $(x, y') \in g$ implies that $y = y'$.

Given two compatible maps, $f, g : A \to B$, we now want to consider if we can create a map that combines f and g. Such a map needs to have certain properties:

Definition 3.3.3. Given $\mathbb{R}$ is a restriction category with zero maps, then $\mathbb{R}$ is said to have *joins* [34] whenever there is an operator $\vee$

$$\frac{A \overset{f}{\underset{g}{\rightrightarrows}} B,\ f \smile g}{A \xrightarrow{f \vee g} B}$$

such that:

(i) $f \leq f \vee g$ and $g \leq f \vee g$,

(ii) $\overline{f \vee g} = \overline{f} \vee \overline{g}$,

(iii) $f, g \leq h$ implies that $f \vee g \leq h$ and

(iv) $h(f \vee g) = hf \vee hg$.

Example 3.3.4 (Joins in PAR)**.** In the restriction category PAR, the join for two compatible maps is given by:

$$(f \vee g)(x) = \begin{cases} f(x)(= g(x)) & \text{when both } f \text{ and } g \text{ are defined;} \\ f(x) & \text{when only } f \text{ is defined;} \\ g(x) & \text{when only } g \text{ is defined;} \\ \uparrow & \text{when both } f \text{ and } g \text{ are undefined.} \end{cases}$$

Note that the first line of the definition requires $f \smile g$.

Showing that the conditions of Definition 3.3.3 hold is straightforward. For example, $\overline{f}(f \vee g) = f(x)$ when f is defined and is undefined otherwise, giving $f \leq f \vee g$ and similarly for $g \leq f \vee g$.

Example 3.3.5 (Joins in $\text{T}\textsc{op}_p$)**.** Recall from Example 3.1.9 that the map $f : A \to B$ is a continuous partial function on some open subset of A. $\overline{f}$ is the identity map on the open subset of A where f is defined, and as such, may be identified with that open subset. As the intersection of open subsets of A is again an open subset of A, given $f, g : A \to B$, define

$$f \vee g(x) = \begin{cases} f(x) & \text{when } x \in \overline{f} \cap \overline{g}, \\ f(x) & \text{when } x \in \overline{f} \setminus \overline{g}, \\ g(x) & \text{when } x \in \overline{g} \setminus \overline{f}, \\ \uparrow & \text{otherwise.} \end{cases}$$

Note this is similar to the definition of $\vee$ in $\text{P}\textsc{ar}$ and similar reasoning may be used to show it is a join.

When $\mathbb{R}$ is a restriction category with joins, this means that $\mathbb{R}(X, Y)$ is now a join semi-lattice. Joins are related to the coproduct by the following:

Theorem 3.3.6 (Cockett-Guo)**.** *Given a restriction category $\mathbb{R}$ with joins, then*

$$A \xrightarrow{\amalg_1} C \xleftarrow{\amalg_2} B$$

is a coproduct if and only if:

(i) $\amalg_1$ *and* $\amalg_2$ *are restriction monics;*

(ii) $\overline{\amalg_1^{(-1)}}\,\overline{\amalg_2^{(-1)}} = \mathbf{0}_{CC}$ *(the zero map required by the definition of join) and*

(iii) $\overline{\amalg_1^{(-1)}} \vee \overline{\amalg_2^{(-1)}} = 1_C$*.*

Proof. See [15]. $\square$

3.4 Meets

Definition 3.4.1. A restriction category has *meets* if there is an operation $\cap$ on parallel maps:

$$\frac{A \overset{f}{\underset{g}{\rightrightarrows}} B}{A \xrightarrow{f \cap g} B}$$

such that $f \cap g \leq f, f \cap g \leq g, f \cap f = f, h(f \cap g) = hf \cap hg$.

Meets were introduced in [17]. Note that, in general, $(f \cap g)h \neq fh \cap gh$. In fact equality only holds when h is a partial monic, as in Definition 3.5.1 below. We give the following basic results on meets:

Lemma 3.4.2. *In a restriction category $\mathbb{X}$ with meets, where f, g, h are maps in $\mathbb{X}$, the following are true:*

(i) $f \leq g$ *and* $f \leq h \iff f \leq g \cap h$*;*

(ii) $f \cap g = g \cap f$*;*

(iii) $\overline{f \cap 1} = f \cap 1$*;*

(iv) $(f \cap g) \cap h = f \cap (g \cap h)$*;*

(v) $r(f \cap g) = rf \cap g$ *where* $r = \overline{r}$ *is a restriction idempotent;*

(vi) $(f \cap g)r = fr \cap g$ *where* $r = \overline{r}$ *is a restriction idempotent;*

(vii) $\overline{f \cap g} \leq \overline{f}$ *(and therefore* $\overline{f \cap g} \leq \overline{g}$*);*

(viii) $(f \cap 1)f = f \cap 1$*;*

(ix) $e(e \cap 1) = e$ *where* e *is idempotent.*

Proof.

(i) $f \le g$ and $f \le h$ means precisely $f = \overline{f}g$ and $f = \overline{f}h$. Therefore,

$$\overline{f}(g \cap h) = \overline{f}g \cap \overline{f}h = f \cap f = f$$

and so $f \le g \cap h$. Conversely, given $f \le g \cap h$, we have $f = \overline{f}(g \cap h) = \overline{f}g \cap \overline{f}h \le \overline{f}g$. But $f \le \overline{f}g$ means $f = \overline{f}\,\overline{f}g = \overline{f}g$ and therefore $f \le g$. Similarly, $f \le h$.

(ii) From (i), as by definition, $f \cap g \le g$ and $f \cap g \le f$.

(iii) $f \cap 1 = \overline{f \cap 1}(f \cap 1) = (\overline{f \cap 1}f) \cap (\overline{f \cap 1}) \le \overline{f \cap 1}$ from which the result follows.

(iv) By definition and transitivity, $(f \cap g) \cap h \le f, g, h$ therefore by (i) $(f \cap g) \cap h \le f \cap (g \cap h)$. Similarly, $f \cap (g \cap h) \le (f \cap g) \cap h$ giving the equality.

(v) Given $rf \cap g \le rf$, calculate:

$$rf \cap g = \overline{rf \cap g}rf = \overline{r(rf \cap g)}f = \overline{rrf \cap rg}f = \overline{r(f \cap g)}f = r\overline{f \cap g}f = r(f \cap g).$$

(vi) Using the previous point with the restriction idempotent $\overline{fr}$,

$$\begin{aligned} fr \cap g = f\overline{r} \cap g = \overline{fr}f \cap g = \overline{fr}(f \cap g) &= \overline{fr}\,\overline{f \cap g}f \\ = \overline{f \cap g}\,\overline{fr}f = \overline{f \cap g}f\overline{r} &= (f \cap g)r. \end{aligned}$$

(vii) For the first claim,

$$\overline{f \cap g}\,\overline{f} = \overline{\overline{f}(f \cap g)} = \overline{(\overline{f}f) \cap g} = \overline{f \cap g}.$$

The second claim then follows by (ii).

(viii) Given $f \cap 1 \le f$:

$$f \cap 1 \le f \iff \overline{f \cap 1}f = f \cap 1 \iff (f \cap 1)f = f \cap 1$$

where the last step is by item (iii) of this lemma.

(ix) As e is idempotent, $e(e \cap 1) = (ee \cap e) = e$.

□

Additionally, when a restriction category has both meets and joins, we have:

Lemma 3.4.3. *If $\mathbb{R}$ is a meet restriction category with joins, then the meet distributes over the join, i.e.,*

$$h \cap (f \vee g) = (h \cap f) \vee (h \cap g).$$

Proof.

$$\begin{aligned}
h \cap (f \vee g) &= \overline{(f \vee g)} h \cap (f \vee g) \\
&= (\overline{f} \vee \overline{g}) h \cap (f \vee g) \\
&= (\overline{f}(h \cap (f \vee g))) \vee (\overline{g}(h \cap (f \vee g))) \\
&= (h \cap \overline{f}(f \vee g)) \vee (h \cap \overline{g}(f \vee g))) \\
&= (h \cap (f \vee g)) \vee (h \cap (f \vee g))).
\end{aligned}$$

□

Example 3.4.4 (Meets in PINJ and PAR)**.** The restriction category PINJ has meets given by the intersection of the sets defining the maps. First, we note that the hom-set ordering for PINJ is given by set inclusion. We immediately have

$$f \cap g \subseteq f$$

$$f \cap g \subseteq g$$

$$f \cap f = f$$

by the properties of sets and intersections. For the final requirement,

$$\begin{aligned}
h(f \cap g) &= \{(x, z) | \exists y.(x, y) \in h, (y, z) \in f \cap g\} \\
&= \{(x, z) | \exists y.(x, y) \in h, (y, z) \in f, (y, z) \in g\} \\
&= \{(x, z) | (x, z) \in hf, (x, z) \in hg\} = hf \cap hg.
\end{aligned}$$

Thus, intersection is a meet in PINJ.

Note that the calculations above apply immediately to PAR as well, therefore intersection is a meet in PAR.

3.5 Partial monics and isomorphisms

Partial isomorphisms play a central role in this thesis. Below we present some of their basic properties.

Definition 3.5.1. In a restriction category $\mathbb{X}$, a map f may have some of the following properties:

- f is a *partial isomorphism* when there is a *partial inverse*, written $f^{(-1)}$ with $ff^{(-1)} = \overline{f}$ and $f^{(-1)}f = \overline{f^{(-1)}}$;
- f is a *partial monic* if $hf = kf$ implies $h\overline{f} = k\overline{f}$;
- f is a *restriction monic* if it is a section s with a retraction r such that $rs = \overline{rs}$.

For example, consider the following maps in PAR, $f_1, f_2, f_3 : \{1, 2\} \to \{a, b, c\}$ where

$$f_1(1) = a, f_1(2) = b; \quad f_2(1) = a, f_2(2)\uparrow; \quad f_3(1) = f_3(2) = a.$$

Then, f_1 is a total partial isomorphism and a partial monic and a restriction monic. f_2 is a partial isomorphism and is a partial monic but is not a restriction monic as it is not a section, i.e., there is no map ${f_2}^{\diamond}$ such that $f_2{f_2}^{\diamond} = 1$. f_3 is none of the items in Definition 3.5.1. Finally, in the category TOP_p, we shall see in Example 3.10.2 that the diagonal map, $\Delta : a \mapsto (a, a)$, does not have a partial inverse unless the topological space is discrete. But as Δ is monic and total, it is a partial monic.

Note that restriction monic is a stronger notion than that of section. In fact, restriction monics are the partial isomorphisms which are total.

Lemma 3.5.2. *In a restriction category:*

(i) f and g partial monic implies fg is partial monic;

(ii) The partial inverse of f, when it exists, is unique;

(iii) If f and g have partial inverses and fg exists, then fg has a partial inverse;

(iv) A restriction monic s is a partial isomorphism.

Proof.

(i) Suppose $hfg = kfg$. As g is partial monic, $hf\overline{g} = kf\overline{g}$. Therefore:

$$\begin{aligned} h\overline{fg}f &= k\overline{fg}f && \textbf{[R.4]} \\ h\overline{fg}\,\overline{f} &= k\overline{fg}\,\overline{f} && f \text{partial monic} \\ h\overline{fg} &= k\overline{fg} && \text{Lemma 3.1.3, (ii).} \end{aligned}$$

(ii) Suppose both $f^{(-1)}$ and $f^{\diamond}$ are partial inverses of f. Then,

$$\begin{aligned} f^{(-1)} &= \overline{f^{(-1)}}f^{(-1)} = f^{(-1)}ff^{(-1)} = f^{(-1)}\overline{f} = f^{(-1)}ff^{\diamond} = f^{(-1)}f\overline{f^{\diamond}}f^{\diamond} \\ &= \overline{f^{(-1)}}\,\overline{f^{\diamond}}f^{\diamond} = \overline{f^{\diamond}}\,\overline{f^{(-1)}}f^{\diamond} = f^{\diamond}f\overline{f^{(-1)}}f^{\diamond} = f^{\diamond}ff^{(-1)}ff^{\diamond} = f^{\diamond}ff^{\diamond} = f^{\diamond}. \end{aligned}$$

(iii) For $f : A \to B,\ g : B \to C$ with partial inverses $f^{(-1)}$ and $g^{(-1)}$ respectively, the partial inverse of fg is $g^{(-1)}f^{(-1)}$. Calculating $fgg^{(-1)}f^{(-1)}$ using all the restriction identities:

$$fgg^{(-1)}f^{(-1)} = f\overline{g}f^{(-1)} = \overline{fg}ff^{(-1)} = \overline{fg}\,\overline{f} = \overline{f}\,\overline{fg} = \overline{\overline{f}fg} = \overline{fg}.$$

The calculation of $g^{(-1)}f^{(-1)}fg = \overline{g^{(-1)}f^{(-1)}}$ is similar.

(iv) The partial inverse of s is $\overline{rs}\,r$. First, note that $\overline{\overline{rs}\,r} = \overline{rs}\,\overline{r} = \overline{r}\,\overline{rs} = \overline{\overline{r}\,rs} = \overline{rs}$. Then, it follows that $(\overline{rs}\,r)s = rs\ = \overline{rs} = \overline{\overline{rs}r}$ and $s(\overline{rs}\,r) = sr\overline{s} = \overline{s}$.

□

3.6 Range categories

Corresponding to Definition 3.1.1 for restriction, which axiomatizes the concept of a domain of definition, we now introduce range categories [16, 17, 34] which algebraically axiomatize the concept of the range for a function, in the presence of a restriction. Note this is different from a corestriction category $\mathbb{Y}$, which has a single operator, the corestriction, which is a restriction in $\mathbb{Y}^{op}$. In general, the range is weaker than a corestriction in that it may fail [**R.4**].

Definition 3.6.1. A restriction category $\mathbb{X}$ is a *range category* when it has an operator on all maps

$$\frac{f : A \to B}{\hat{f} : B \to B}$$

where the operator satisfies the following:

[**RR.1**] $\overline{\hat{f}} = \hat{f}$ [**RR.2**] $f\hat{f} = f$

[**RR.3**] $\widehat{f\overline{g}} = \hat{f}\overline{g}$ [**RR.4**] $\widehat{\hat{f}g} = \widehat{fg}$

whenever the compositions are defined.

Lemma 3.6.2. *In a range category $\mathbb{X}$, the following hold:*

(i) $\hat{g}\hat{f} = \hat{f}\hat{g}$;

(ii) $\overline{f}\hat{g} = \hat{g}\overline{f}$;

(iii) $\widehat{f\hat{g}} = \hat{f}\hat{g}$;

(iv) $\hat{f} = 1$ *when* f *is epic, hence* $\hat{1} = 1$;

(v) $\hat{f}\hat{f} = \hat{f}$;

(vi) $\hat{\hat{f}} = \hat{f}$;

(vii) $\hat{\overline{f}} = \overline{f}$;

(viii) $\hat{g}\widehat{fg} = \widehat{fg}$;

(ix) $\widehat{\hat{f}\hat{g}} = \hat{f}\hat{g}$.

Proof. See, e.g., [34]. □

Lemma 3.6.3. *In a range category:*

(i) $\widehat{hf} \leq \hat{f}$;

(ii) $f' \leq f$ *implies* $\hat{f'} \leq \hat{f}$.

Proof.

(i) Noting that $\overline{\widehat{hf}}\hat{f} = \widehat{hf}\hat{f} = \widehat{hf\hat{f}} = \widehat{hf}$, we see $\widehat{hf} \leq \hat{f}$.

(ii) Calculating $\overline{\hat{f}'}\hat{f} = \hat{f}'\hat{f} = \widehat{\overline{f'}f}\hat{f} = \widehat{\overline{f'}f\hat{f}} = \widehat{\overline{f'}f} = \hat{f}'$, we see $\hat{f}' \leq \hat{f}$.

$\square$

Note that unlike restrictions, a range is a *property* of a restriction category. To see this, assume we have two ranges $\widehat{(\_)}$ and $\widetilde{(\_)}$. Then,

$$\hat{f} = \widehat{f\tilde{f}} = \hat{f}\tilde{f} = \tilde{f}\hat{f} = \widetilde{f\hat{f}} = \tilde{f}.$$

Example 3.6.4. In $\textsc{Pinj}$, $\hat{f} = \{(y,y)|\exists x.(x,y) \in f\}$.

For a further example, see Section 4.1.

3.7 Split restriction categories

The Karoubi envelope of a restriction category, $\mathrm{K}_E(\mathbb{X})$ as defined in Definition 2.2.2 is a restriction category.

Note that for $f : (A,d) \to (B,e)$, by definition, in $\mathbb{X}$ we have $f = dfe$, giving

$$df = d(dfe) = ddfe = dfe = f \quad \text{and} \quad fe = (dfe)e = dfee = dfe = f.$$

When $\mathbb{X}$ is a restriction category, there is an immediate candidate for a restriction in $\mathrm{K}_E(\mathbb{X})$. If $f \in \mathrm{K}_E(\mathbb{X})$ is $e_1 f e_2$ in $\mathbb{X}$, then define $\overline{f}$ as given by $e_1\overline{f}$ in $\mathbb{X}$. Note that for $f : (A,d) \to (B,e)$, in $\mathbb{X}$ we have:

$$d\overline{f} = \overline{df}d = \overline{f}d.$$

Proposition 3.7.1. *If $\mathbb{X}$ is a restriction category and E is a set of idempotents, then the restriction as defined above makes $K_E(\mathbb{X})$ a restriction category.*

Proof. The restriction takes $f : (A,e_1) \to (B,e_2)$ to an endomorphism of (A,e_1). The restriction is in $\mathrm{K}_E(\mathbb{X})$ as

$$e_1(e_1\overline{f})e_1 = e_1\overline{f}e_1 = \overline{e_1 f}e_1 e_1 = \overline{e_1 f}e_1 = e_1\overline{f}.$$

Checking the 4 restriction axioms:

[**R.1**] $e_1\overline{f}f = e_1 f = f$.

[**R.2**] $e_1\overline{g}e_1\overline{f} = e_1e_1\overline{g}\,\overline{f} = e_1e_1\overline{f}\,\overline{g} = e_1\overline{f}e_1\overline{g}$.

[**R.3**] $e_1(\overline{e_1\overline{f}g}) = \overline{e_1e_1\overline{f}g}e_1 = \overline{e_1\overline{f}g}e_1 = e_1\overline{\overline{f}g} = e_1\overline{f}\,\overline{g} = e_1e_1\overline{f}\,\overline{g} = e_1\overline{f}e_1\overline{g}$.

[**R.4**] $fe_2\overline{g} = \overline{fe_2g}fe_2 = (\overline{e_1fg}e_1)f = e_1\overline{fg}f$.

$\square$

Given this, provided all identity maps are in E, $\mathrm{K}_E(\mathbb{X})$ is a restriction category with $\mathbb{X}$ as a full sub-restriction category, via the embedding defined by taking an object A in $\mathbb{X}$ to the object $(A, 1)$ in $\mathrm{K}_E(\mathbb{X})$.

Proposition 3.7.2. *In a restriction category $\mathbb{X}$ with meets, let R be the set of restriction idempotents. Then, $K(\mathbb{X}) \cong K_R(\mathbb{X})$. That is, splitting over all the idempotents is equivalent to splitting over just the restriction idempotents. Furthermore, $K_R(\mathbb{X})$ has meets.*

Proof. The proof first shows the equivalence of the two categories, then addresses the claim that $\mathrm{K}_R(\mathbb{X})$ has meets.

For equivalence, we require two functors,

$$U : \mathrm{K}_R(\mathbb{X}) \to \mathrm{K}(\mathbb{X}) \text{ and } V : \mathrm{K}(\mathbb{X}) \to \mathrm{K}_R(\mathbb{X}),$$

with:

$$UV \cong I_{\mathrm{K}_R(\mathbb{X})} \tag{3.1}$$

$$VU \cong I_{\mathrm{K}(\mathbb{X})}. \tag{3.2}$$

U is the standard inclusion functor. V will take the object (A, e) to $(A, e \cap 1)$ and the map $f : (A, e_1) \to (B, e_2)$ to $(e_1 \cap 1)f$.

V is a functor as:

Well Defined: If $f : (A, e_1) \to (B, e_2)$, then $(e_1 \cap 1)f$ is a map in $\mathbb{X}$ from A to B and $(e_1 \cap 1)(e_1 \cap 1)f(e_2 \cap 1) = (e_1 \cap 1)(fe_2 \cap f) = (e_1 \cap 1)(f \cap f) = (e_1 \cap 1)f$, therefore, $V(f) : V((A, e_1)) \to V((B, e_2))$.

Identities: $V(e) = (e \cap 1)e = e \cap 1$ by lemma 3.4.2.

Composition: $V(f)V(g) = (e_1 \cap 1)f(e_2 \cap 1)g = (e_1 \cap 1)fe_2(e_2 \cap 1)g = (e_1 \cap 1)f(e_2 \cap e_2)g = (e_1 \cap 1)fe_2g = (e_1 \cap 1)fg = V(fg)$.

By Lemma 3.4.2 $(e \cap 1)$ is a restriction idempotent. Using this fact, the commutativity of restriction idempotents and Lemma 3.4.2, the composite functor UV is the identity on $\mathrm{K}_R(\mathbb{X})$. This is because when e is a restriction idempotent, $e = e(e \cap 1) = (e \cap 1)e = (e \cap 1)$.

For the other direction, note that for a particular idempotent $e : A \to A$, this gives the maps $e : (A, e) \to (A, e \cap 1)$ and $e \cap 1 : (A, e \cap 1) \to (A, e)$, again by 3.4.2. These maps give the natural isomorphism between I and VU as

$$\begin{tikzcd} (A,e) \arrow[r, "e"] \arrow[dr, "e"'] & (A, e\cap 1) \arrow[d, "e\cap 1"] \\ & (A,e) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (A,e\cap 1) \arrow[r, "e\cap 1"] \arrow[dr, "e\cap 1"'] & (A, e) \arrow[d, "e"] \\ & (A,e\cap 1) \end{tikzcd}$$

both commute. Therefore, $UV = I$ and $VU \cong I$, giving an equivalence of the categories.

For the rest of this proof, functions in bold type, e.g., $\mathbf{f}$, are in $\mathrm{K}_R(\mathbb{X})$. Functions in normal slanted type, e.g., f are in $\mathbb{X}$.

To show that $\mathrm{K}_R(\mathbb{X})$ has meets, designate the meet in $\mathrm{K}_R(\mathbb{X})$ as $\cap_{\mathrm{K}}$ and define $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}$ as the map given by the $\mathbb{X}$ map $f \cap g$, where $\mathbf{f}, \mathbf{g} : (A, d) \to (B, e)$ in $\mathrm{K}_R(\mathbb{X})$ and $f, g : A \to B$ in $\mathbb{X}$. This is a map in $\mathrm{K}_R(\mathbb{X})$ as $d(f \cap g)e = (df \cap dg)e = (f \cap g)e = (fe \cap g) = f \cap g$ where the penultimate equality is by 3.4.2. By definition $\overline{\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}}$ is $d\overline{f \cap g}$.

It is necessary to show $\cap_{\mathrm{K}}$ satisfies the four meet properties.

- $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g} \le \mathbf{f}$: We need to show $\overline{\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}}\mathbf{f} = \mathbf{f} \cap_{\mathrm{K}} \mathbf{g}$. Calculating now in $\mathbb{X}$:

$$d\overline{f \cap g}f = \overline{d(f \cap g)}df = \overline{df \cap dg}df = \overline{f \cap g}f = f \cap g$$

which is the definition of $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}$.

- $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g} \leq \mathbf{g}$: Similarly and once again calculating in $\mathbb{X}$,

$$d\overline{f \cap g}g = \overline{d(f \cap g)}dg = \overline{df \cap dg}dg = \overline{f \cap g}g = f \cap g$$

  which is the definition of $\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}$.

- $\mathbf{f} \cap_{\mathrm{K}} \mathbf{f} = \mathbf{f}$: From the definition, this is $f \cap f = f$ which is just $\mathbf{f}$.

- $\mathbf{h}(\mathbf{f} \cap_{\mathrm{K}} \mathbf{g}) = \mathbf{hf} \cap_{\mathrm{K}} \mathbf{hg}$: From the definition, this is given in $\mathbb{X}$ by $h(f \cap g) = hf \cap hg$ which in $\mathrm{K}_R(\mathbb{X})$ is $\mathbf{hf} \cap_{\mathrm{K}} \mathbf{hg}$.

$\square$

Consider two objects A, B in a restriction category where we have $m : A \to B$, $r : B \to A$ with $mr = 1_A$. In this case A is called a *retract* of B, which we will write as $A \triangleleft B$. As m and r need not be unique, we will also write $A \triangleleft_m^r B$ when the specific section and retraction are to be emphasized. Since m is a section, it is a monic and therefore total. The map rm is idempotent on B as $rmrm = r1m = rm$. A is referred to as a *splitting* of the idempotent rm. Note there is no requirement that $rm = \overline{rm}$ when m is simply monic.

3.8 Partial map categories

In [21], it is shown that split restriction categories are equivalent to *partial map categories*. The main definitions and results related to partial map categories are given below.

Definition 3.8.1. A collection $\mathcal{M}$ of monics is *a stable system of monics* when:

(i) it includes all isomorphisms;

(ii) it is closed under composition;

(iii) it is pullback stable.

Stable in this definition means that if $m : A \to B$ is in $\mathcal{M}$, then for arbitrary b with codomain B, the pullback

$$\begin{tikzcd}
A' \arrow[r, "a"] \arrow[d, "m'"'] & A \arrow[d, "m"] \\
B' \arrow[r, "b"'] & B
\end{tikzcd}$$

exists and $m' \in \mathcal{M}$. A category that has a stable system of monics is referred to as an $\mathcal{M}$-category.

Lemma 3.8.2. *If $nm \in \mathcal{M}$, a stable system of monics, and m is monic, then $n \in \mathcal{M}$.*

Proof. The commutative square

$$\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "n"'] & A \arrow[d, "nm"] \\
A' \arrow[r, "m"'] & B
\end{tikzcd}$$

is a pullback. $\square$

Given a category $\mathbb{B}$ and a stable system of monics $\mathcal{M}$, the *partial map category*, $\mathrm{Par}(\mathbb{B}, \mathcal{M})$ is:

Objects: $A \in \mathbb{B}$;

Equivalence Classes of Maps: $(m, f) : A \to B$ with $m : A' \to A$ is in $\mathcal{M}$ and $f : A' \to B$ is a map in $\mathbb{B}$. i.e.,

$$\begin{tikzcd}
& A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\
A & & B
\end{tikzcd};$$

Identity: $1_A, 1_A : A \to A$;

Composition: via a pullback, $(m, f)(m', g) = (m''m, f'g)$ where

$$\begin{tikzcd}
& & A'' \arrow[dl, "m''"'] \arrow[dr, "f'"] & & \\
& A' \arrow[dl, "m"'] \arrow[dr, "f"'] \arrow[rr, phantom, "\text{(pb)}"] & & B' \arrow[dl, "m'"] \arrow[dr, "g"] & \\
A & & B & & C
\end{tikzcd};$$

Restriction: $\overline{(m, f)} = (m, m)$.

For the maps, $(m, f) \sim (m', f')$ when there is an isomorphism $\gamma : A'' \to A'$ such that $\gamma m' = m$ and $\gamma f' = f$.

The proof that this is a restriction category is given by Proposition 3.1 in [21].

In [22], it is shown that:

Theorem 3.8.3 (Cockett-Lack)**.** *Every restriction category is a full subcategory of a partial map category.*

3.9 Restriction products and Cartesian restriction categories

Restriction categories have analogues of products and terminal objects.

Definition 3.9.1. In a restriction category $\mathbb{X}$, a *restriction product* of two objects X, Y is an object $X \times Y$ equipped with *total* projections $\pi_0 : X \times Y \to X, \pi_1 : X \times Y \to Y$ where:

$\forall f : Z \to X, g : Z \to Y, \quad \exists$ a unique $\langle f, g\rangle : Z \to X \times Y$ such that

- $\langle f, g\rangle\pi_0 \leq f$,
- $\langle f, g\rangle\pi_1 \leq g$ and
- $\overline{\langle f, g\rangle} = \overline{f}\,\overline{g}(= \overline{g}\,\overline{f})$.

Definition 3.9.2. In a restriction category $\mathbb{X}$, a *restriction terminal object* is an object $\top$ such that for all objects X, there is a unique total map $!_X : X \to \top$ and the diagram

$$\begin{tikzcd} X \arrow[r, "\overline{f}"] \arrow[d, "f"] & X \arrow[r, "!_X"] & \top \\ Y \arrow[urr, "!_Y"'] & & \end{tikzcd}$$

commutes. That is, $f\,!_Y = \overline{f}\,!_X$. Note this implies that a restriction terminal object is unique up to a unique isomorphism.

For example, in $\textsc{Par}$, the restriction terminal object is the one object set, $\{*\}$. The product of two sets is the standard Cartesian product, with π_0 mapping $(x, y) \mapsto x$ and $\pi_1 : (x, y) \mapsto y$. The product map $\langle f, g\rangle : Z \to X \times Y$ is given as:

$$\langle f, g\rangle(z) = \begin{cases} (x, y) & f(z) = x \text{ and } g(z) = y \\ \uparrow & f(z)\uparrow \text{ or } g(y)\uparrow . \end{cases}$$

Definition 3.9.3. A restriction category $\mathbb{X}$ is *Cartesian* if it has all restriction products and a restriction terminal object.

3.10 Discrete Cartesian restriction categories

Definition 3.10.1. An object A in a Cartesian restriction category is *discrete* when the diagonal map

$$\Delta : A \to A \times A$$

is a partial isomorphism. A Cartesian restriction category where all objects are discrete is called a *discrete* Cartesian restriction category.

Example 3.10.2 ($\textsc{Top}_p$ is not discrete)**.** In any topological space T, the only way that Δ can have a continuous inverse is when the topology is the discrete topology. This example is the motivating example for our terminology of discrete.

The topology of $T \times T$ is generated by open sets $U \times V$ where U, V are open sets of T. We see that $\Delta \cap U \times V = \Delta \cap (U \cap V) \times (U \cap V)$, so if $\Delta \subseteq \cup_i U_i \times V_i$, then $\Delta \subseteq \cup_i (U_i \cap V_i) \times (U_i \cap V_i)$. Thus, any open cover of Δ has a subcover of the form $\cup_i U_i \times U_i$. For $\Delta^{(-1)}$ to be a continuous map, that means the diagonal must be an open set. But if Δ is open, then $\Delta = \cup_i U_i \times V_i$ if and only if $\Delta \subseteq \cup_i (U_i \cap V_i) \times (U_i \cap V_i) \subseteq \cup_i U_i \times U_i \subseteq \Delta$. So, $\Delta = \cup_i U_i \times U_i$, which gives us $U_i \times U_i \subseteq \Delta$, but this can only happen when $U_i = \{x\}$, a singleton set. Therefore, T has the discrete topology.

Example 3.10.3 (Par is discrete)**.** In Par,

$$\Delta : x \mapsto (x,x) \text{ and } \Delta^{(-1)} : (x,y) \mapsto \begin{cases} x & x = y, \\ \uparrow & x \neq y. \end{cases}$$

Thus, Par is a discrete Cartesian restriction category.

Further examples of discrete and non-discrete Cartesian restriction categories are given at the end of the section.

Theorem 3.10.4. *A Cartesian restriction category $\mathbb{X}$ is discrete if and only if it has meets.*

Proof. If $\mathbb{X}$ has meets, then

$$\Delta(\pi_0 \cap \pi_1) = \Delta\pi_0 \cap \Delta\pi_1 = 1 \cap 1 = 1.$$

As $\langle \pi_0, \pi_1 \rangle$ is identity,

$$\begin{aligned} \overline{\pi_0 \cap \pi_1} &= \overline{\pi_0 \cap \pi_1}\langle \pi_0, \pi_1 \rangle \\ &= \langle \overline{\pi_0 \cap \pi_1}\pi_0, \overline{\pi_0 \cap \pi_1}\pi_1 \rangle \\ &= \langle \pi_0 \cap \pi_1, \pi_0 \cap \pi_1 \rangle \\ &= (\pi_0 \cap \pi_1)\Delta \end{aligned}$$

and therefore, $\pi_0 \cap \pi_1$ is $\Delta^{(-1)}$.

To show the other direction, we set $f \cap g = \langle f, g \rangle \Delta^{(-1)}$. By the definition of the restriction product:

$$f \cap g = \langle f, g \rangle \Delta^{(-1)} = \langle f, g \rangle \Delta^{(-1)} \Delta \pi_0 = \langle f, g \rangle \overline{\Delta^{(-1)}} \pi_0 \leq \langle f, g \rangle \pi_0 \leq f.$$

Then, substituting π_1 for π_0 above, this gives us $f \cap g \leq g$.

For the left distributive law,

$$h(f \cap g) = h\langle f, g \rangle \Delta^{(-1)} = \langle hf, hg \rangle \Delta^{(-1)} = hf \cap hg.$$

The intersection of a map with itself is

$$f \cap f = \langle f, f\rangle \Delta^{(-1)} = (f\Delta)\Delta^{(-1)} = f\overline{\Delta} = f$$

as Δ is total. This shows that $\cap$ as defined above is a meet for the Cartesian restriction category $\mathbb{X}$.

$\square$

Definition 3.10.5. In a Cartesian restriction category, a map $A \xrightarrow{f} B$ is called *graphic* when the maps

$$A \xrightarrow{\langle f,1\rangle} B \times A \qquad \text{and} \qquad A \xrightarrow{\langle \overline{f},1\rangle} A \times A$$

have partial inverses. A Cartesian restriction category is *graphic* when all of its maps are graphic.

Lemma 3.10.6. *In a Cartesian restriction category:*

(i) Graphic maps are closed under composition;

(ii) Graphic maps are closed under the restriction;

(iii) An object is discrete if and only if its identity map is graphic.

Proof.

(i) To show closure, it is necessary to show for graphic maps $f : A \to B$ and $g : B \to C$ that $\langle fg, 1\rangle$ has a partial inverse. By Lemma 3.5.2, the uniqueness of the partial inverse gives

$$(\langle f, 1\rangle(\langle g, 1\rangle \times 1))^{(-1)} = (\langle g, 1\rangle^{(-1)} \times 1)\langle f, 1\rangle^{(-1)}.$$

By the definition of the restriction product, we have $\overline{\langle fg, 1\rangle} = \overline{fg}$. Additionally, a straightforward calculation shows that $\overline{\langle f, 1\rangle(\langle g, 1\rangle \times 1)} = \overline{\langle f\langle g, 1\rangle, 1\rangle} = \overline{f\langle g, 1\rangle} = \overline{\langle fg, f\rangle} = \overline{fg}\,\overline{f} = \overline{fg}$ where the last equality is from Lemma 3.1.3.

Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "\langle f,1\rangle"] \arrow[drr, "\langle fg,1\rangle"'] & B \times A \arrow[r, "\langle g,1\rangle\times 1"] & C \times B \times A \\
& & C \times A. \arrow[u, "1\times\langle f,1\rangle"']
\end{tikzcd}$$

Thus,

$$\begin{aligned}
&\langle fg,1\rangle(1\times\langle f,1\rangle)(\langle g,1\rangle^{(-1)}\times 1)\langle f,1\rangle^{(-1)} \\
&\qquad = \langle f,1\rangle(\langle g,1\rangle\times 1)(\langle g,1\rangle^{(-1)}\times 1)\langle f,1\rangle^{(-1)} \\
&\qquad = \langle f,1\rangle(\overline{g\times 1})\langle f,1\rangle^{(-1)} \\
&\qquad = \overline{\langle f,1\rangle(g\times 1)}\langle f,1\rangle\langle f,1\rangle^{(-1)} \\
&\qquad = \overline{\langle f,1\rangle(g\times 1)}\,\overline{\langle f,1\rangle} \\
&\qquad = \overline{\langle f,1\rangle}\,\overline{\langle f,1\rangle(g\times 1)} \\
&\qquad = \overline{\langle f,1\rangle(g\times 1)} \\
&\qquad = \overline{\langle fg,1\rangle}(=\overline{fg})
\end{aligned}$$

showing that $1\times\langle f,1\rangle(\langle g,1\rangle^{(-1)}\times 1)\langle f,1\rangle^{(-1)}$ is a right inverse for $\langle fg,1\rangle$. For the other direction, note that in general $(hk)^{(-1)} = k^{(-1)}h^{(-1)}$ and that we have $\langle fg,1\rangle = \langle f,1\rangle(\langle g,1\rangle\times 1)(1\times\langle f,1\rangle^{(-1)})$, thus $(1\times\langle f,1\rangle)(\langle g,1\rangle^{(-1)}\times 1)\langle f,1\rangle^{(-1)}$ will also be a left inverse and $\langle fg,1\rangle$ is a partial isomorphism.

(ii) This follows from the definition of graphic and that $\overline{\langle f,1\rangle} = \overline{f} = \overline{\overline{f}} = \overline{\langle\overline{f},1\rangle}$.

(iii) Given a discrete object A, the identity on A is graphic as $\langle 1,1\rangle = \Delta$ and therefore $\langle 1,1\rangle^{(-1)} = \Delta^{(-1)}$. Conversely, if $\langle 1,1\rangle = \Delta$ has an inverse, A is discrete by definition.

□

Lemma 3.10.7. *A discrete Cartesian restriction category $\mathbb{D}$ is precisely a graphic Cartesian restriction category.*

Proof. The requirement is that for all $f \in \mathbb{D}_m$ both $\langle f, 1\rangle$ and $\langle \overline{f}, 1\rangle$ have partial inverses. For $\langle f, 1\rangle$ the inverse is $\overline{(1 \times f)\Delta^{(-1)}}\pi_1$.

To show this, calculate the two compositions. First,

$$\langle f, 1\rangle \overline{1 \times f\Delta^{(-1)}}\pi_1 = \overline{\langle f, f\rangle\Delta^{(-1)}}\langle f, 1\rangle\pi_1 = \overline{f\Delta\Delta^{(-1)}}\langle f, 1\rangle\pi_1 = \overline{f}\langle f, 1\rangle\pi_1 = \overline{f}.$$

The other direction is:

$$\begin{aligned}
\overline{(1 \times f)\Delta^{(-1)}}\pi_1\langle f, 1\rangle &= \langle \overline{(1 \times f)\Delta^{(-1)}}\pi_1 f, \overline{(1 \times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle \overline{(1 \times f)\Delta^{(-1)}}(1 \times f)\pi_1, \overline{(1 \times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle (1 \times f)\overline{\Delta^{(-1)}}\pi_1, \overline{(1 \times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle (1 \times f)\overline{\Delta^{(-1)}}\pi_0, \overline{(1 \times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle \overline{(1 \times f)\Delta^{(-1)}}(1 \times f)\pi_0, \overline{(1 \times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \langle \overline{(1 \times f)\Delta^{(-1)}}\pi_0, \overline{(1 \times f)\Delta^{(-1)}}\pi_1\rangle \\
&= \overline{(1 \times f)\Delta^{(-1)}}\langle \pi_0, \pi_1\rangle \\
&= \overline{(1 \times f)\Delta^{(-1)}}.
\end{aligned}$$

The above follows in a discrete Cartesian restriction category, as we have

$$\overline{\Delta^{(-1)}}\pi_1 = \Delta^{(-1)}\Delta\pi_1 = \Delta^{(-1)} = \Delta^{(-1)}\Delta\pi_0 = \overline{\Delta^{(-1)}}\pi_0.$$

For $\langle \overline{f}, 1\rangle$, the inverse is $\overline{(1 \times \overline{f})\Delta^{(-1)}}\pi_1$. Similarly to above,

$$\langle \overline{f}, 1\rangle \overline{1 \times \overline{f}\Delta^{(-1)}}\pi_1 = \overline{\langle \overline{f}, \overline{f}\rangle\Delta^{(-1)}}\langle \overline{f}, 1\rangle\pi_1 = \overline{\overline{f}\Delta\Delta^{(-1)}}\langle \overline{f}, 1\rangle\pi_1 = \overline{\overline{f}}\langle \overline{f}, 1\rangle\pi_1 = \overline{f}.$$

The other direction follows the same pattern as for $\langle f, 1\rangle$. $\square$

To conclude this section, we give a few examples of Cartesian restriction categories, of both the discrete and non-discrete variety.

Example 3.10.8 (Terminal object is discrete)**.** In any Cartesian restriction category, the terminal object, 1, is discrete as $1 \times 1 \cong 1$.

Example 3.10.9 (Semi-lattice is discrete)**.** As the product is the meet of the semi-lattice and $A \wedge A = A$, we have $\Delta = 1$ and, therefore, is always invertible. Note that a total discrete Cartesian restriction category must be a semi-lattice. Also, we see that any Cartesian restriction category which is a restriction preorder will also be discrete.

Example 3.10.10 (Non-discrete Cartesian restriction categories)**.** Besides the example of Top_p given at the beginning of this section, the following are not discrete:

(i) Any total non-trivial (i.e., not a semi-lattice) Cartesian category is not discrete.

(ii) $\text{Par}(\mathbb{X}, \mathcal{M})$ is not discrete unless $\Delta : X \to X \times X$ is in $\mathcal{M}$.

(iii) stabLat^{op} is not discrete.

We will give the details for stabLat^{op}. Recall stabLat is the category of meet semi-lattices whose maps preserve the meet, but do not necessarily preserve the top, $\top$. The corestriction of $f : L_1 \to L_2$ is given by $\overline{f} : L_2 \to L_2$, $f(y) = y \wedge f(\top)$. Hence, the total maps are those which preserve the top element.

The restriction product in stabLat^{op} is given by the coproduct of the semi-lattices, which is also the product as the category has biproducts. In stabLat, we have the maps

$$in_0 : L_0 \to L_0 \times L_1 \quad \text{and} \quad in_1 : L_2 \to L_1 \times L_2$$

where

$$in_0 : \ell \mapsto (\ell, \top) \quad \text{and } in_1 : m \mapsto (\top, m).$$

Then, $\pi_i = {in_i}^{op}$ are the projections in stabLat^{op}. Again considering stabLat, for $f : L_0 \to L_2$, $g : L_1 \to L_2$, there is the map $[f, g] : L_0 \times L_1 \to L_2$ where $[f, g] : (\ell, m) \mapsto f(\ell) \wedge g(m)$. The product map in stabLat^{op}, $\langle f^{op}, g^{op} \rangle$ is $[f, g]^{op} : L_2 \to L_0 \times L_1$.

Using the standard notation of ∇ for the map $[1,1]$ in $\textsc{stabLat}$, we see $\Delta : L_1 \to L_1 \times L_1$ in $\textsc{stabLat}^{op}$ is ∇^{op}. For this to have an inverse, we need a map $h : L_1 \to L_1 \times L_1$ in $\textsc{stabLat}$ such that $\Delta h^{op} = 1_{L_1}$, i.e., $h\nabla = 1_{L_1}$ in $\textsc{stabLat}$. In $\textsc{stabLat}$, we may write h as having two components, that is, $h(x) = (h_1(x), h_2(x))$.

As we know that $x \wedge y = \nabla(h(x \wedge y)$, we have that $h(\top) \wedge (x, y) = h(x \wedge y)$. But this gives us $(h_1(\top) \wedge x, h_2(\top) \wedge y) = (h_1(x \wedge y), h_2(x \wedge y))$. Therefore:

$$h_1(x \wedge y) = h_1(\top) \wedge x \qquad \text{and}$$
$$h_2(x \wedge y) = h_2(\top) \wedge y.$$

But then we have

$$x \wedge y = h_1(x \wedge y) \wedge h_2(x \wedge y) = h_1(\top) \wedge h_2(\top) \wedge x \wedge y$$

which means that $h_1(\top) \wedge h_2(\top) = \top$, which gives $h_1(x \wedge y) = x$ and $h_2(x \wedge y) = y$. In any lattice where there is a non top element p, this produces a contradiction as $h_1(p) = h_1(p \wedge \top) = p$ and $h_1(p) = h_1(\top \wedge p) = \top$ which can only happen when $\top = p$. This shows that the only discrete object in $\textsc{stabLat}^{op}$ is the one element semi-lattice, $\{\top\}$.

Thus, the map h does not exist in general and $\textsc{stabLat}^{op}$ is not discrete.

Chapter 4

Inverse categories and products

This chapter will introduce inverse categories. We first give a few results about inverse categories and then proceed to show an inverse category which has restriction products is a restriction preorder.

Given this fact, the chapter then focuses on adding product-like structures to an inverse category, which we call *inverse products*. These will be defined below in Subsection 4.3.1. Inverse products are given by a natural structure on a tensor product which includes a diagonal but lacks projections. The diagonal map is required to give a natural Frobenius structure on each object.

4.1 Inverse categories

Definition 4.1.1. A restriction category in which every map is a partial isomorphism is called an *inverse category*.

Lemma 4.1.2. *In an inverse category, all idempotents are restriction idempotents.*

Proof. Given an idempotent e,

$$\overline{e} = ee^{(-1)} = eee^{(-1)} = e\overline{e} = \overline{\overline{e}e} = \overline{e}e = e.$$

$\square$

Lemma 4.1.3. *An inverse category $\mathbb{X}$ is a range category (Definition 3.6.1), where $\hat{f} = f^{(-1)}f = \overline{f^{(-1)}}$.*

Proof.

[RR.1] $\overline{\hat{f}} = \overline{\overline{f^{(-1)}}} = \overline{f^{(-1)}} = \hat{f}$;

[RR.2] $f\hat{f} = f\overline{f^{(-1)}} = ff^{(-1)}f = \overline{f}f = f$;

[RR.3] $\widehat{f\overline{g}} = \overline{(f\overline{g})^{(-1)}} = \overline{\overline{g}^{(-1)}f^{(-1)}} = \overline{\overline{g}f^{(-1)}} = \overline{g}\,\overline{f^{(-1)}} = \overline{f^{(-1)}}\,\overline{g} = \hat{f}\overline{g}$;

[RR.4] $\widehat{\hat{f}g} = \overline{(\overline{f^{(-1)}}g)^{(-1)}} = \overline{g^{(-1)}\overline{f^{(-1)}}^{(-1)}} = \overline{g^{(-1)}\overline{\overline{f^{(-1)}}}} = \overline{g^{(-1)}f^{(-1)}} = \overline{(fg)^{(-1)}} = \widehat{fg}$.

□

For an inverse category $\mathbb{X}$, as $()^{(-1)}$ is an involution, the range $(\hat{\_})$ is in fact more than just a range, it is a restriction in $\mathbb{X}^{op}$.

The property of being an inverse category is preserved by splitting.

Lemma 4.1.4. *When $\mathbb{X}$ is an inverse category, $K_E(X)$ is an inverse category.*

Proof. First recall that in the split category, for $f : (A, e_1) \to (B, e_2)$, we have $f = e_1 f e_2 = e_1 f = f e_2$ and by Proposition 3.7.1 the restriction of $f : (A, e_1) \to (B, e_2)$ is $e_1\overline{f}$. Then, the inverse of $f : (A, e_1) \to (B, e_2)$ in $\mathrm{K}_E(\mathbb{X})$ is $f^{(-1)}$ as

$$f^{(-1)} = e_1 f {e_2}^{(-1)} = {e_2}^{(-1)} f^{(-1)} {e_1}^{(-1)} = e_2 f^{(-1)} e_1.$$

Note the last equality uses Lemma 4.1.2. Additionally, we have

$$ff^{(-1)} = e_1 ff^{(-1)} = e_1\overline{f} \text{ and } f^{(-1)}f = e_2 f^{(-1)} f = e_2\overline{f^{(-1)}}.$$

□

Example 4.1.5 (Pinj is an inverse category)**.** For any map f, $f^{(-1)} = \{(y, x)|(x, y) \in f\}$. Note that $f^{(-1)}$ is a map in Pinj due to the two dual conditions on maps as given in Example 2.6.5.

Example 4.1.6 (Par is not an inverse category)**.** Par, while it is a restriction category, is not an inverse category. For example, let $A = \{1, 2\}$, $B = \{1\}$ and $f = \{(1, 1), (2, 1)\}$ in Par. The restriction of f is $\overline{f} = \{(1, 1), (2, 2)\} = 1_A$. There is no partial function $g : B \to A$ such that $fg = 1_A$.

Example 4.1.7. Generally, let $\mathbb{R}$ be a restriction category, and $\mathbf{INV}(\mathbb{R})$ the subcategory of $\mathbb{R}$ having the same objects as $\mathbb{R}$ but only the partial isomorphisms as maps. Then, $\mathbf{INV}(\mathbb{R})$ is an inverse category.

Example 4.1.8. A groupoid, which is a category in which every map is an isomorphism, is an inverse category. As all maps in the groupoid are total, the partial isomorphisms are all isomorphisms.

As well, we note that for $\mathbb{X}$ any inverse category, $Total(\mathbb{X})$ is a groupoid.

Example 4.1.9. Given a category $\mathbb{P}$, create a partial map category as in Section 3.8, where the stable system of monics, $\mathcal{M}$, contains all of the isomorphisms in $\mathbb{P}$. Then the partial isomorphisms are the maps of the form $\begin{tikzcd}[sep=small] & A' \arrow[dl, "m"'] \arrow[dr, "m'"] & \\ A & & B \end{tikzcd}$, where $m' \in \mathcal{M}$. Its inverse is $\begin{tikzcd}[sep=small] & A' \arrow[dl, "m'"'] \arrow[dr, "m"] & \\ B & & A \end{tikzcd}$. The composition is $\begin{tikzcd}[sep=small] & A \arrow[dl, "m^{-1}m"'] \arrow[dr, "m^{-1}m"] & \\ A & & A \end{tikzcd}$, which is the identity when m is an isomorphism. The partial isomorphisms will have either m or m' in $\mathcal{M}$, but may not be an isomorphism. Taking just the partial isomorphisms gives us an inverse category.

Example 4.1.10. A semigroup [57] is a set with an associative binary operation. A semigroup need not have an identity. An inverse semigroup is a semigroup where each element x has an associated element x^* such that:

$$x = xx^*x \quad \text{and} \quad x^* = x^*xx^*.$$

In much the same way that a group may be viewed as a one object category, an inverse semigroup is a one object inverse category, where the elements are the maps. For any map x, we have $x^{(-1)} = x^*$.

Example 4.1.11 (Equivalence relations)**.** Equivalence relations of finite sets are representable as a pair of surjective functions onto another set. That is,

$$A \overset{E_{[f,g]}}{\rightsquigarrow} B := \begin{tikzcd} & C & \\ A \arrow[ur, two heads, "f"] & & B \arrow[ul, two heads, "g"'] \end{tikzcd} \iff \begin{tikzcd} C \\ A + B. \arrow[u, two heads, "{[f,g]}"] \end{tikzcd}$$

Define the category EQR with objects being finite sets and maps being equivalence classes of the relations $E_{[f,g]} : A \rightsquigarrow B$. The equivalence classes are given by the following:

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] \arrow[dr, "f'"'] & & B \arrow[ul, "g"'] \arrow[dl, "g'"] \\ & C' \arrow[uu, leftrightarrow, "b"'] & \end{tikzcd} \iff E_{[f,g]} \cong E_{[f',g']}.$$

That is, whenever there is a bijection $b : C \leftrightarrow C'$ such that the above diagram commutes, then the relations are equivalent.

The identity is given by $E_{[1,1]}$ and composition is by pushout. The restriction of a relation is given by restricting it to the first element only, i.e., $\overline{E_{[f,g]}} = E_{[f,f]}$. This is an inverse category with the partial inverse of a relation given by swapping the maps f, g. That is, ${E_{[f,g]}}^{(-1)} = E_{[g,f]}$. The total maps are those where f is a bijection.

Example 4.1.12 (Partial isometries)**.** In a finite dimensional Hilbert space, endomaps which are self-adjoint and idempotent are called *projectors*. Endomaps in FDHILB also form a lattice where the meet of two maps, $f \wedge g$ is given by $\lim_{n\to\infty}(fg)^n$.

If we are given a map $f : H_1 \to H_2$ between finite dimensional Hilbert spaces, then when $ff^* : H_1 \to H_1$ is a projector, then f is called a *partial isometry* [36]. In general, partial isometries are not closed under composition. However, if composition of $f : H_1 \to H_2$ and $g : H_2 \to H_3$ is defined as:

$$fg = f(f^*f \wedge gg^*)g,$$

then Hilbert spaces with partial isometries form an inverse category. This is shown in Theorem 9.5.4 of [36]. The construction of the composition for this inverse category is an application of a general construction for inverse semigroups given by Lawson [46].

4.2 Inverse categories with restriction products

We start by showing that an inverse category with restriction products is a restriction preorder and thus, is a very restrictive notion.

Proposition 4.2.1. *Given an inverse category $\mathbb{X}$, if it has restriction products, it is a restriction preorder as in Definition 3.3.1. That is,*

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \implies f \smile g.$$

Proof. Notice that for $\pi_1 : A \times A \to A$, ${\pi_1}^{(-1)} = \Delta \pi_1 {\pi_1}^{(-1)} = \Delta \overline{\pi_1} = \Delta$. This gives $\overline{{\pi_1}^{(-1)}} = 1$ and therefore π_1 (and similarly, π_0) is an isomorphism.

Starting with the product map $\langle f, g \rangle$,

$$\dfrac{\dfrac{\dfrac{\dfrac{\langle f, g\rangle = \langle f, g\rangle}{\langle f, g\rangle \pi_1 {\pi_1}^{(-1)} = \langle f, g\rangle \pi_0 {\pi_0}^{(-1)}}}{\overline{f} g {\pi_1}^{(-1)} = \overline{g} f {\pi_0}^{(-1)}}}{\overline{f} g \Delta = \overline{g} f \Delta}}{\overline{f} g = \overline{g} f}$$

which shows that f and g are compatible. □

Corollary 4.2.2. $\mathbb{X}$ *is an inverse category with restriction products iff Total(K($\mathbb{X}$)) is a meet preorder.*

Proof. Total($\mathbb{X}$), the subcategory of total maps on $\mathbb{X}$, has products and therefore every pair of parallel maps is compatible. As total compatible maps are equal, there is at most one map between any two objects. Hence, Total($\mathbb{X}$) is a preorder with the meet being the product.

By Lemma 4.1.4 and Lemma 4.1.2, K($\mathbb{X}$), the split of $\mathbb{X}$ over all idempotents, is an inverse category.

Similarly, from [21] and [23], Total(K($\mathbb{X}$)) has products and is therefore also a meet preorder. This shows the "only if" side of the corollary.

For the other direction, if Total(K($\mathbb{X}$)) is a meet preorder, then the product is the meet of the maps and the terminal object is the supremum of all maps. □

Corollary 4.2.3. *Every inverse category with restriction products is a full subcategory of a partial map category of a meet semi-lattice.*

4.3 Inverse products

4.3.1 Inverse product definition

Definition 4.3.1. An *inverse product* on an inverse category $\mathbb{X}$ is given by a symmetric tensor product, based on a restriction bi-functor, $\_ \otimes \_ : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$. Recall the structural maps of the tensor are the following natural isomorphisms:

$$
\begin{array}{ll}
1 : \mathbf{1} \to \mathbb{X} & \\
u^l_\otimes : 1 \otimes A \xrightarrow{\cong} A & u^r_\otimes : A \otimes 1 \xrightarrow{\cong} A \\
a_\otimes : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C) & c_\otimes : A \otimes B \xrightarrow{\cong} B \otimes A.
\end{array}
$$

The tensor makes $\mathbb{X}$ a symmetric monoidal category as per Definition 2.8.1 and there is a natural diagonal map Δ, which is canonical. If an inverse category has inverse products, it is called a *discrete inverse category*.

The diagonal map $\Delta_A : A \to A \otimes A$ must be total and create a cosemigroup. It must satisfy Diagrams (4.1), (4.2), (4.3) and (4.4) below.

$$
\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & A \otimes A \arrow[d, "c_\otimes"] \\
& A \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[r, "\Delta^{(-1)}"] \arrow[d, "c_\otimes"] & A \\
A \otimes A \arrow[ur, "\Delta^{(-1)}"'] &
\end{tikzcd}
\tag{4.1}
$$

Cocommutative and Commutative;

$$
\begin{tikzcd}
A \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] & & A \otimes A \arrow[d, "1\otimes\Delta"] \\
A \otimes A \arrow[dr, "\Delta\otimes 1"'] & & A \otimes (A \otimes A) \\
& (A \otimes A) \otimes A \arrow[ur, "a_\otimes"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes (A \otimes A) \arrow[rr, "1\otimes\Delta^{(-1)}"] \arrow[d, "{a_\otimes}^{(-1)}"'] & & A \otimes A \arrow[d, "\Delta^{(-1)}"] \\
(A \otimes A) \otimes A \arrow[dr, "\Delta^{(-1)}\otimes 1"'] & & A \\
& A \otimes A \arrow[ur, "\Delta^{(-1)}"'] &
\end{tikzcd}
\tag{4.2}
$$

Coassociative and Associative;

$$\begin{tikzcd}
A \otimes A \arrow[rr, "(\Delta\otimes 1)a_{\otimes}"] \arrow[dd, "(1\otimes\Delta){a_{\otimes}}^{(-1)}"'] \arrow[dr, "\Delta^{(-1)}"] & & A \otimes (A \otimes A) \arrow[dd, "1\otimes\Delta^{(-1)}"] \\
& A \arrow[dr, "\Delta"] & \\
(A \otimes A) \otimes A \arrow[rr, "\Delta^{(-1)}\otimes 1"'] & & A \otimes A
\end{tikzcd} \tag{4.3}$$

Δ is Semi-Frobenius.

If we define the map:

$$ex_{\otimes} = a_{\otimes}(1 \otimes {a_{\otimes}}^{(-1)})(1 \otimes (c_{\otimes} \otimes 1))(1 \otimes a_{\otimes}){a_{\otimes}}^{(-1)}) : (A \otimes B) \otimes (C \otimes D) \to (A \otimes C) \otimes (B \otimes D)$$

then we have:

$$\begin{tikzcd}
A \otimes B \arrow[rr, "\Delta\otimes\Delta"] \arrow[dr, "\Delta"'] & & (A \otimes A) \otimes (B \otimes B) \arrow[dl, "ex_{\otimes}"] \\
& (A \otimes B) \otimes (A \otimes B) &
\end{tikzcd} \tag{4.4}$$

Δ is canonical.

Thus, Δ is a cocommutative, coassociative map which together with $\Delta^{(-1)}$ forms a special semi-Frobenius algebra. We use the prefix "semi-" since we are not requiring the unit laws of a Frobenius algebra as presented in Definition 10.1.10.

Note also, cocommutativity implies commutativity, i.e., that $c_{\otimes}\Delta^{(-1)} = \Delta^{(-1)}$. One can see this as:

$$\Delta(c_{\otimes}\Delta^{(-1)}) = (\Delta c_{\otimes})\Delta^{(-1)} = \Delta\Delta^{(-1)} = \overline{\Delta} \text{ and}$$
$$(c_{\otimes}\Delta^{(-1)})\Delta = (c_{\otimes}\Delta^{(-1)})(\Delta c_{\otimes}) = \overline{c_{\otimes}\Delta^{(-1)}}.$$

This means that both $\Delta^{(-1)}$ and $c_{\otimes}\Delta^{(-1)}$ are partial inverses for Δ and are therefore equal.

Similarly, coassociativity implies associativity, in that

$$(1 \otimes \Delta^{(-1)})\Delta^{(-1)} = (\Delta(1 \otimes \Delta))^{(-1)} = {a_{\otimes}}^{(-1)}(\Delta^{(-1)} \otimes 1)\Delta^{(-1)}.$$

This can be seen from the following calculation:

$$\begin{aligned}\Delta(1\otimes\Delta){a_\otimes}^{(-1)}(\Delta^{(-1)}\otimes 1)\Delta^{(-1)} &= \Delta(\Delta\otimes 1)a_\otimes {a_\otimes}^{(-1)}(\Delta^{(-1)}\otimes 1)\Delta^{(-1)}\\ &= \Delta(\Delta\otimes 1)(\Delta^{(-1)}\otimes 1)\Delta^{(-1)}\\ &= \Delta 1\Delta^{(-1)} = 1.\end{aligned}$$

Example 4.3.2 (PINJ is a discrete inverse category)**.** In the inverse category PINJ (see Examples 2.6.5 and 4.1.5), we saw in Example 2.8.3 that the Cartesian product is a symmetric tensor.

Define $\Delta_A = \{(a,(a,a))|a\in A\}$. Then PINJ is a discrete inverse category with the inverse product of $\otimes$. The required properties of cocommutativity, coassociativity and exchange are immediately obvious. To show the Frobenius rule for Δ, first note that $\Delta^{(-1)}$ is defined only on the elements of $A\otimes A$ which agree in the first and second coordinate. We show the upper triangle of the Frobenius diagram in detail. Equation (4.5) shows the result of applying $\Delta^{(-1)}$ followed by Δ.

$$\Delta(\Delta^{(-1)}(A\otimes A)) = \Delta(\{a|(a,a)\in A\otimes A\}) = \{(a,a)|(a,a)\in A\otimes A\}. \tag{4.5}$$

Applying $(\Delta\otimes 1)a_\otimes$ to $A\otimes A$ is shown in Equation (4.6).

$$a_\otimes(\Delta\otimes 1(A\otimes A)) = a_\otimes(\{((a,a),a')|(a,a')\in A\otimes A\} = \{(a,(a,a'))|(a,a')\in A\otimes A\}. \tag{4.6}$$

Finally, applying $1\otimes\Delta^{(-1)}$ to the result of Equation (4.6) gives us Equation (4.7).

$$(1\otimes\Delta^{(-1)})(\{(a,(a,a'))|(a,a')\in A\otimes A\} = \{(a,a)|(a,a)\in A\otimes A\}. \tag{4.7}$$

Thus, we have $\Delta^{(-1)}\Delta = (\Delta\otimes 1)a_\otimes(1\otimes\Delta^{(-1)})$ and the Frobenius condition is satisfied.

Example 4.3.3 (TOP$_p$ does not give a discrete inverse category)**.** Recalling TOP$_p$ from Example 3.1.9, we know that the partial isomorphisms of TOP$_p$ form an inverse category — **INV**(TOP$_p$). Additionally, TOP$_p$ has a product, given by the standard Cartesian product. However, as noted in Example 3.10.2, TOP$_p$ is not a discrete Cartesian restriction category.

The product of $\textsc{Top}_p$ does work as a tensor in $\mathbf{INV}(\textsc{Top}_p)$, but Δ is not a map in $\mathbf{INV}(\textsc{Top})$ and hence we do not have a discrete inverse category.

Inverse products are extra structure on an inverse category, rather than a property. An example to demonstrate this is given next.

Example 4.3.4 (Inverse products are additional structure)**.**

Any discrete category (i.e., a category with only the identity arrows) is a trivial inverse category. To create an inverse product on a discrete category, add a commutative, associative, idempotent multiplication, with a unit.

Let $\mathbb{D}$ be the discrete category with four objects a, b, c and d. Then, define two different inverse product tensors, $\otimes$ and $\odot$, with d the unit of each as shown in Table 4.1.

$\otimes$	a	b	c	d
a	a	a	a	a
b	a	b	**b**	b
c	a	**b**	c	c
d	a	b	c	d

$\odot$	a	b	c	d
a	a	a	a	a
b	a	b	**a**	b
c	a	**a**	c	c
d	a	b	c	d

Table 4.1: Two different inverse products on the same category.

As $\mathbb{D}$ is discrete, Δ is forced to be the identity. One can check easily that each of the conditions for being an inverse product are satisfied by $\otimes$ and by $\odot$ with the trivial diagonal.

4.3.2 Diagrammatic language

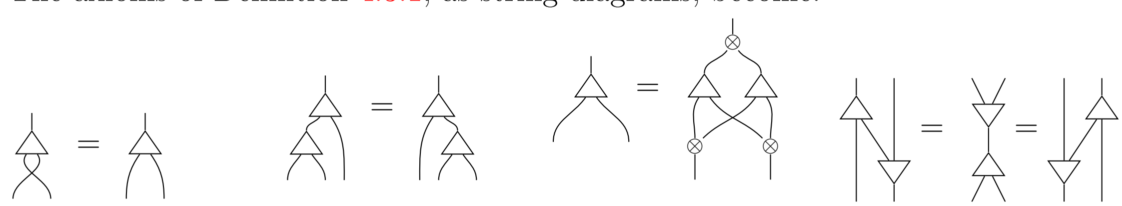
While it is certainly possible to prove results about inverse products using direct algebraic manipulation, it is much more understandable to use circuits or string diagrams. See [65] for a comparison of various graphical languages for monoidal categories. As shown in [41], diagrammatic reasoning is equivalent to reasoning algebraically for symmetric monoidal categories.

In the diagrams, we will use the following representations:

- Δ will be represented by an upward pointing triangle: $\triangle$.
- $\Delta^{(-1)}$ by a downward triangle: $\triangledown$.
- Maps by a rectangle with the name of the map inside: $\boxed{f}$.
- Use of the tensor: $\otimes$.
- Unit introduction (often referred to as an η map): $\circ$.
- Unit removal (often referred to as an ϵ map): $\bullet$.

String diagrams in this thesis are to be read from top to bottom. Note that unit introduction and unit removal maps are not required in a discrete inverse category. However, when they are present they will be represented diagrammatically as above.

The axioms of Definition 4.3.1, as string diagrams, become:



Cocommutativity, coassociativity, exchange, Frobenius.

The diagram for commutativity is obtained by flipping the diagram of cocommutativity vertically. Similarly, the diagram for associativity is obtained by flipping the diagram for coassociativity vertically.

4.3.3 Properties of discrete inverse categories

We now present some properties of discrete inverse categories.

Lemma 4.3.5. *In a discrete inverse category $\mathbb{X}$ with the inverse product $\otimes$ and Δ, where $e = \overline{e}$ is a restriction idempotent and f, g, h are arrows in $\mathbb{X}$, the following are true:*

(i) $e = \Delta(e \otimes 1)\Delta^{(-1)}$;

(ii) $e\Delta(f \otimes g) = \Delta(ef \otimes g)$ *(and* $= \Delta(f \otimes eg)$ *and* $= \Delta(ef \otimes eg)$*);*

(iii) $(f \otimes ge)\Delta^{(-1)} = (f \otimes g)\Delta^{(-1)}e$ *(and* $= (fe \otimes g)\Delta^{(-1)}$ *and* $= (fe \otimes ge)\Delta^{(-1)}$*);*

(iv) $\overline{\Delta(f \otimes g)\Delta^{(-1)}} = \Delta(1 \otimes g f^{(-1)})\Delta^{(-1)}$*;*

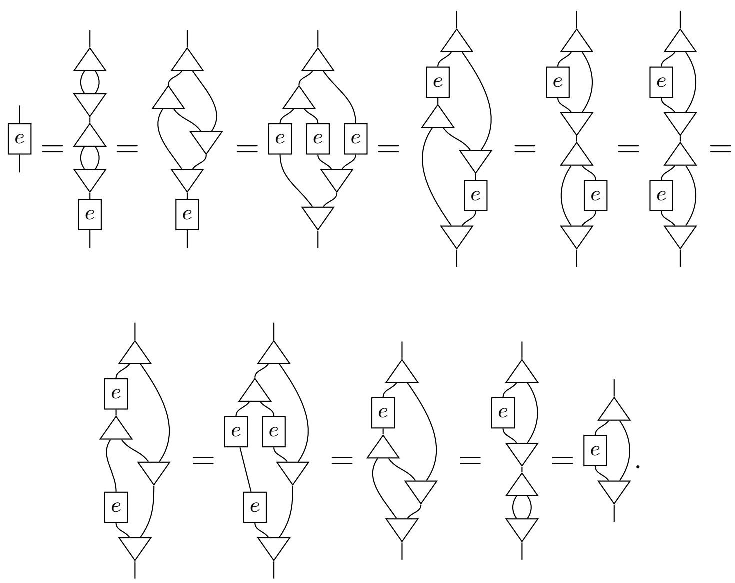
(v) If $\Delta(h \otimes g)\Delta^{(-1)} = \overline{\Delta(h \otimes g)\Delta^{(-1)}}$ *then* $(\Delta(h \otimes g)\Delta^{(-1)})h = \Delta(h \otimes g)\Delta^{(-1)}$*;*

(vi) $\Delta(f \otimes 1) = \Delta(g \otimes 1) \implies f = g$*;*

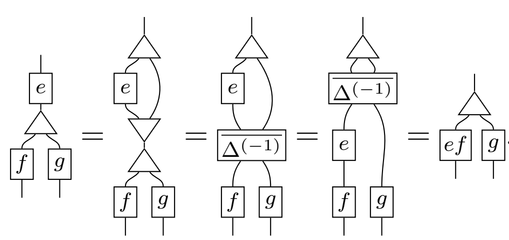
(vii) $(f \otimes 1) = (g \otimes 1) \implies f = g$*.*

Proof.

(i)

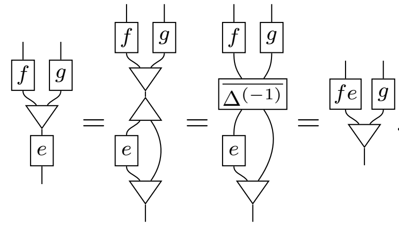


(ii) This equality uses the previous equality, the commutativity of restriction idempotents ([**R.2**]) and the identity $\Delta\overline{\Delta^{(-1)}} = \Delta$.



The second equality ($e\Delta(f \otimes g) = \Delta(f \otimes eg)$) follows by cocommutativity. The third equality, ($e\Delta(f \otimes g) = \Delta(ef \otimes eg)$) follows by naturality of Δ.

(iii) As in (ii), details are only given for the first equality. This proof is obtained by reversing the diagrams of (ii).

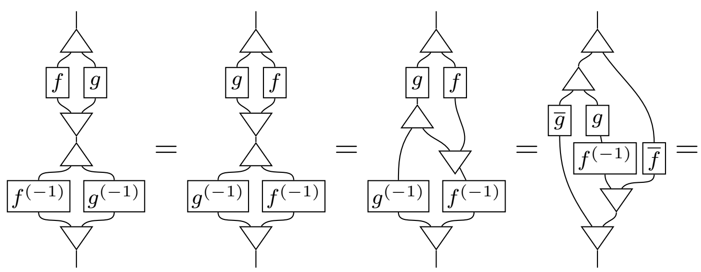


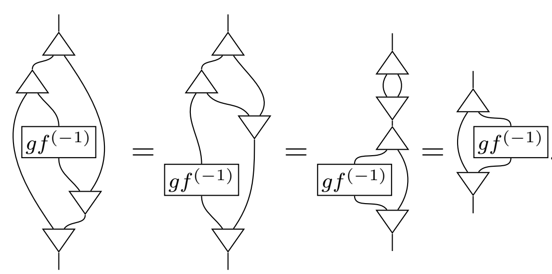
The other equalities follow for the same reasons as in (ii).

(iv) Here, we start by using the fact that all maps have a partial inverse, therefore we have:

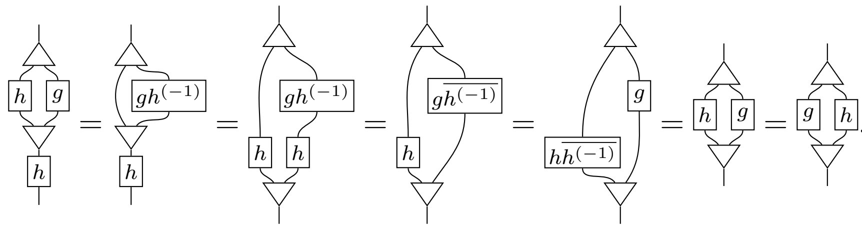
$$\overline{\Delta(f \otimes g)\Delta^{(-1)}} = \Delta(f \otimes g)\Delta^{(-1)}\Delta(f^{(-1)} \otimes g^{(-1)})\Delta^{(-1)}.$$

Now, we proceed with showing the rest of the equality via diagrams.

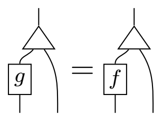


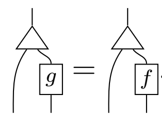


(v) Beginning with the assumption that $\Delta(h \otimes g)\Delta^{(-1)}$ equals its restriction and by item (iv), we have:

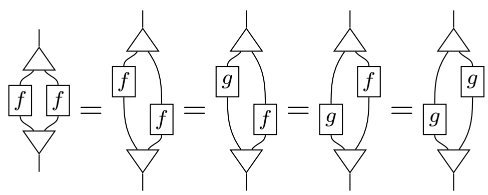


(vi) Our assumption is that:



and by cocommutativity,


Hence,



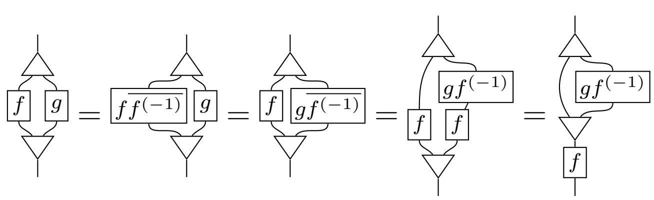
$= g.$

(vii) Use the same diagrammatic argument as in item (vi).

$\square$

Proposition 4.3.6. *A discrete inverse category has meets, where* $f \cap g = \Delta(f \otimes g)\Delta^{(-1)}$.

Proof. $f \cap g \leq f$:

$f \cap g =$


$= \overline{f \cap g} f.$

$f \cap f = f$:

$$f \cap f = \Delta(f \otimes f)\Delta^{(-1)} = f\Delta\Delta^{(-1)} = f.$$

$h(f \cap g) = hf \cap hg$:

$$\begin{aligned}
h(f \cap g) &= h\Delta(f \otimes g)\Delta^{(-1)} && \text{Definition of } \cap \\
&= \Delta(h \otimes h)(f \otimes g)\Delta^{(-1)} && \Delta \text{ natural} \\
&= \Delta(hf \otimes hg)\Delta^{(-1)} && \text{compose maps} \\
&= hf \cap hg && \text{Definition of } \cap .
\end{aligned}$$

□

4.3.4 The inverse subcategory of a discrete Cartesian restriction category

Given a discrete Cartesian restriction category, one can pick out the maps which are partial isomorphisms. Using results from Subsection 4.3.3 and from Section 3.10, we will show that these maps form a subcategory which is a discrete inverse category.

Proposition 4.3.7. *Given $\mathbb{X}$ is a discrete Cartesian restriction category, the partial isomorphisms of $\mathbb{X}$, together with the objects of $\mathbb{X}$ form a sub-restriction category which is a discrete inverse category. For the restriction category $\mathbb{X}$, we denote this subcategory by* $\mathbf{INV}(\mathbb{X})$.

Proof. As shown in Lemma 3.5.2, partial isomorphisms are closed under composition. The identity maps are in $\mathbf{INV}(\mathbb{X})$ and restrictions of partial isomorphisms are also partial isomorphisms.

The product on the discrete Cartesian restriction category $\mathbb{X}$ becomes the tensor product of the restriction category $\mathbf{INV}(\mathbb{X})$. Table 4.2 shows how each of the elements of the tensor are defined. Note that the last definition makes explicit use of the fact we are in a discrete Cartesian restriction category and hence the Δ of $\mathbb{X}$ possesses a partial inverse.

The monoid coherence diagrams follow directly from the characteristics of the product in $\mathbb{X}$. Similarly, Δ is total as it is total in $\mathbb{X}$. It remains to show cocommutativity, coassociativity and the Frobenius condition.

$\mathbb{X}$	$\mathbf{INV}(\mathbb{X})$	Inverse map
$A\times B$	$A\otimes B$	
$\top$	1	
$\pi_1{:}\top\times A\to A$	$u_\otimes^l{:}1\otimes A\to A$	$\langle !,1\rangle$
$\pi_0{:}A\times\top\to A$	$u_\otimes^r{:}A\otimes 1\to A$	$\langle 1,!\rangle$
$a_\mathbb{X}=\langle\pi_0\pi_0,\langle\pi_0\pi_1,\pi_1\rangle\rangle{:}(A\times B)\times C\to A\times(B\times C)$	$a_\otimes{:}(A\otimes B)\otimes C\to A\otimes(B\otimes C)$	$\langle\langle\pi_0,\pi_1\pi_0\rangle,\pi_1\pi_1\rangle$
$c_\mathbb{X}=\langle\pi_1,\pi_0\rangle{:}A\times B\to B\times A$	$c_\otimes{:}A\otimes B\to B\otimes A$	$\langle\pi_1,\pi_0\rangle$
$\Delta_\mathbb{X}{:}A\to A\times A$	$\Delta{:}A\to A\otimes A$	${\Delta_\mathbb{X}}^{(-1)}$

Table 4.2: Structural maps for the tensor in $\mathbf{INV}(\mathbb{X})$

Cocommutativity requires $\Delta c_\otimes = c_\otimes$. We have

$$\Delta_\mathbb{X}\langle\pi_1,\pi_0\rangle = \langle\Delta_\mathbb{X}\pi_1,\Delta_\mathbb{X}\pi_0\rangle = \langle 1,1\rangle = \Delta_\mathbb{X},$$

giving us the required cocommutativity.

Coassociativity requires $\Delta(1\otimes\Delta) = \Delta(\Delta\otimes 1)a_\otimes$. Expressing this in $\mathbb{X}$, it is the requirement that

$$\Delta_\mathbb{X}(1\times\Delta_\mathbb{X}) = \Delta_\mathbb{X}(\Delta_\mathbb{X}\times 1)a_\mathbb{X}.$$

Recalling that $f\times g\pi_0 = \pi_0 f$ and $f\times g\pi_1 = \pi_1 g$, we have:

$$\begin{aligned}\Delta_\mathbb{X}(\Delta_\mathbb{X}\times 1)a_\mathbb{X} &= \Delta_\mathbb{X}(\Delta_\mathbb{X}\times 1)\langle\pi_0\pi_0,\langle\pi_0\pi_1,\pi_1\rangle\rangle\\ &= \langle\Delta_\mathbb{X}(\Delta_\mathbb{X}\times 1)\pi_0\pi_0,\langle\Delta_\mathbb{X}(\Delta_\mathbb{X}\times 1)\pi_0\pi_1,\Delta_\mathbb{X}(\Delta_\mathbb{X}\times 1)\pi_1\rangle\rangle\\ &= \langle\Delta_\mathbb{X}\pi_0\Delta_\mathbb{X}\pi_0,\langle\Delta_\mathbb{X}\pi_0\Delta_\mathbb{X}\pi_1,\Delta_\mathbb{X}\pi_1 1\rangle\rangle\\ &= \langle 1,\langle 1,1\rangle\rangle = \Delta_\mathbb{X}(1\times\Delta_\mathbb{X})\end{aligned}$$

and shows that we have coassociativity.

The semi-Frobenius requirement is two-fold:

$$\Delta^{(-1)}\Delta = (\Delta\otimes 1)a_\otimes(1\otimes\Delta^{(-1)}), \tag{4.8}$$

$$\Delta^{(-1)}\Delta = (1\otimes\Delta){a_\otimes}^{(-1)}(\Delta^{(-1)}\otimes 1). \tag{4.9}$$

In $\mathbb{X}$, these become:

$$\Delta_{\mathbb{X}}{}^{(-1)}\Delta_{\mathbb{X}} = (\Delta_{\mathbb{X}} \times 1)\langle \pi_0\pi_0, \langle \pi_0\pi_1, \pi_1\rangle\rangle(1 \times \Delta_{\mathbb{X}}{}^{(-1)}), \tag{4.10}$$

$$\Delta_{\mathbb{X}}{}^{(-1)}\Delta_{\mathbb{X}} = (1 \times \Delta_{\mathbb{X}})\langle\langle \pi_0, \pi_1\pi_0\rangle, \pi_1\pi_1\rangle(\Delta_{\mathbb{X}}{}^{(-1)} \times 1). \tag{4.11}$$

We will give the details of the proof for Equation (4.10). Proving Equation (4.11) is similar.

Note first that $\Delta(1{\times}!)$ (and $\Delta(!{\times}1)$) is the identity. Second, we see that maps to a product of objects may be expressed as a pairing — i.e. if $f : A \to B \times B$, then $f = \langle f(1{\times}!), f(!{\times}1)\rangle$.

Using this we see that the left hand side of Equation (4.10) may be computed as follows:

$$\Delta_{\mathbb{X}}{}^{(-1)}\Delta_{\mathbb{X}} = \langle \Delta_{\mathbb{X}}{}^{(-1)}\Delta_{\mathbb{X}}(1{\times}!), \Delta_{\mathbb{X}}{}^{(-1)}\Delta_{\mathbb{X}}(! \times 1)\rangle = \langle \Delta_{\mathbb{X}}{}^{(-1)}, \Delta_{\mathbb{X}}{}^{(-1)}\rangle.$$

Similarly, removing the associativity maps, the right hand side of the same equation becomes:

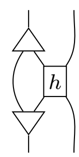
$$\begin{aligned}
(\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)}) &= \langle (\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})(1{\times}!), (\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})(! \times 1)\rangle \\
&= \langle (\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})(1{\times}!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle (\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})(1 \times \Delta_{\mathbb{X}})(1{\times}!{\times}!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle (\Delta_{\mathbb{X}} \times 1)(1 \times \overline{\Delta_{\mathbb{X}}{}^{(-1)}})(1{\times}!{\times}!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle (\Delta_{\mathbb{X}} \times 1)\overline{1 \times \Delta_{\mathbb{X}}{}^{(-1)}}(1{\times}!{\times}!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle \overline{(\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})}(\Delta_{\mathbb{X}} \times 1)(1{\times}!{\times}!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle \overline{(\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})}(1{\times}!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle \overline{(\Delta_{\mathbb{X}} \times 1)(1 \times \Delta_{\mathbb{X}}{}^{(-1)})(! \times 1)}(1{\times}!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle \overline{\Delta_{\mathbb{X}}{}^{(-1)}}(1{\times}!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle \\
&= \langle \Delta_{\mathbb{X}}{}^{(-1)}\Delta_{\mathbb{X}}(1{\times}!), \Delta_{\mathbb{X}}{}^{(-1)}\rangle = \langle \Delta_{\mathbb{X}}{}^{(-1)}, \Delta_{\mathbb{X}}{}^{(-1)}\rangle
\end{aligned}$$

and therefore we see that the first equation for the Frobenius condition is satisfied. Thus, **INV**($\mathbb{X}$) is a discrete inverse category. $\square$

4.4 The "slice" construction on a discrete inverse category

Throughout this section, we will assume $\mathbb{X}$ is a discrete inverse category.

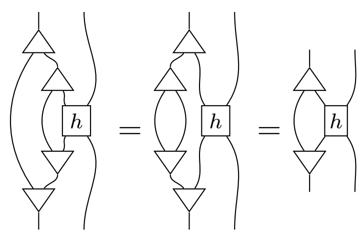
In a discrete inverse category, suppose we are given a map $h : A \otimes B \to A \otimes C$. We define $h_{\nabla}^{\Delta} : A \otimes B \to A \otimes C$ as the composite $(\Delta \otimes 1)(1 \otimes h)(\Delta^{(-1)} \otimes 1)$. We want to consider those maps where $h = h_{\nabla}^{\Delta}$. In our graphical language, this means

$h =$  .

Maps of this form satisfy a variety of closure properties, as shown in the lemmas below.

Lemma 4.4.1. *For any map h in a discrete inverse category, ${h_{\nabla}^{\Delta}}_{\nabla}^{\Delta} = h_{\nabla}^{\Delta}$.*

Proof.

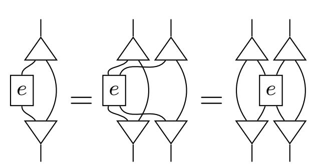
${h_{\nabla}^{\Delta}}_{\nabla}^{\Delta} =$  $= h_{\nabla}^{\Delta}$.

□

The self-closure aspect of maps where $h = h_{\nabla}^{\Delta}$ allows us to show all restriction idempotents have this property:

Corollary 4.4.2. *When $e = \overline{e} : A \otimes Y \to A \otimes Y$, then $e = e_{\nabla}^{\Delta}$.*

Proof. Using Lemma 4.3.5 and the exchange rule, when $e = \overline{e} : A \otimes Y \to A \otimes Y$, we have

$e =$  . (4.12)

But then the same graphical argument as shown in Lemma 4.4.1 applied to the right hand term of Equation (4.12) gives $e = e_{\nabla}^{\Delta}$. □

There are a variety of other closure properties:

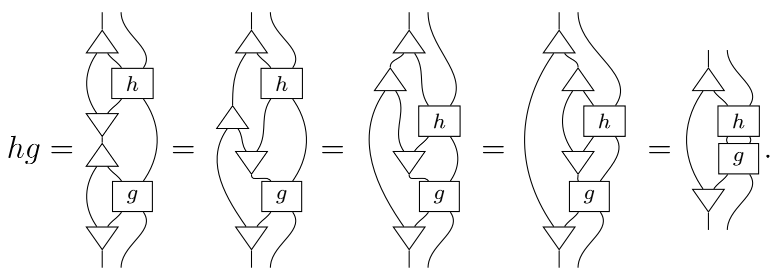
Lemma 4.4.3. *In a discrete inverse category $\mathbb{X}$ with the object A define A^{Δ}_{∇} as the set of maps $h : A \otimes Y \to A \otimes Z$ where $h = h^{\Delta}_{\nabla}$. Then A^{Δ}_{∇} has the following properties:*

- *(i) It is closed under partial inverses;*
- *(ii) it is closed under composition;*
- *(iii) it contains all maps of the form $1 \otimes k$ where $k : Y \to Z$;*
- *(iv) it contains $\Delta : A \otimes B \to A \otimes B \otimes A \otimes B$.*

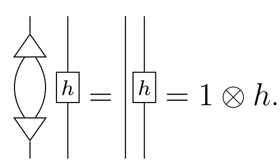
Proof. For (i), if $h = h^{\Delta}_{\nabla}$, then

$$h^{(-1)} = ((\Delta \otimes 1)(1 \otimes h)(\Delta^{(-1)} \otimes 1))^{(-1)} =$$
$$(\Delta^{(-1)} \otimes 1)^{(-1)}(1 \otimes h)^{(-1)}(\Delta \otimes 1)^{(-1)} = (\Delta \otimes 1)(1 \otimes h^{(-1)})(\Delta^{(-1)} \otimes 1).$$

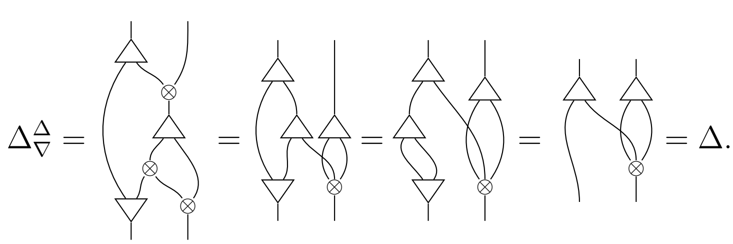
To show (ii), we compose h and g:



For (iii), this follows immediately from



To show (iv), recalling the exchange rule, we have



$\square$

Because of these closure rules, rather than stating $h = h^{\Delta}_{\nabla}$ we may equivalently say $h \in A^{\Delta}_{\nabla}$ when $h = h^{\Delta}_{\nabla} : A \otimes X \to A \otimes Y$.

From Lemma 4.4.3, we see that we will be able to form a category based on maps h such that $h = h^{\Delta}_{\nabla}$. Of course, this category is dependent upon the choice of the object A, hence we will label it $\mathbb{X}[A]$, as it is reminiscent of the simple slice category of Example 2.4.3. We make this precise in the following proposition:

Proposition 4.4.4. *Given a discrete inverse category $\mathbb{X}$, define $\mathbb{X}[A]$ as the restriction category:*

Objects: *The objects of $\mathbb{X}$;*

Maps: *A map $h = h^{\Delta}_{\nabla} : A \otimes X \to A \otimes Y$ in $\mathbb{X}$ is a map from X to Y in $\mathbb{X}[A]$;*

Identity: $1 \otimes 1$ *in $\mathbb{X}$;*

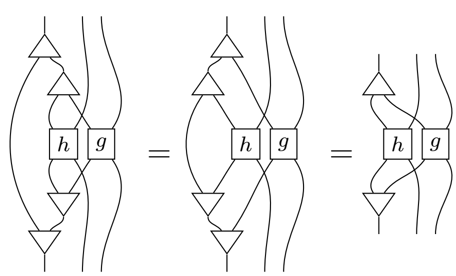
Composition: *Composition in $\mathbb{X}$;*

Restriction: $\overline{h}$ *in $\mathbb{X}[A]$ is given by $\overline{h}$ in $\mathbb{X}$.*

Then, $\mathbb{X}[A]$ is a discrete inverse category.

Proof. Given Lemma 4.4.3, we see immediately that $\mathbb{X}[A]$ is a category. By Corollary 4.4.2, we know $\overline{h} = \overline{h}^{\Delta}_{\nabla}$. We must show that $\mathbb{X}[A]$ has a tensor and a Frobenius Δ.

The tensor of objects $X \otimes Y$ in $\mathbb{X}[A]$ is the element $A \otimes X \otimes Y$ in $\mathbb{X}$. For two maps h, g in $\mathbb{X}[A]$, $h \otimes g$ is given by the $\mathbb{X}$ map $(\Delta \otimes 1 \otimes 1)(1 \otimes c_{\otimes} \otimes 1)(h \otimes g)(1 \otimes c_{\otimes} \otimes 1)(\Delta^{(-1)} \otimes 1 \otimes 1)$. This is a map in $\mathbb{X}[A]$:

$(h \otimes g)^{\Delta}_{\nabla} =$  $= h \otimes g.$

The Δ in $\mathbb{X}[A]$ is given by the map $1 \otimes \Delta$ in $\mathbb{X}$. The various identities required of Δ hold in $\mathbb{X}[A]$ as they hold in $\mathbb{X}$, therefore $\mathbb{X}[A]$ is a discrete inverse category. $\square$

We note that there are functors between $\mathbb{X}$ and $\mathbb{X}[A]$, given by:

$$G : \mathbb{X} \to \mathbb{X}[A]; \quad G : B \mapsto B; \quad G : f \mapsto 1 \otimes f$$

and

$$F : \mathbb{X}[A] \to \mathbb{X}; \quad F : B \mapsto A \otimes B; \quad F : f \mapsto f.$$

However, these do not form an adjoint pair as the relation is

$$\frac{\mathbb{X}(A \otimes X, A \otimes Y)}{\mathbb{X}[A](X, Y)} \quad \text{that is,} \quad \frac{\mathbb{X}(F(X), F(Y))}{\mathbb{X}[A](X, G(Y))}$$

rather than the required

$$\frac{\mathbb{X}(F(X), Y)}{\mathbb{X}[A](X, G(Y))}.$$

Example 4.4.5. In Pinj, note that the slice over the object $\{*\}$ just gives us a category where the maps are in bijective correspondence with the maps of Pinj, as $\{*\}$ is the identity for the inverse product.

4.4.1 The interpretation of the slice construction in resource theory

Although resource theory is generally not in the scope of this thesis, the slice construction of this section may be examined from the viewpoint of resource theory as given in Coecke, Fritz and Spekkens [25]. Coecke, Fritz and Spekkens describe a mathematical theory of resources inspired, in part, by the pragmatic approach exemplified in chemistry — understanding something means being able to make use of it. Specifically in chemistry, this means understanding chemicals as resources. The way to model this mathematically is with a symmetric monoidal category:

Definition 4.4.6 ([25], Definition 2.1)**.** A *resource theory* is a symmetric monoidal category $(\mathbb{R}, \otimes)$, where the objects $\mathbb{R}_o$ represent the actual resources and the morphisms $f : A \to B$ represent the transformation of resource A into resource B which can be done with no cost.

In this definition, composition of maps then represent sequential composition of transformation, the operation $\otimes$ represents parallel composition of transformation and the identity I of $\otimes$ denotes an empty resource.

An important question in such a theory is "Given a resource A, is it possible to transform it to a resource B?" Of course, answering such a question will depend upon the details of the category and the difficulty of answering such a question may vary greatly. This may be reduced to a question of resource convertibility:

Definition 4.4.7 ([25], Definition 4.1)**.** A *theory of resource convertibility* $(R, +, \succeq, 0)$ is a set R with a binary operation $+$, element $0 \in R$ such that when $a \simeq b := a \succeq b$ and $b \succeq a$, then

$$a + (b + c) \simeq (a + b) + c \qquad a + b \simeq b + a \qquad a + 0 \simeq 0 + a$$

and

$$a \succeq b,\ c \succeq d \implies a + c \succeq b + d.$$

When $\mathbb{R}$ is a resource theory, then if we set $R := \mathbb{R}_o$, $+ := \otimes$, $a \succeq b := \mathbb{R}(a, b) \neq \emptyset$ and $0 := I$, then $(R, +, \succeq, 0)$ is a theory of resource convertibility.

From this, it is possible to define what is meant by a catalyst:

Definition 4.4.8 ([25], Definition 4.8)**.** Given R is a theory of resource convertibility, then a resource $a \in R$ is a *catalyst* for $b, c \in R$ when $b \not\succeq c$ but $a + b \succeq a + c$.

Lifting this definition to a resource theory says the object A is a catalyst for objects B, C when there is a map $f : A \otimes B \to A \otimes C$, but no map $g : B \to C$. We might call such maps f, *catalytic transformations* and objects A, *catalytic objects*.

Here, we begin to see the connection to the slice category in a discrete inverse category. As a discrete inverse category is also a symmetric monoidal category, it is therefore also a resource theory and hence may generate a theory of resource convertibility.

However, given $\mathbb{X}$ is a discrete inverse category, the slice category $\mathbb{X}[A]$ does not correspond directly to the catalytic transformations as defined above for two reasons:

(i) For two objects B, C in $\mathbb{X}[A]$ with $f : B \to C$ in $\mathbb{X}[A]$, there is no guarantee that we do *not* have a map $g : B \to C$ in $\mathbb{X}$.

(ii) There may be maps $f : A \oplus B \to A \oplus C$ in $\mathbb{X}$, which are not in $\mathbb{X}[A]$ since they do not preserve A.

Continuing with the inspiration of chemistry, we note that $\mathbb{X}[A]$ may perhaps be considered as an alternate way of describing "the catalytic transformations" for A. In chemistry, one of the requirements is that the catalyst is not changed by the transformation in any way. The theory of resource convertibility from Definition 4.4.7 handles this by considering the resource as an indivisible item. In a resource theory, one way to specify that the catalytic object has not changed is to identify a subcategory of "reversible" transformations and then to require the catalytic transformation be in $\mathbb{X}[A]$.

Chapter 5

Constructing a Cartesian restriction category from a discrete inverse category

The purpose of this chapter is to prove that the category of discrete inverse categories is equivalent to the category of discrete Cartesian restriction categories. We will show how to construct a discrete Cartesian restriction category, $\widetilde{\mathbb{X}}$, from a discrete inverse category, $\mathbb{X}$. The construction, to be called the "Cartesian Completion", is reminiscent of the technique generally used to make a computation reversible — one adds a history to the function to make it reversible: Here we know how to add the history to make it invertible!

5.1 The restriction category $\widetilde{\mathbb{X}}$

We begin by giving the construction of $\widetilde{\mathbb{X}}$.

Definition 5.1.1 (Cartesian Completion)**.** When $\mathbb{X}$ is a discrete inverse category, define $\widetilde{\mathbb{X}}$, the Cartesian Completion of $\mathbb{X}$, as:

Objects: objects as in $\mathbb{X}$;

Maps: A map $(f, C) : A \to B$ in $\widetilde{\mathbb{X}}$ is the equivalence class of the map $f : A \to B \otimes C$ in $\mathbb{X}$ (detailed below in Definition 5.1.2). We have the following relationship between maps in $\widetilde{\mathbb{X}}$ and $\mathbb{X}$:

$$\frac{A \xrightarrow{(f,C)} B \text{ in } \widetilde{\mathbb{X}}}{A \xrightarrow{f} B \otimes C \text{ in } \mathbb{X};}$$

Identity: by

$$\frac{A \xrightarrow{(u_\otimes^{r\,(-1)},1)} A}{A \xrightarrow{u_\otimes^{r\,(-1)}} A \otimes 1;}$$

Composition: given by

$$
\begin{array}{c}
A \xrightarrow{(f,B')} B \xrightarrow{(g,C')} C \\
\hline
A \xrightarrow{f} B \otimes B',\ B \xrightarrow{g} C \otimes C' \\
\hline
A \xrightarrow{f(g\otimes 1)a_\otimes} C \otimes (C' \otimes B') \\
\hline
A \xrightarrow{(f(g\otimes 1)a_\otimes, C'\otimes B')} C.
\end{array}
$$

When considering an $\widetilde{\mathbb{X}}$ map $(f, C) : A \to B$ in $\mathbb{X}$, we occasionally use the notation $f : A \to B_{|C}$ ($\equiv f : A \to B \otimes C$).

5.1.1 Equivalence classes of maps in $\mathbb{X}$

Definition 5.1.2. In a discrete inverse category $\mathbb{X}$, the map f is equivalent to f' in $\mathbb{X}$ when $\overline{f} = \overline{f'}$ in $\mathbb{X}$ and Figure 5.1 is a commutative diagram for some map $h \in B^{\Delta}_{\nabla}$.

$$
\begin{tikzcd}
 & B \otimes C \arrow[dd, dotted, "h"] \\
A \arrow[ur, "f"] \arrow[dr, "f'"'] & \\
 & B \otimes C'
\end{tikzcd}
$$

Figure 5.1: Equivalence diagram for constructing maps in $\widetilde{\mathbb{X}}$.

Notation 5.1.3. When f is equivalent to g as in Definition 5.1.2 via the mediating map h, this is written as:

$$f \overset{h}{\simeq} g.$$

Lemma 5.1.4. *Definition 5.1.2 gives a symmetric, reflexive equivalence class of maps in $\mathbb{X}$.*

Proof.

Reflexivity: Choose h as the identity map.

Symmetry: Suppose $f \overset{h}{\simeq} g$. Then, $\overline{f} = \overline{g}$ and $fh = g$. Applying $h^{(-1)}$, we have

$$gh^{(-1)} = fhh^{(-1)} = f\overline{h} = \overline{fh}f = \overline{g}f = \overline{f}f = f.$$

Thus, $g \overset{h^{(-1)}}{\simeq} f$.

Transitivity: Suppose $f \overset{h}{\simeq} f'$ and $f' \overset{k}{\simeq} f''$, i.e., $fh = f'$ and $f'k = f''$. Therefore, $fhk = f'k = f''$ and by Lemma 4.4.3, we know $hk = (hk)^{\Delta}_{\nabla}$ and therefore we have an equivalence, $f \overset{hk}{\simeq} f''$.

□

Although Definition 5.1.2 above does not require a unique h, we may always find a minimal such h.

Lemma 5.1.5. *Suppose $f \overset{h}{\simeq} g$. Then $f \overset{\hat{f}h\hat{g}}{\simeq} g$ and $\hat{f}h\hat{g} \leq h$.*

Proof. We know that $h = h^{\Delta}_{\nabla}$. But by Lemma 4.4.3 and Corollary 4.4.2, we may pre-compose and post-compose h with restriction idempotents, thus, $\hat{f}h\hat{g} = (\hat{f}h\hat{g})^{\Delta}_{\nabla}$. By $f \overset{h}{\simeq} g$, we have $fh = g$. Therefore,

$$f(\hat{f}h\hat{g}) = fh\hat{g} = g\hat{g} = g.$$

This gives us that $f \overset{\hat{f}h\hat{g}}{\simeq} g$.

The inequality $\hat{f}h\hat{g} \leq h$ follows by applying Lemma 3.1.3 twice and then [**R.4**]:

$$\overline{\hat{f}h\hat{g}}h = \overline{\overline{f^{(-1)}}h\overline{g^{(-1)}}}h = \overline{\overline{f^{(-1)}}hg^{(-1)}}h = \overline{\overline{f^{(-1)}}hg^{(-1)}}\,\overline{f^{(-1)}}h = \overline{f^{(-1)}}h\overline{g^{(-1)}} = \hat{f}h\hat{g}.$$

□

Corollary 5.1.6. *Given f, g in a discrete inverse category, if $f \overset{h}{\simeq} g$, then $f \overset{f^{(-1)}g}{\simeq} g$ and $f^{(-1)}g$ is minimal among all h such that $f \overset{h}{\simeq} g$.*

Proof. First, it is necessary to show that $f^{(-1)}g = (f^{(-1)}g)^{\Delta}_{\nabla}$. We have $fh = g$ and $h = h^{\Delta}_{\nabla}$. This means $f^{(-1)}fh = f^{(-1)}g$. As $f^{(-1)}f = \hat{f}$ is a restriction idempotent, we have $f^{(-1)}g = (f^{(-1)}g)^{\Delta}_{\nabla}$. As $ff^{(-1)}g = \overline{f}g = \overline{g}g = g$, the required diagram of Definition 5.1.2 commutes.

Given another h such that $f \overset{h}{\simeq} g$, we have $\hat{f}h\hat{g} \leq h$ and

$$\overline{f^{(-1)}g}\hat{f}h\hat{g} = \overline{f^{(-1)}g}f^{(-1)}fh\hat{g} = f^{(-1)}\overline{g}fh\hat{g} = f^{(-1)}\overline{g}g\hat{g} = f^{(-1)}g.$$

Thus, by transitivity $f^{(-1)}g \leq h$ and is minimal among all h such that $f \overset{h}{\simeq} g$. □

5.1.2 $\widetilde{\mathbb{X}}$ is a restriction category

Lemma 5.1.7. *$\widetilde{\mathbb{X}}$ as defined above is a category.*

Proof. The maps are well defined, as shown in Lemma 5.1.4. As $\mathbb{X}$ is a discrete inverse category, ${u^r_\otimes}^{(-1)}$ is defined and therefore the $({u^r_\otimes}^{(-1)}, 1)$ is a map in $\widetilde{\mathbb{X}}$.

It remains to show the composition is associative and that $({u^r_\otimes}^{(-1)}, 1)$ acts as an identity in $\widetilde{\mathbb{X}}$. For all of these, we will make use of Lemma 4.4.3 (iii), which states $1 \otimes f \in A^{\Delta}_{\nabla}$ for all f.

Associativity: Consider

$$A \xrightarrow{(f,B')} B \xrightarrow{(g,C')} C \xrightarrow{(h,D')} D.$$

To show the associativity of this in $\widetilde{\mathbb{X}}$, we need to show in $\mathbb{X}$ that

$$\overline{(f(g \otimes 1)a_\otimes)(h \otimes 1)a_\otimes} = \overline{f(((g(h \otimes 1)a_\otimes) \otimes 1)a_\otimes)}$$

and that there exists a mediating map between the two of them.

To see that the restrictions are equal, first note that by the functorality of $\otimes$, for any two maps u and v, we have $uv \otimes 1 = (u \otimes 1)(v \otimes 1)$. Second, the naturality of $a_\otimes$ gives us that $a_\otimes(h \otimes 1) = ((h \otimes 1) \otimes 1)a_\otimes$. Thus,

$$\begin{aligned}
\overline{f(g \otimes 1)a_\otimes(h \otimes 1)a_\otimes} &= \overline{f(g \otimes 1)a_\otimes(h \otimes 1)\overline{a_\otimes}} && \text{Lemma 3.1.3}\\
&= \overline{f(g \otimes 1)a_\otimes(h \otimes 1)} && \overline{a_\otimes} = 1\\
&= \overline{f(g \otimes 1)((h \otimes 1) \otimes 1)a_\otimes} && a_\otimes \text{ natural}\\
&= \overline{f(g \otimes 1)((h \otimes 1) \otimes 1)} && \text{Lemma 3.1.3}\\
&= \overline{f(g \otimes 1)((h \otimes 1) \otimes 1)(a_\otimes \otimes 1)} && \text{Lemma 3.1.3}\\
&= \overline{f((g(h \otimes 1)a_\otimes) \otimes 1)} && \text{see above}\\
&= \overline{f((g(h \otimes 1)a_\otimes) \otimes 1)a_\otimes} && \overline{a_\otimes} = 1.
\end{aligned}$$

For the mediating map, see the diagram below, where the calculation is in $\mathbb{X}$. The path, starting in the top left at A and going right to $D_{|D' \otimes (C' \otimes B')}$, is grouping parentheses to the left. The path which starts with A, then goes down to $(D_{|D'\otimes C'})_{|B'}$, followed by continuing right to $D_{|(D'\otimes C')\otimes B'}$ is grouping parentheses to the right. The commutativity of the diagram is shown by the commutativity of the internal portions, which all follow from the standard coherence diagrams for the tensor and naturality of association.

$$\begin{tikzcd}
A \arrow[r, "f(g\otimes 1)a_\otimes"] \arrow[d, "f"'] \arrow[dr, "f(g\otimes 1)"] & C_{|C'\otimes B'} \arrow[r, "h\otimes 1"] & (D_{|D'})_{|C'\otimes B'} \arrow[r, "a_\otimes"] & D_{|D'\otimes(C'\otimes B')} \arrow[dd, dotted, "1\otimes a_\otimes{}^{(-1)}"] \\
B_{|B'} \arrow[r, "g\otimes 1"] \arrow[d, "(g(h\otimes 1)a_\otimes)\otimes 1"'] & (C_{|C'})_{|B'} \arrow[u, "a_\otimes"'] \arrow[r, "(h\otimes 1)\otimes 1"] & ((D_{|D'})_{|C'})_{|B'} \arrow[u, "a_\otimes"'] \arrow[dll, "a_\otimes\otimes 1"'] & \\
(D_{|D'\otimes C'})_{|B'} \arrow[rrr, "a_\otimes"'] & & & D_{|(D'\otimes C')\otimes B'}.
\end{tikzcd}$$

Therefore, we can conclude

$$(f(g\otimes 1)a_\otimes)(h\otimes 1)a_\otimes \overset{1\otimes a_\otimes{}^{(-1)}}{\simeq} f(((g(h\otimes 1)a_\otimes)\otimes 1)a_\otimes)$$

which gives us that composition in $\widetilde{\mathbb{X}}$ is associative.

Identity: This requires:

$$(f,C)(u^r_\otimes{}^{(-1)},1) = (f,C) = (u^r_\otimes{}^{(-1)},1)(f,C)$$

for all maps $A \xrightarrow{(f,C)} B$ in $\widetilde{\mathbb{X}}$. In $\mathbb{X}$, this corresponds to requiring:

$$f \simeq f(u^r_\otimes{}^{(-1)}\otimes 1)a_\otimes \text{ and} \tag{5.1}$$

$$f \simeq u^r_\otimes{}^{(-1)}(f\otimes 1)a_\otimes. \tag{5.2}$$

For Equation (5.1), by Lemma 3.1.3 we have $\overline{f(u^r_\otimes{}^{(-1)}\otimes 1)a_\otimes} = \overline{f}$. Then, calculating in $\mathbb{X}$, we have a mediating map of $1 \otimes u_\otimes^l$ as shown below:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[drrr, "f"'] & B \otimes C \arrow[r, "{u_\otimes^r}^{(-1)} \otimes 1"] \arrow[rr, bend right, "1 \otimes {u_\otimes^l}^{(-1)}"'] \arrow[drr, equal] & (B \otimes 1) \otimes C \arrow[r, "a_\otimes"] & B \otimes (1 \otimes C) \arrow[d, dotted, "1 \otimes u_\otimes^l"] \\
 & & & B \otimes C.
\end{tikzcd}$$

Similarly, for Equation (5.2), $\overline{{u_\otimes^r}^{(-1)}(f \otimes 1)a_\otimes} = \overline{f}$ by Lemma 3.1.3 and ${u_\otimes^r}^{(-1)}$ is natural. The diagram

$$\begin{tikzcd}
A \arrow[r, "{u_\otimes^r}^{(-1)}"] \arrow[dr, "f"'] \arrow[dd, equal] & A \otimes 1 \arrow[r, "f \otimes 1"] & (B \otimes C) \otimes 1 \arrow[r, "a_\otimes"] & B \otimes (C \otimes 1) \arrow[dd, dotted, "1 \otimes u_\otimes^r"] \\
 & B \otimes C \arrow[ur, "{u_\otimes^r}^{(-1)}"] \arrow[urr, "1 \otimes {u_\otimes^r}^{(-1)}"'] \arrow[drr, equal] & & \\
A \arrow[rrr, "f"] & & & B \otimes C
\end{tikzcd}$$

shows our mediating map is $1 \otimes u_\otimes^r$.

Hence we have shown both Equations (5.1) and (5.2) hold and therefore $({u_\otimes^r}^{(-1)}, 1)$ is the identity map in $\widetilde{\mathbb{X}}$. $\square$

Now that we have shown $\widetilde{\mathbb{X}}$ is a category, we may define a restriction:

Lemma 5.1.8. *The category $\widetilde{\mathbb{X}}$ with restriction defined as*

$$\overline{(f, C)} := (\overline{f}{u_\otimes^r}^{(-1)}, 1)$$

is a restriction category.

Proof. First we note that for $(f, C) : A \to B$, we have $\overline{(f, C)} : A \to A$. The four restriction axioms must now be checked. For the remainder of this proof, all diagrams will be in $\mathbb{X}$. We make use of Lemma 4.4.3 (iii), which states $1 \otimes f \in A_{\triangledown}^{\triangle}$ for all f.

[**R.1**] $(\overline{(f,C)}(f,C) = (f,C))$. Calculating the restriction of the left hand side in $\mathbb{X}$, we have:

$$
\begin{aligned}
\overline{\overline{f}{u^r_\otimes}^{(-1)}(f\otimes 1)a_\otimes} &= \overline{\overline{f}{u^r_\otimes}^{(-1)}(f\otimes 1)} && \text{Lemma 3.1.3} \\
&= \overline{\overline{f}f{u^r_\otimes}^{(-1)}} && {u^r_\otimes}^{(-1)} \text{ natural} \\
&= \overline{f{u^r_\otimes}^{(-1)}} && [\textbf{R.1}] \text{ in } \mathbb{X} \\
&= \overline{f} && \text{Lemma 3.1.3.}
\end{aligned}
$$

Then, the following diagram

$$
\begin{tikzcd}
A \arrow[r, "\overline{f}{u^r_\otimes}^{(-1)}"] \arrow[drr, "\overline{f}f"'] \arrow[ddrrr, bend right, "f"'] & A\otimes 1 \arrow[r, "f\otimes 1"] & (B\otimes C)\otimes 1 \arrow[r, "a_\otimes"] \arrow[ddr, "u^r_\otimes"] & B\otimes(C\otimes 1) \arrow[dd, dotted, "1\otimes u^r_\otimes"] \\
 & & B\otimes C \arrow[u, "{u^r_\otimes}^{(-1)}"] \arrow[dr, equal] & \\
 & & & B\otimes C
\end{tikzcd}
$$

shows $\overline{f}{u^r_\otimes}^{(-1)}(f\otimes 1)a_\otimes \overset{1\otimes u^r_\otimes}{\simeq} f$ in $\mathbb{X}$ and therefore $\overline{(f,C)}(f,C) = (f,C)$ in $\widetilde{\mathbb{X}}$.

[**R.2**] $(\overline{(g,D)}\,\overline{(f,C)} = \overline{(f,C)}\,\overline{(g,D)})$. We must show

$$\overline{f}{u^r_\otimes}^{(-1)}((\overline{g}{u^r_\otimes}^{(-1)})\otimes 1))a_\otimes \simeq \overline{g}{u^r_\otimes}^{(-1)}((\overline{f}{u^r_\otimes}^{(-1)})\otimes 1)a_\otimes. \tag{5.3}$$

The restriction of the left hand side equals the restriction of the right hand side as seen below:

$$
\begin{aligned}
\overline{\overline{f}{u^r_\otimes}^{(-1)}((\overline{g}{u^r_\otimes}^{(-1)})\otimes 1))a_\otimes} &= \overline{\overline{f}(\overline{g}{u^r_\otimes}^{(-1)}){u^r_\otimes}^{(-1)}a_\otimes} && {u^r_\otimes}^{(-1)} \text{ natural} \\
&= \overline{\overline{g}\,\overline{f}{u^r_\otimes}^{(-1)}{u^r_\otimes}^{(-1)}a_\otimes} && [\textbf{R.2}] \text{ in } \mathbb{X} \\
&= \overline{\overline{g}{u^r_\otimes}^{(-1)}((\overline{f}{u^r_\otimes}^{(-1)})\otimes 1)a_\otimes} && {u^r_\otimes}^{(-1)}\text{natural.}
\end{aligned}
$$

The below diagram commutes by the naturality of $u_\otimes^r$ and the tensor coherence,

$$\begin{tikzcd}
A \arrow[r, "\overline{g}u_\otimes^{r\,(-1)}"] \arrow[d, "\overline{f}u_\otimes^{r\,(-1)}"'] \arrow[dr, "\overline{g}\overline{f}", "\overline{f}\overline{g}"'] & A \otimes 1 \arrow[r, "(\overline{f}u_\otimes^{r\,(-1)})\otimes 1"] & (A \otimes 1) \otimes 1 \arrow[r, "a_\otimes"] \arrow[dl, "u_\otimes^r u_\otimes^r"'] & A \otimes (1 \otimes 1) \arrow[dd, dotted, "1\otimes id"'] \\
A \otimes 1 \arrow[d, "(\overline{g}u_\otimes^{r\,(-1)})\otimes 1"'] & A \arrow[ur, bend right, "u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}"'] & & \\
(A \otimes 1) \otimes 1 \arrow[ur, "u_\otimes^r u_\otimes^r"] \arrow[rrr, "a_\otimes"'] & & & A \otimes (1 \otimes 1)
\end{tikzcd}$$

which allows us to conclude $\overline{(g,D)(f,C)} = \overline{(f,C)(g,D)}$ in $\widetilde{\mathbb{X}}$.

[**R.3**] $(\overline{\overline{(f,C)}(g,D)} = \overline{(f,C)(g,D)}\ )$. We must show

$$\overline{(\overline{f}u_\otimes^{r\,(-1)})(g \otimes 1)a_\otimes u_\otimes^{r\,(-1)}} \simeq (\overline{f}u_\otimes^{r\,(-1)})(\overline{g}u_\otimes^{r\,(-1)} \otimes 1)a_\otimes. \tag{5.4}$$

As above, the first step is to show that the restrictions of each side of Equation (5.4) are the same. Computing the restriction of the left hand side in $\mathbb{X}$:

$$\begin{aligned}
\overline{\overline{(\overline{f}u_\otimes^{r\,(-1)})(g \otimes 1)a_\otimes}\,u_\otimes^{r\,(-1)}} &= \overline{\overline{(\overline{f}u_\otimes^{r\,(-1)})(g \otimes 1)a_\otimes}} && \text{Lemma 3.1.3} \\
&= \overline{(\overline{f}u_\otimes^{r\,(-1)})(g \otimes 1)a_\otimes} && \text{Lemma 3.1.3} \\
&= \overline{\overline{f}gu_\otimes^{r\,(-1)}a_\otimes} && u_\otimes^{r\,(-1)} \text{ natural} \\
&= \overline{\overline{f}g} && \text{Lemma 3.1.3} \\
&= \overline{f}\,\overline{g} && [\textbf{R.3}] \text{ in } \mathbb{X}.
\end{aligned}$$

The restriction of the right hand side of Equation (5.4) computes as:

$$\begin{aligned}
&\overline{(\overline{f}u_\otimes^{r\,(-1)})(\overline{g}u_\otimes^{r\,(-1)} \otimes 1)a_\otimes} \\
&= \overline{(\overline{f}u_\otimes^{r\,(-1)})(\overline{g}u_\otimes^{r\,(-1)} \otimes 1)} && \text{Lemma 3.1.3} \\
&= \overline{\overline{f}\,\overline{g}u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}} && u_\otimes^{r\,(-1)} \text{ natural} \\
&= \overline{\overline{f}\,\overline{g}} && \text{Lemma 3.1.3} \\
&= \overline{f}\,\overline{g} && \text{Lemma 3.1.3.}
\end{aligned}$$

Additionally, we see $\overline{\overline{(f,C)}(g,D)}$ in $\widetilde{\mathbb{X}}$ is expressed in $\mathbb{X}$ as:

$$
\begin{aligned}
&\overline{(\overline{f}u_\otimes^{r\,(-1)})(g\otimes 1)a_\otimes} u_\otimes^{r\,(-1)} && \\
&= \overline{\overline{f}gu_\otimes^{r\,(-1)}a_\otimes} u_\otimes^{r\,(-1)} && u_\otimes^{r\,(-1)} \text{ natural} \\
&= \overline{\overline{f}g}u_\otimes^{r\,(-1)} && u_\otimes^{r\,(-1)},\ a_\otimes \text{ isomorphisms} \\
&= \overline{f}\overline{g}u_\otimes^{r\,(-1)} && [\mathbf{R.3}].
\end{aligned}
$$

The following diagram in $\mathbb{X}$ follows the right hand side of Equation (5.4) with the top curved arrow and the left hand side of Equation (5.4) with the bottom curved arrow, using the fact that $\overline{(\overline{f}u_\otimes^{r\,(-1)})(g\otimes 1)a_\otimes} u_\otimes^{r\,(-1)} = \overline{f}\overline{g}u_\otimes^{r\,(-1)}$ as shown above.

$$
\begin{tikzcd}
A \arrow[r, "\overline{f}"] \arrow[dr, "\overline{f}"'] \arrow[rrrrr, bend left, "\overline{f}u_\otimes^{r\,(-1)}(\overline{g}u_\otimes^{r\,(-1)}\otimes 1)a_\otimes"] \arrow[dddddrrrrr, bend right, "\overline{f}\,\overline{g}u_\otimes^{r\,(-1)}"'] & A \arrow[r, "u_\otimes^{r\,(-1)}"] \arrow[d, equal] & A\otimes 1 \arrow[r, "\overline{g}\otimes 1"] \arrow[dd, "u_\otimes^r"'] & A\otimes 1 \arrow[r, "u_\otimes^{r\,(-1)}\otimes 1"] \arrow[ddd, "u_\otimes^r"'] & (A\otimes 1)\otimes 1 \arrow[r, "a_\otimes"] \arrow[dddd, "u_\otimes^r\otimes 1"'] & A\otimes(1\otimes 1) \arrow[ddddd, dotted, "1\otimes u_\otimes^r"] \\
& A \arrow[dr, equal] & & & & \\
& & A \arrow[dr, "\overline{g}"'] & & & \\
& & & A \arrow[dr, "u_\otimes^{r\,(-1)}"'] & & \\
& & & & A\otimes 1 \arrow[dr, equal] & \\
& & & & & A\otimes 1.
\end{tikzcd}
$$

Hence, in $\mathbb{X}$, $\overline{(\overline{f}u_\otimes^{r\,(-1)})(g\otimes 1)a_\otimes} u_\otimes^{r\,(-1)} \overset{1\otimes u_\otimes^r}{\simeq} (\overline{f}u_\otimes^{r\,(-1)})(\overline{g}u_\otimes^{r\,(-1)}\otimes 1)a_\otimes$ and therefore $\overline{\overline{f}g} = \overline{f}\overline{g}$ in $\widetilde{\mathbb{X}}$.

[**R.4**] $((f,C)\overline{(g,D)} = \overline{(f,C)(g,D)}(f,C))$. We must show

$$
f(\overline{g}u_\otimes^{r\,(-1)}\otimes 1)a_\otimes \simeq \overline{f(g\otimes 1)}u_\otimes^{r\,(-1)}(f\otimes 1)a_\otimes. \tag{5.5}
$$

The restriction of the left hand side of Equation (5.5) is:

$$
\begin{aligned}
\overline{f(\overline{g}{u^r_\otimes}^{(-1)}\otimes 1)a_\otimes} &= \overline{f(\overline{g}{u^r_\otimes}^{(-1)}\otimes 1)} && \text{Lemma 3.1.3}\\
&= \overline{f\overline{g}{u^r_\otimes}^{(-1)}}\otimes\overline{f} && \otimes\text{ restriction functor}\\
&= \overline{f\overline{g}}\otimes\overline{f} && \text{Lemma 3.1.3}\\
&= \overline{f(\overline{g}\otimes 1)}
\end{aligned}
$$

and the restriction of the right hand side of Equation (5.5) is:

$$
\begin{aligned}
\overline{\overline{f(g\otimes 1)}{u^r_\otimes}^{(-1)}(f\otimes 1)a_\otimes} &= \overline{\overline{f(g\otimes 1)}{u^r_\otimes}^{(-1)}(f\otimes 1)} && \text{Lemma 3.1.3}\\
&= \overline{\overline{f(g\otimes 1)}f{u^r_\otimes}^{(-1)}} && {u^r_\otimes}^{(-1)}\text{ natural}\\
&= \overline{f\overline{(g\otimes 1)}{u^r_\otimes}^{(-1)}} && [\mathbf{R.4}]\text{ for }\mathbb{X}\\
&= \overline{f(\overline{g}\otimes 1){u^r_\otimes}^{(-1)}} && \otimes\text{ a restriction functor}\\
&= \overline{f(\overline{g}\otimes 1)} && \text{Lemma 3.1.3.}
\end{aligned}
$$

Computing the right hand side of Equation (5.5) in $\mathbb{X}$,

$$
\begin{aligned}
\overline{f(g\otimes 1)}a_\otimes{u^r_\otimes}^{(-1)}(f\otimes 1)a_\otimes &= \overline{f(g\otimes 1)}f{u^r_\otimes}^{(-1)}a_\otimes && {u^r_\otimes}^{(-1)}\text{ natural,}\\
&= f(\overline{g}\otimes 1){u^r_\otimes}^{(-1)}a_\otimes && [\mathbf{R.4}],\ \otimes\text{ a restriction functor.}
\end{aligned}
$$

Thus,

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f"'] & B\otimes C \arrow[rr, "\overline{g}{u^r_\otimes}^{(-1)}\otimes 1"] \arrow[d, equal] & & (B\otimes 1)\otimes C \arrow[r, "a_\otimes"] \arrow[d, "a_\otimes(1\otimes c_\otimes){a_\otimes}^{(-1)}"] & B\otimes(1\otimes C) \arrow[d, dotted, "1\otimes c_\otimes"] \\
& B\otimes C \arrow[r, "\overline{g}\otimes 1"'] & B\otimes C \arrow[ur, "{u^r_\otimes}^{(-1)}\otimes 1"] \arrow[r, "{u^r_\otimes}^{(-1)}"'] & (B\otimes C)\otimes 1 \arrow[r, "a_\otimes"'] & B\otimes(C\otimes 1)
\end{tikzcd}
$$

giving us $\overline{f(g\otimes 1)a_\otimes}{u^r_\otimes}^{(-1)}(f\otimes 1)a_\otimes = f(\overline{g}\otimes 1){u^r_\otimes}^{(-1)}a_\otimes \overset{1\otimes c_\otimes}{\simeq} f(\overline{g}{u^r_\otimes}^{(-1)}\otimes 1)a_\otimes$ and hence, $\widetilde{\mathbb{X}}$ is a restriction category. $\square$

5.1.3 $\widetilde{\mathbb{X}}$ is a discrete Cartesian restriction category

Lemma 5.1.9. *The unit of the inverse product in $\mathbb{X}$, a discrete inverse category, is the terminal object in $\widetilde{\mathbb{X}}$.*

Proof. The unique map to the terminal object for any object A in $\widetilde{\widetilde{\mathbb{X}}}$ is the equivalence class of maps given by $({u^l_\otimes}^{(-1)}, A)$ and is designated as $!_A$. For this to be a terminal object, the diagram

$$\begin{tikzcd} X \arrow[r, "\overline{(f,C)}"] \arrow[d, "(f,C)"] & X \arrow[r, "!_X"] & \top \\ Y \arrow[urr, "!_Y"'] & & \end{tikzcd}$$

must commute for all choices of f. Translating this to $\mathbb{X}$, this is the same as requiring

$$\begin{tikzcd} X \arrow[r, "\overline{f}"] \arrow[d, "f"] & X \arrow[r, "{u^r_\otimes}^{(-1)}"] & X \otimes 1 \arrow[r, "{u^l_\otimes}^{(-1)}"] & 1 \otimes X \otimes 1 \arrow[d, dotted, "1\otimes(u^r_\otimes f)"] \\ Y \otimes C \arrow[rrr, "{u^l_\otimes}^{(-1)}"'] & & & 1 \otimes Y \otimes C \end{tikzcd}$$

to commute, which is true by [**R.1**] and from the coherence diagrams for the inverse product tensor. $\square$

Next, we show that the category $\widetilde{\widetilde{\mathbb{X}}}$ has restriction products, given by the action of the Cartesian Completion on the $\otimes$ tensor in $\mathbb{X}$.

First, define total maps π_0, π_1 in $\widetilde{\widetilde{\mathbb{X}}}$ by:

$$\pi_0 : \qquad A \otimes B \xrightarrow{(1,B)} A, \tag{5.6}$$

$$\pi_1 : \qquad A \otimes B \xrightarrow{(c_\otimes,A)} B. \tag{5.7}$$

Definition 5.1.10. Given a discrete inverse category $\mathbb{X}$, suppose we are given the maps $Z \xrightarrow{(f,C)} A$ and $Z \xrightarrow{(g,C')} B$ in $\widetilde{\widetilde{\mathbb{X}}}$. Then define $\langle (f,C),(g,C')\rangle$as

$$Z \xrightarrow{(\Delta(f\otimes g)(1\otimes c_\otimes \otimes 1), C\otimes C')} A \otimes B \tag{5.8}$$

where associativity is assumed as needed. Note that with the associativity maps, this is actually:

$$Z \xrightarrow{(\Delta(f\otimes g)a_\otimes(1\otimes {a_\otimes}^{(-1)})(1\otimes(c_\otimes\otimes 1))(1\otimes a_\otimes){a_\otimes}^{(-1)}, C\otimes C')} A \otimes B. \tag{5.9}$$

Lemma 5.1.11. *For $\mathbb{X}$ a discrete inverse category, $\otimes$ is a restriction product in $\widetilde{\widetilde{\mathbb{X}}}$ with projections π_0, π_1 and the product of maps f, g being $\langle f, g\rangle$.*

Proof. To show that the tensor is a product in $\widetilde{\mathbb{X}}$, we must show three things:

(i) The maps π_0 and π_1 are total;

(ii) $\overline{\langle (f,C),(g,D)\rangle} = \overline{(f,C)}\,\overline{(g,D)}$;

(iii) $\langle (f,C),(g,D)\rangle \pi_0 \leq (f,C)$ and $\langle (f,C),(g,D)\rangle \pi_1 \leq (g,D)$.

For (i), as 1 and $c_\otimes$ are isomorphisms, the maps π_0, π_1 are total.

For (ii), to show $\overline{\langle (f,C),(g,D)\rangle} = \overline{(f,C)}\,\overline{(g,D)}$, first reduce the left hand side in $\mathbb{X}$:

$$\begin{aligned}
\overline{\langle (f,C),(g,D)\rangle} &= \overline{\Delta(f\otimes g)(1\otimes c_\otimes \otimes 1)}u_\otimes^{r\,(-1)} && \text{in } \mathbb{X},\ \text{definition of restriction}\\
&= \overline{\Delta(f\otimes g)}u_\otimes^{r\,(-1)} \\
&= \overline{\Delta\overline{(f\otimes g)}}u_\otimes^{r\,(-1)} && \text{from Lemma 3.1.3}\\
&= \overline{\Delta(\overline{f}\otimes\overline{g})}u_\otimes^{r\,(-1)} && \otimes \text{ is a restriction functor}\\
&= \overline{\overline{f}\,\overline{g}\,\Delta(1\otimes 1)}u_\otimes^{r\,(-1)} && \text{Lemma 4.3.5((ii)) twice}\\
&= \overline{\overline{f}}\,\overline{g}u_\otimes^{r\,(-1)} && \text{Lemma 3.1.3}\\
&= \overline{f}\,\overline{g}u_\otimes^{r\,(-1)} && \text{Lemma 3.1.3.}
\end{aligned}$$

Then, the right hand side of (ii) reduces as:

$$\begin{aligned}
\overline{(f,C)(g,D)} &= \overline{f}u_\otimes^{r\,(-1)}(\overline{g}u_\otimes^{r\,(-1)}\otimes 1)a_\otimes && \text{in } \mathbb{X} \text{ by definitions}\\
&= \overline{f}\overline{g}u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}a_\otimes && u_\otimes^{r\,(-1)} \text{ natural.}
\end{aligned}$$

Thus, to show the equality of (ii) in $\widetilde{\mathbb{X}}$, we need only show that $\overline{f}\,\overline{g}u_\otimes^{r\,(-1)}$ is equivalent to $\overline{f}\overline{g}u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}a_\otimes$ in $\mathbb{X}$. The restriction of both of these, in $\mathbb{X}$, is $\overline{\overline{f}\overline{g}}$, via Lemma 3.1.3. Thus, using the mediating map $1\otimes u_\otimes^r$, we have

$$\overline{f}\,\overline{g}u_\otimes^{r\,(-1)} \overset{1\otimes u_\otimes^r}{\simeq} \overline{f}\overline{g}u_\otimes^{r\,(-1)}u_\otimes^{r\,(-1)}a_\otimes.$$

This shows $\overline{\langle (f,C),(g,D)\rangle} = \overline{(f,C)(g,D)}$ in $\widetilde{\mathbb{X}}$.

Finally, for (iii), to show $\langle (f,C),(g,D)\rangle \pi_0 \leq (f,C)$ (and $\langle (f,C),(g,D)\rangle \pi_1 \leq (g,D)$), it is required to show $\overline{\langle (f,C),(g,D)\rangle \pi_0}(f,C) = \langle (f,C),(g,D)\rangle \pi_0$. Calculating the left side, we see:

$$\begin{aligned}
\overline{\langle (f,C),(g,D)\rangle \pi_0}(f,C) &= \overline{\langle (f,C),(g,D)\rangle \overline{\pi_0}}(f,C) && \text{Lemma 3.1.3}\\
&= \overline{\langle (f,C),(g,D)\rangle}(f,C) && \pi_0 \text{ is total}\\
&= \overline{(f,C)}\,\overline{(g,D)}\,(f,C) && \text{by above}\\
&= \overline{(g,D)}\overline{(f,C)}(f,C) && [\mathbf{R.2}]\\
&= \overline{(g,D)}(f,C) && [\mathbf{R.1}].
\end{aligned}$$

Now, turning to the right hand side:

$$\langle (f,C),(g,D)\rangle \pi_0 = \Delta(f\otimes g)(1\otimes c_\otimes \otimes 1)1 \qquad \text{in } \mathbb{X}\text{, by definition.}$$

To show these are equal in $\widetilde{\mathbb{X}}$, we need to first show the restrictions are the same in $\mathbb{X}$ and then give a mediating map. For $\overline{(g,D)}(f,C)$, in $\mathbb{X}$, this is $\overline{g}{u_\otimes^r}^{(-1)}(f\otimes 1)$. Calculating the restriction, we have

$$\overline{\overline{g}{u_\otimes^r}^{(-1)}(f\otimes 1)} = \overline{\overline{g}f{u_\otimes^r}^{(-1)}} = \overline{\overline{g}f} = \overline{f}\,\overline{g}.$$

For the right hand side, calculate in $\mathbb{X}$:

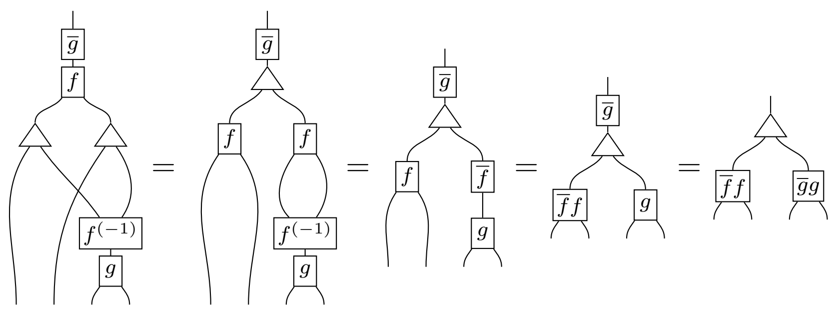
$$\begin{aligned}
\overline{\Delta(f\otimes g)(1\otimes c_\otimes \otimes 1)} &= \overline{\Delta(f\otimes g)} && \text{Lemma 3.1.3}\\
&= \Delta(f\otimes g)(f^{(-1)}\otimes g^{(-1)})\Delta^{(-1)} && \mathbb{X} \text{ is an inverse category}\\
&= \Delta(\overline{f}\otimes \overline{g})\Delta^{(-1)} &&\\
&= \overline{f}\,\overline{g}\Delta\Delta^{(-1)} && \text{Lemma 4.3.5(ii) twice}\\
&= \overline{f}\,\overline{g}. &&
\end{aligned}$$

The diagram below shows the required mediating map. By Lemma 4.4.3, $\Delta \in A^{\Delta}_{\nabla}$,

$1 \otimes k \in A_{\nabla}^{\Delta}$ and A_{∇}^{Δ} is closed under composition:

$$\begin{tikzcd}
 & & A \otimes C \arrow[d, dotted, "\Delta"] \\
 & Z \arrow[ur, "f"] & A \otimes C \otimes A \otimes C \arrow[d, dotted, "1\otimes f^{(-1)}"] \\
Z \arrow[ur, "\overline{g}"] \arrow[dr, "\Delta(f\otimes g)"'] & & A \otimes C \otimes Z \arrow[d, dotted, "1\otimes 1\otimes g"] \\
 & A \otimes C \otimes B \otimes C' \arrow[dr, "1\otimes c_{\otimes}\otimes 1"'] & A \otimes C \otimes B \otimes C' \arrow[d, dotted, "1\otimes c_{\otimes}\otimes 1"] \\
 & & A \otimes B \otimes C \otimes C'.
\end{tikzcd}$$

To see that this commutes, we will use the diagram language introduced in SubSection 4.3.2, where we drop the common term of $1 \otimes c_{\otimes} \otimes 1$. Then we have $\overline{g} f \Delta (1 \otimes f^{(-1)})(1 \otimes 1 \otimes g) =$



$= \Delta(f \otimes g).$

$\square$

At this point, we have shown that $\widetilde{\mathbb{X}}$ is a restriction category with restriction products. This leads us to the following theorem:

Theorem 5.1.12. *For any discrete inverse category $\mathbb{X}$, the category $\widetilde{\mathbb{X}}$ is a discrete Cartesian restriction category.*

Proof. The fact that $\widetilde{\mathbb{X}}$ is a Cartesian restriction category is immediate from Lemmas 5.1.7, 5.1.8, 5.1.9 and 5.1.11.

To show that it is discrete, we need only show that the map $(\Delta {u_{\otimes}^{r}}^{(-1)}, 1)$ is in the same equivalence class as $\widetilde{\mathbb{X}}$'s $\Delta (= \langle 1, 1 \rangle = \langle ({u_{\otimes}^{r}}^{(-1)}, 1), ({u_{\otimes}^{r}}^{(-1)}, 1) \rangle$. As both Δ and ${u_{\otimes}^{r}}^{(-1)}$

are total, the restriction of each side is the same, namely 1. The diagram below uses Lemma 4.4.3 (iii) and shows that the two maps are in the same equivalence class:

$$\begin{tikzcd}
 & & A \otimes A \otimes 1 \arrow[d, dotted, "1\otimes u_{\otimes}^{r\,(-1)}"] \\
A \arrow[urr, "\Delta u_{\otimes}^{r\,(-1)}"] \arrow[rr, "\Delta(u_{\otimes}^{r\,(-1)}\otimes u_{\otimes}^{r\,(-1)})(1\otimes c_{\otimes}\otimes 1)"'] & & A \otimes A \otimes 1 \otimes 1.
\end{tikzcd}$$

□

5.2 Equivalence between the category **DInv** and the category **DCartRest**

This section will show that **DInv**, the category of discrete inverse categories with functors that preserve the inverse product, is equivalent to **DCartRest**, the category of discrete Cartesian restriction categories with restriction functors which preserve the product.

Here are the steps:

1. Give a functor **INV** from discrete Cartesian restriction categories to discrete inverse categories and show that it is full and faithful.

2. In

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "\eta"] \arrow[dr, "F"'] & \mathbf{INV}(\widetilde{\mathbb{X}}) \arrow[d, "\mathbf{INV}(F^{\#})"] \\
 & \mathbf{INV}(\mathbb{D})
\end{tikzcd} \tag{5.10}$$

   we show that there exists a functor $F^{\#} : \widetilde{\mathbb{X}} \to \mathbb{D}$ which makes the diagram commute. As we have **INV** full and faithful, we may conclude it is unique and that Diagram (5.10) is a universal diagram.

3. Show that η in Diagram (5.10) is an isomorphism. Note this also follows directly from the previous point by Proposition 2.2.6 in [19].

Once we have completed these steps, we may then conclude that there is an equivalence.

In the following, $\mathbb{X}$ will always be a discrete inverse category, $\mathbb{D}$ and $\mathbb{B}$ will be discrete Cartesian restriction categories.

The functor **INV** maps a discrete Cartesian restriction category to its inverse subcategory. It maps a functor between discrete Cartesian restriction categories to a functor having the same action on the partial inverses. That is, given $G : \mathbb{B} \to \mathbb{D}$, then:

$$\begin{aligned}
&\mathbf{INV}(G) : \mathbf{INV}(\mathbb{B}) \to \mathbf{INV}(\mathbb{D}) \\
&\mathbf{INV}(G)(A) = GA && \text{(all objects of } \mathbb{D} \text{ are in } Inv(\mathbb{D})) \\
&\mathbf{INV}(G)(f) = G(f) && \text{(restriction functors preserve partial inverse).}
\end{aligned}$$

Lemma 5.2.1. *The functor* **INV** *from the category of discrete Cartesian restriction categories to the category of discrete inverse categories is full and faithful.*

Proof. To show fullness, we must show **INV** is surjective on hom-sets. Given a functor between two categories in the image of **INV**, i.e., $G : \mathbf{INV}(\mathbb{B}) \to \mathbf{INV}(\mathbb{D})$, construct a functor $H : \mathbb{B} \to \mathbb{D}$ as follows:

Action on objects: $H(A) = G(A)$,

Objects on maps: $H(f) = G(\langle f, 1\rangle)\pi_0$, where $\langle f, 1\rangle$ is the product map of f and 1 in $\mathbb{B}$.

H is well defined as we know $\langle f, 1\rangle$ is an invertible map and therefore in the domain of G. To see H is a functor:

$$\begin{aligned}
H(1) &= G(\langle 1, 1\rangle)\pi_0 = \Delta_{\mathbb{D}}\pi_0 = 1, \\
H(fg) &= G(\langle fg, 1\rangle)\pi_0 = \langle G(fg), 1\rangle\pi_0 = G(f)\langle G(g, 1)\rangle\pi_0 \\
&= \langle G(f), 1\rangle\pi_0\langle G(g), 1\rangle\pi_0 = G(\langle f, 1\rangle)\pi_0 G(\langle g, 1\rangle)\pi_0 = H(f)H(g).
\end{aligned}$$

But on any invertible map, $H(f) = G(\langle f, 1\rangle)\pi_0 = \langle G(f), 1\rangle\pi_0 = G(f)$ and therefore $\mathbf{INV}(H) = G$, so **INV** is full.

Next, assume we have $F, G : \mathbb{B} \to \mathbb{D}$ with $\mathbf{INV}(F) = \mathbf{INV}(G)$. Considering $F(f)$ and $F(g)$, we know $F(\langle f, 1\rangle) = G(\langle f, 1\rangle)$ as $\langle f, 1\rangle$ is invertible. Thus, as the functors preserve the product structure, i.e., $F(\pi_0) = \pi_0 = G(\pi_0)$, we have

$$F(f) = F(\langle f, 1\rangle\pi_0) = F(\langle f, 1\rangle)F(\pi_0) = G(\langle f, 1\rangle)G(\pi_0) = G(f).$$

Thus, $\mathbf{INV}$ is faithful. $\square$

Next, define $\eta : \mathbb{X} \to \mathbf{INV}(\widetilde{\mathbb{X}})$ as an identity on objects functor. For a map f in $\mathbb{X}$, $\eta : f \mapsto (f {u^r_\otimes}^{(-1)}, 1)$. This is a functor as

$$\eta(1) = ({u^r_\otimes}^{(-1)}, 1) \quad \text{and}$$
$$\eta(fg) = (fg{u^r_\otimes}^{(-1)}, 1) \simeq (f{u^r_\otimes}^{(-1)}, 1)(g{u^r_\otimes}^{(-1)}, 1) = \eta(f)\eta(g).$$

Now, we may define the functor $F^\# : \widetilde{\mathbb{X}} \to \mathbb{D}$. Recall that $\mathbf{INV}(\mathbb{D})$ is a subcategory of $\mathbb{D}$ having the same objects, but only the invertible maps. Given a functor $F : \mathbb{X} \to \mathbf{INV}(\mathbb{D})$ define $F^\#$ as follows:

On objects: $F^\# : A \mapsto F(A) \in \mathbb{D}_o$.

On maps: $F^\# : (f, C) \mapsto F(f)\pi_0 \in \mathbb{D}_m$.

We will now show (5.10) is a universal diagram.

Lemma 5.2.2. *Diagram (5.10) above commutes and is a universal diagram. That is,*

$$\eta\mathbf{INV}(F^\#) = F$$

and $F^\#$ is unique.

Proof. Using our definitions above, given a map f in $\mathbb{X}$, then:

$$\begin{aligned}
\mathbf{INV}(F^{\#})(\eta(f)) &= \mathbf{INV}(F^{\#})((f{u^r_\otimes}^{(-1)}, 1)) \\
&= F^{\#}((f{u^r_\otimes}^{(-1)}, 1)) \\
&= F(f{u^r_\otimes}^{(-1)})\pi_0 \\
&= F(f).
\end{aligned}$$

As η is identity on the objects, Diagram (5.10) commutes. The uniqueness of $F^{\#}$ follows immediately from Lemma 5.2.1, i.e., **INV** is faithful. □

Corollary 5.2.3. *The category $\widetilde{\widetilde{\mathbb{X}}}$ and functor $\eta : \mathbb{X} \to \mathbf{INV}(\widetilde{\widetilde{\mathbb{X}}})$ is a universal pair for the functor* **INV**.

Proof. Immediate by Lemma 5.2.2. □

We may now proceed to show η is an isomorphism, but we need a lemma first showing that all invertible maps in $\widetilde{\widetilde{\mathbb{X}}}$ are the equivalence class of the form $(f{u^r_\otimes}^{(-1)}, 1)$ for some f.

Lemma 5.2.4. *For any discrete inverse category $\mathbb{X}$, all invertible maps $(g, C) : A \to B$ in $\widetilde{\widetilde{\mathbb{X}}}$ are in the equivalence class of $(f{u^r_\otimes}^{(-1)}, 1)$ for some $f : A \to B$.*

Proof. As (g, C) is invertible in $\widetilde{\widetilde{\mathbb{X}}}$, the map $(g, C)^{(-1)} : B \to A$ exists. The map $(g, C)^{(-1)}$ must be in the equivalence class of some map $k : B \to A \otimes D$. By construction, the map $\overline{(k, D)}$ is in the equivalence class of the map $\overline{k}{u^r_\otimes}^{(-1)} : B \to B \otimes 1$ in $\mathbb{X}$. This means, in $\mathbb{X}$, there is an n such that

$$\begin{tikzcd}
B \arrow[r, "k"] \arrow[drr, "\overline{k}{u^r_\otimes}^{(-1)}"'] & A \otimes D \arrow[r, "g \otimes 1"] & B \otimes C \otimes D \arrow[d, dotted, "n"] \\
& & B \otimes 1
\end{tikzcd}$$

commutes.

Starting with $g : A \to B \otimes C$, construct the map f in $\mathbb{X}$ with the following diagram:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[ddddddr, dotted, "f"'] & B \otimes C \arrow[d, "\Delta\otimes 1"] \\
& B \otimes B \otimes C \arrow[d, "1\otimes k\otimes 1"] \\
& B \otimes A \otimes D \otimes C \arrow[d, "1\otimes g\otimes 1\otimes 1"] \\
& B \otimes B \otimes C \otimes D \otimes C \arrow[d, "\Delta^{(-1)}\otimes 1\otimes c_\otimes"] \\
& B \otimes C \otimes C \otimes D \arrow[d, "1\otimes \Delta^{(-1)}\otimes 1"] \\
& B \otimes C \otimes D \arrow[d, "nu^l_\otimes"] \\
& B.
\end{tikzcd}$$

By its construction, $f : A \to B$ in $\mathbb{X}$ and $(fu_\otimes^{r\,(-1)}, 1)$ are in the same equivalence class as (g, C).

$\square$

Lemma 5.2.5. *The functor $\eta : \mathbb{X} \to \mathbf{INV}(\widetilde{\mathbb{X}})$ is an isomorphism.*

Proof. As η is an identity on objects functor, we need only show that it is full and faithful. Referring to Lemma 5.2.4 above, we immediately see that η is full. For faithful, if we assume $(fu_\otimes^{r\,(-1)}, 1)$ is equal in $\widetilde{\mathbb{X}}$ to $(gu_\otimes^{r\,(-1)}, 1)$. This means in $\mathbb{X}$, that $\overline{f} = \overline{g}$ and there is a $h \in B\frac{\Delta}{\nabla}$ such that

$$\begin{tikzcd}
& B \otimes 1 \arrow[dd, dotted, "h"] \\
A \arrow[ur, "fu_\otimes^{r\,(-1)}"] \arrow[dr, "gu_\otimes^{r\,(-1)}"'] & \\
& B \otimes 1.
\end{tikzcd} \tag{5.11}$$

But as $h = (\Delta \otimes 1)(1 \otimes h)(\Delta^{(-1)} \otimes 1)$, and letting $\ell = u_\otimes^{r\,(-1)} h u_\otimes^r$, Diagram (5.11) equates to $g = f\Delta(1 \otimes \ell)\Delta^{(-1)}$. But by Lemma 4.3.5(iv), $\Delta(1 \otimes \ell)\Delta^{(-1)} = \overline{\Delta(1 \otimes \ell)\Delta^{(-1)}}$. Setting $\Delta(1 \otimes \ell)\Delta^{(-1)}$ as k, we have $g = f\overline{k}$. This gives us:

$$g = f\overline{k} = \overline{fk}f = \overline{f\overline{k}}f = \overline{g}f = \overline{f}f = f.$$

This shows η is faithful and hence an isomorphism between $\mathbb{X}$ and $\mathbf{INV}(\widetilde{\mathbb{X}})$. $\square$

Theorem 5.2.6. *The category **DInv** of discrete inverse categories is equivalent to the category **DCartRest** of discrete Cartesian restriction categories.*

Proof. Letting $\mathbf{T} : \mathbb{X} \to \widetilde{\mathbb{X}}$ be the functor that takes $\mathbb{X}$ to its Cartesian Completion, then from the above lemmas, we have shown that we have an adjoint:

$$(\eta, \varepsilon) : \mathbf{T} \vdash \mathbf{INV} : \mathbf{DInv} \to \mathbf{DCartRest}. \tag{5.12}$$

By Lemma 5.2.5 we know η is an isomorphism. But this means the functor $\mathbf{T}$ is full and faithful, as shown in, e.g., Proposition 2.2.6 of [19]. From lemma 5.2.1 we know that $\mathbf{INV}$ is full and faithful. But again by the previous reference, this means ε is an isomorphism. Thus, by Corollary 5.2.3 and Proposition 2.2.7 of [19] we have the equivalence of the two categories. $\square$

$$\begin{tikzcd}
\mathbf{DCartRest} \arrow[r, hook] \arrow[d, leftrightarrow] \arrow[r, phantom, shift left=2, "\top"] & \mathbf{CartRest} \arrow[l, bend right] \\
\mathbf{DInv} &
\end{tikzcd}$$

Figure 5.2: Functors between Cartesian restriction categories and inverse categories.

Thus, we may now draw out the relationship between Cartesian restriction categories, discrete Cartesian restriction categories and discrete inverse categories in Figure 5.2. In the figure, the arrow from discrete Cartesian restriction categories to Cartesian restriction categories is the standard embedding and the reverse arrow picks out the discrete objects in the Cartesian restriction category. Of course, the terminal object is always discrete, as noted in Example 3.10.8.

5.3 Examples of the Cartesian Construction

Example 5.3.1 (Different inverse products produce different $\widetilde{\mathbb{X}}$)**.**

Continuing from Example 4.3.4, recall the discrete category of 4 elements with two different tensors. Completing these gives two different lattices: The straight line lattice and the diamond semi-lattice. Below are the details of these constructions.

Recall $\mathbb{D}$ has four elements a, b, c and d, and there are two possible inverse product tensors, given in Table 4.1. (Repeated here for your convenience).

$\otimes$	a	b	c	d
a	a	a	a	a
b	a	b	**b**	b
c	a	**b**	c	c
d	a	b	c	d

$\odot$	a	b	c	d
a	a	a	a	a
b	a	b	**a**	b
c	a	**a**	c	c
d	a	b	c	d

Table 5.1: Two different inverse products on the same category.

Define Δ as the identity map. Then, for the first tensor, $\otimes$ of Table 4.1, $\widetilde{\mathbb{D}}$ has the following maps

$$\begin{array}{cccc}
a \xrightarrow{(id,a)\ (\equiv(id,b)\equiv(id,c)\equiv(id,d))} a, & a \xrightarrow{(id,a)} b, & a \xrightarrow{(id,a)} c, & a \xrightarrow{(id,a)} d \\
 & b \xrightarrow{(id,b)\ (\equiv(id,c)\equiv(id,d))} b, & b \xrightarrow{(id,b)} c, & b \xrightarrow{(id,b)} d \\
 & & c \xrightarrow{(id,c)\ (\equiv(id,d))} c, & c \xrightarrow{(id,c)} d \\
 & & & d \xrightarrow{(id,d)} d.
\end{array}$$

This shows $\widetilde{\mathbb{D}}$ is the straight-line $(a \to b \to c \to d)$ lattice. The tensor in $\mathbb{D}$ becomes the meet and hence is a categorical product in $\widetilde{\mathbb{D}}$. Note that the only partial inverses in $\widetilde{\mathbb{D}}$ are the identity functions and that for all maps f, $\langle f, 1\rangle = id$.

With the second tensor, $\odot$ from Table 4.1, we have:

$$\begin{array}{cccc}
a \xrightarrow{(id,a)\ (\equiv(id,b)\equiv(id,c)\equiv(id,d))} a, & a \xrightarrow{(id,a)} b, & a \xrightarrow{(id,a)} c, & a \xrightarrow{(id,a)} d \\
 & & b \xrightarrow{(id,b)\ (\equiv(id,d))} b, & b \xrightarrow{(id,b)} d \\
 & & c \xrightarrow{(id,c)\ (\equiv(id,d))} c, & c \xrightarrow{(id,c)} d \\
 & & & d \xrightarrow{(id,d)} d.
\end{array}$$

In this case, $\widetilde{\mathbb{D}}$ results in the "diamond" lattice, $\begin{tikzcd}[row sep=tiny, column sep=tiny] & b \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & d \\ & c \arrow[ur] & \end{tikzcd}$. Once again, the tensor in $\mathbb{D}$ becomes the meet.

Example 5.3.2 (Lattice completion)**.** Suppose we have a set together with an idempotent, commutative, associative operation $\wedge$ on the set, giving us a lattice, $\mathbb{L}$. Further suppose the set is partially ordered via $\leq$ with the order being compatible with $\wedge$.

Then, we may create a pullback square for any $x' \leq x,\ y' \leq x$ with

$$\begin{tikzcd} & x & \\ x' \arrow[ur, "\leq"'] & & y' \arrow[ul, "\leq"] \\ & x' \wedge y'. \arrow[ul, "\leq"] \arrow[ur, "\leq"'] & \end{tikzcd}$$

Considering $\mathbb{L}$ as a category, we see that all maps are monic and therefore, we may create a partial map category $\mathrm{Par}(\mathbb{L}, \mathcal{M})$ where the stable system of monics are all the maps.

Then $\widetilde{\mathrm{Par}(\mathbb{L}, \mathcal{M})}$ becomes the completion of the lattice over $\wedge$.

Example 5.3.3 ($\widetilde{\text{PINJ}}$ is PAR)**.** Noting that the objects of both PINJ and PAR are sets, we simply need to show that any partial function is in the equivalence class of some f, a map in PINJ.

Suppose we are given $g : A \to B = \{(a, b)|a \in A, b \in B\}$, a partial function in sets. Of course, if it is a partial injective function, then g is in the equivalence class of $(g, \{*\})$ and we are done. If it is not injective, that means there are one or more elements of B which appear in the right hand element of g multiple times. Construct a new function h as follows:

$$h := \{(a, (b, a))|(a, b) \in g\}. \tag{5.13}$$

By its definition, $h : A \to B \otimes A$ is injective, $(h, A) : A \to B$ coincides with g and therefore we see that using the Cartesian Construction on PINJ results in PAR.

Chapter 6

Disjointness in inverse categories

This chapter explores coproduct like structure in inverse categories. It starts by showing, that similar to the product, having coproducts is too restrictive a notion for inverse categories: An inverse category with a coproduct is a preorder. Nonetheless it is possible to define coproduct like structures in an inverse category. To introduce this structure we define a "disjointness" relation between parallel maps of an inverse category and whence a "disjoint join" for disjoint maps. The next chapter will then show how a tensor satisfying certain specific conditions gives rise to both a disjointness relation and a disjoint join. Such a tensor provides the replacement for "coproducts" in an inverse category.

6.1 Coproducts in inverse categories

A restriction category can have coproducts and an initial object. For example, Par (sets and partial functions) has coproducts.

Definition 6.1.1. In a restriction category $\mathbb{R}$, a coproduct is a *restriction coproduct* when the embeddings $\amalg_1$ and $\amalg_2$ are total.

Lemma 6.1.2. *A restriction coproduct* $+$ *in* $\mathbb{R}$ *satisfies:*

(i) $\overline{f+g} = \overline{f} + \overline{g}$ *which means* $+$ *is a restriction functor.*

(ii) $\nabla : A + A \to A$ *is total.*

(iii) $? : 0 \to A$ *is total, where* 0 *is the initial object in the restriction category.*

Proof.

(i) $+$ **is a restriction functor.** Consider the diagram:

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\amalg_1"'] & & A' \arrow[dr, "\amalg_1'"] & \\
& A+B \arrow[rr, dotted, "f+g"] & & A'+B' \\
B \arrow[rr, "g"'] \arrow[ur, "\amalg_2"] & & B'. \arrow[ur, "\amalg_2'"'] &
\end{tikzcd}$$

In order to show $\overline{f+g} = \overline{f}+\overline{g}$, it suffices to show that $\amalg_1\overline{f+g} = \amalg_1(\overline{f}+\overline{g}) = \overline{f}\amalg_1$.

$$\begin{aligned}
\amalg_1\overline{f+g} &= \overline{\amalg_1(f+g)}\amalg_1 && [\mathbf{R.4}] \\
&= \overline{f\amalg_1'}\amalg_1 && \text{coproduct diagram} \\
&= \overline{f\overline{\amalg_1'}}\amalg_1 && \text{Lemma 3.1.3[}(iii)\text{]} \\
&= \overline{f}\amalg_1 && \amalg_1' \text{ total.}
\end{aligned}$$

(ii) $\nabla : A+A \to A$ **is total.** By the definition of ∇ $(= \langle 1|1\rangle)$ and the coproduct, the following diagram commutes,

$$\begin{tikzcd}
& A+A \arrow[d, "\nabla"] & \\
A \arrow[ur, "\amalg_1"] \arrow[r, equal] & A \arrow[r, equal] & A \arrow[ul, "\amalg_2"']
\end{tikzcd}$$

resulting in:

$$\amalg_1\overline{\nabla} = \overline{\amalg_1\nabla}\amalg_1 = \overline{1}\amalg_1 = \amalg_1.$$

Similarly, $\amalg_2\overline{\nabla} = \amalg_2$, hence, the restriction of ∇ is 1 and therefore ∇ is total.

(iii) $? : 0 \to A$ **is total.** This follows from

$$\begin{tikzcd}
0 \arrow[r, "\amalg_2"] \arrow[dr, "?"'] & A+0 \arrow[d, equal] \\
& A
\end{tikzcd}$$

so $?$ can be defined as the total coproduct injection.

$\square$

Recall that when an object is both initial and terminal, it is referred to as a zero object and denoted as $\mathbf{0}$. This gives rise to the zero map $\mathbf{0}_{A,B} : A \to \mathbf{0} \to B$ between any two objects. Note that for all $f : D \to A, g : B \to C$, we have $f\mathbf{0}_{A,B}g = \mathbf{0}_{D,C}$ as this factors through the zero object.

Definition 6.1.3. Given a restriction category $\mathbb{X}$ with a zero object, then $\mathbf{0}$ is a *restriction zero* when for each object A in $\mathbb{X}$, $\overline{\mathbf{0}_{A,A}} = \mathbf{0}_{A,A}$.

Lemma 6.1.4 (Cockett-Lack [23], Lemma 2.7)**.** *For a restriction category $\mathbb{X}$, the following are equivalent:*

(i) $\mathbb{X}$ has a restriction zero;

(ii) $\mathbb{X}$ has an initial object 0 and terminal object 1 and each initial map z_A is a restriction monic;

(iii) $\mathbb{X}$ has a terminal object 1 and each terminal map t_A is a restriction retraction.

6.1.1 Inverse categories with restriction coproducts

Proposition 6.1.5. *An inverse category $\mathbb{X}$ with restriction coproducts is a preorder.*

Proof. By Lemma 6.1.2, ∇ is total and therefore $\nabla\nabla^{(-1)} = 1$. From the coproduct diagrams, $\amalg_1\nabla = 1$ and $\amalg_2\nabla = 1$. But this gives $\nabla^{(-1)}{\amalg_1}^{(-1)} = (\amalg_1\nabla)^{(-1)} = 1$ and similarly $\nabla^{(-1)}{\amalg_2}^{(-1)} = 1$. Hence, $\nabla^{(-1)} = \amalg_1$ *and* $\nabla^{(-1)} = \amalg_2$.

This means for parallel maps $f, g : A \to B$,

$$f = \amalg_1[f, g] = \nabla^{(-1)}[f, g] = \amalg_2[f, g] = g$$

and therefore $\mathbb{X}$ is a preorder. $\square$

6.2 Disjointness in an inverse category

This section and the next add a relation, *disjointness*, and an operation, *disjoint join*, on parallel maps in an inverse category with a restriction zero and zero maps. The disjoint join is evocative of the join as defined in Section 3.3. In later chapters, we shall see how the disjoint join will allow us to add a tensor to a discrete inverse category, which will then become a coproduct via the Cartesian Construction from Chapter 5. This section will begin by defining disjointness on parallel maps and then show this is equivalent to a definition of disjointness on the restriction idempotents.

From this point forward in the thesis, we will work with a number of relations and operations on parallel pairs of maps. Suppose there is a relation $\Diamond$ between maps $f, g : B \to C$, i.e., $f \Diamond g$. Then, $\Diamond$ will be referred to as *stable* whenever given $h : A \to B$, then $hf \Diamond hg$. As well, $\Diamond$ will be referred to as *universal* whenever given $k : C \to D$, then $fk \Diamond gk$.

Definition 6.2.1. In an inverse category $\mathbb{X}$ with zero maps, the relation $\perp$ between two parallel maps $f, g : A \to B$ is called a *disjointness relation* when it satisfies the following properties:

- [**Dis.1**] For all $f : A \to B,\ f \perp 0$; (Zero is disjoint to all maps)
- [**Dis.2**] $f \perp g$ implies $\overline{f} g = 0$; (Disjoint maps have no intersection)
- [**Dis.3**] $f \perp g,\ f' \leq f,\ g' \leq g$ implies $f' \perp g'$; (Disjointness is down closed)
- [**Dis.4**] $f \perp g$ implies $g \perp f$; (Symmetric)
- [**Dis.5**] $f \perp g$ implies $hf \perp hg$; (Stable)
- [**Dis.6**] $f \perp g$ implies $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$; (Closed under range and restriction)
- [**Dis.7**] $\overline{f} \perp \overline{g},\ \hat{h} \perp \hat{k}$ implies $fh \perp gk$ (Determined by restriction/range).

When $f \perp g$, we will say f is *disjoint* from g.

Lemma 6.2.2. *In 6.2.1, given [**Dis.1-5**], the axioms [**Dis.6**] and [**Dis.7**] may be replaced by:*

$$[\textbf{\textit{Dis.6'}}] \quad f \perp g \iff \overline{f} \perp \overline{g} \text{ and } \hat{f} \perp \hat{g}.$$

Proof. First, we show [**Dis.6**] and [**Dis.7**] imply [**Dis.6'**]:

Given [**Dis.6**], the $\implies$ direction of [**Dis.6'**] is immediate. To show the other direction, assume $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$. This also means that $\overline{\overline{f}} \perp \overline{\overline{g}}$. Then, by [**Dis.7**], $\overline{f}f \perp \overline{g}g$ and therefore $f \perp g$. Thus [**Dis.6**] and [**Dis.7**] imply [**Dis.6'**].

Next, we must show [**Dis.6'**] implies [**Dis.6**] and [**Dis.7**]:

Given [**Dis.6'**], [**Dis.6**] follows immediately. To show [**Dis.7**], assume $\overline{f} \perp \overline{g}$, $\hat{h} \perp \hat{k}$. As $\overline{fh} \leq \overline{f}$ and $\overline{gk} \leq \overline{g}$, [**Dis.3**] gives us that $\overline{fh} \perp \overline{gk}$. Similarly, $\widehat{fh} \leq \hat{h}$ and $\widehat{gk} \leq \hat{k}$, giving us $\widehat{fh} \perp \widehat{gk}$. Then, from [**Dis.6'**] we may conclude $fh \perp gk$, showing [**Dis.7**] holds. $\square$

Lemma 6.2.3. *In an inverse category $\mathbb{X}$ with $\perp$ a disjointness relation:*

(i) $f \perp g$ if and only if $f^{(-1)} \perp g^{(-1)}$;

(ii) $f \perp g$ implies $fh \perp gh$ (Universal);

(iii) $f \perp g$ implies $f\hat{g} = 0$;

(iv) if m, n are monic, then $fm \perp gn$ implies $\overline{f} \perp \overline{g}$;

(v) if m, n are monic, then $m^{(-1)}f \perp n^{(-1)}g$ implies $\hat{f} \perp \hat{g}$.

Proof.

(i) Assume $f \perp g$. By [**Dis.6**], $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$. Since $\hat{f} = \overline{f^{(-1)}}$ and $\overline{f} = \widehat{f^{(-1)}}$, this means $\overline{f^{(-1)}} \perp \overline{g^{(-1)}}$ and $\widehat{f^{(-1)}} \perp \widehat{g^{(-1)}}$. By [**Dis.6'**] from Lemma 6.2.2, $f^{(-1)} \perp g^{(-1)}$. The converse follows with a similar argument.

(ii) Assume $f \perp g$. By (i), $f^{(-1)} \perp g^{(-1)}$. By [**Dis.5**], $h^{(-1)}f^{(-1)} \perp h^{(-1)}g^{(-1)}$, giving $(fh)^{(-1)} \perp (gh)^{(-1)}$. Applying (i), we now have $fh \perp gh$.

(iii) Assume $f \perp g$. From (i) and symmetry, we know that $g^{(-1)} \perp f^{(-1)}$ and therefore $\overline{g^{(-1)}}f^{(-1)} = \hat{g}f^{(-1)} = 0$. However, in an inverse category, $0^{(-1)} = 0$ and therefore $0 = (\hat{g}f^{(-1)})^{(-1)} = f\hat{g}^{(-1)} = f\hat{g}$.

(iv) Assume $fm \perp gn$ where m, n are monic. By [**Dis.6**], this gives $\overline{fm} \perp \overline{gn}$. By Lemma 3.1.3, $\overline{fm} = \overline{f\overline{m}} = \overline{f1} = \overline{f}$ and therefore $\overline{f} \perp \overline{g}$.

(v) By assumption, we have $m^{(-1)}f \perp n^{(-1)}g$ and therefore $f^{(-1)}m \perp g^{(-1)}n$. By (iv), this means $\overline{f^{(-1)}} \perp \overline{g^{(-1)}}$ and hence $\hat{f} \perp \hat{g}$.

□

We may equivalently define a disjointness relation by a relation on the restriction idempotents, $\mathcal{O}(A)$.

Definition 6.2.4. Given an inverse category $\mathbb{X}$, a relation $\perp_A \subseteq \mathcal{O}(A) \times \mathcal{O}(A)$ for each $A \in \mathbb{X}_o$, is an *open disjointness* relation when for all $e, e' \in \mathcal{O}(A)$

[$\mathcal{O}$**dis.1**] $1 \perp_A 0$; (Zero disjoint to identity)

[$\mathcal{O}$**dis.2**] $e \perp_A e'$ implies $ee' = 0$; (Disjoint implies no intersection)

[$\mathcal{O}$**dis.3**] $e \perp_A e',\ e_1 \leq e,\ e_1' \leq e'$ implies $e_1 \perp_A e_1'$; (Down closed)

[$\mathcal{O}$**dis.4**] $e \perp_A e'$ implies $e' \perp_A e$; (Symmetric)

[$\mathcal{O}$**dis.5**] $e \perp_A e'$ implies $\overline{fe} \perp_B \overline{fe'}$ for all $f : B \to A$; (Stable).

Note that as we are in an inverse category, [$\mathcal{O}$**dis.5**] immediately implies that $e \perp_A e'$ gives us $\widehat{eg} \perp_C \widehat{e'g}$ for all $g : A \to C$ by simply taking the inverses and recalling that $\hat{g} = \overline{g^{(-1)}}$.

We will normally write $\perp$ rather than $\perp_A$ where the object is clear.

Proposition 6.2.5. *If $\perp$ is a disjointness relation in $\mathbb{X}$, then $\perp = \perp \cap (\cup_{A \in \mathbb{X}} \mathcal{O}(A) \times \mathcal{O}(A))$, the restriction of $\perp$ to the restriction idempotents, is an open disjointness relation.*

Proof.

[$\mathcal{O}$**dis.1**] This follows immediately from [**Dis.1**] by taking $f = 1$.

[$\mathcal{O}$**dis.2**] By [**Dis.2**], $0 = \overline{e}e' = ee'$.

[$\mathcal{O}$**dis.3**] Assuming $e \perp e'$ and $e_1 \leq e,\ e_1' \leq e'$, by [**Dis.3**], $e_1 \perp e_1'$.

[$\mathcal{O}$**dis.4**] Symmetry follows directly from [**Dis.4**].

[$\mathcal{O}$**dis.5**] Given $e \perp e'$, this means $fe \perp fe'$ by [**Dis.5**]. Then, [**Dis.6**] gives a conclusion of $\overline{fe} \perp \overline{fe'}$.

Therefore, $\underline{\perp} = \perp \cap (\cup_{A \in \mathbb{X}} \mathcal{O}(A) \times \mathcal{O}(A))$ acts as an open disjointness relation on $\mathcal{O}(A)^2$.

$\square$

For the converse, if $\underline{\perp}$ is an open disjointness relation in $\mathbb{X}$, then define a relation ${}_A{\perp}_B \subseteq \cup_{A,B \in \mathbb{X}} \mathbb{X}(A, B) \times \mathbb{X}(A, B)$ on parallel maps by

$$f \, {}_A{\perp}_B \, g \iff \overline{f} \, \underline{\perp} \, \overline{g} \text{ and } \hat{f} \, \underline{\perp} \, \hat{g}.$$

Note the use of the $\hat{f} \, \underline{\perp} \, \hat{g}$. By [**RR.1**], $\overline{\hat{f}} = \hat{f}$ and therefore it is a restriction idempotent in $\mathcal{O}(B)$. The relation ${}_A{\perp}_B$ is a disjointness relation:

Proposition 6.2.6. *If $\underline{\perp}$ is an open disjointness relation in $\mathbb{X}$, then*

$$f \, {}_A{\perp}_B \, g \iff \overline{f} \, \underline{\perp} \, \overline{g} \textit{ and } \hat{f} \, \underline{\perp} \, \hat{g}$$

is a disjointness relation in $\mathbb{X}$.

Proof.

[**Dis.1**] This requires showing $f \perp 0$ for any f. The assumption is that $1 \, \underline{\perp} \, 0$ and therefore $\overline{f} \, \underline{\perp} \, 0$ and $\hat{f} \, \underline{\perp} \, 0$, as $\overline{f} \leq 1$ and $\hat{f} \leq 1$. This gives $f \perp 0$.

[**Dis.2**] Assume $f \perp g$, i.e., $\overline{f} \, \underline{\perp} \, \overline{g}$. Then, $\overline{f}g = \overline{f}\,\overline{g}g = 0g = 0$.

[**Dis.3**] This axiom assumes $f \perp g$, $f' \leq f$ and $g' \leq g$. By Lemma 3.2.2[(i)] $\overline{f'} \leq \overline{f}$ and $\overline{g'} \leq \overline{g}$. Then, $\overline{f'} \,\underline{\perp}\, \overline{g'}$ by [$\mathcal{O}$**dis.3**], as $\overline{f} \,\underline{\perp}\, \overline{g}$. By Lemma 3.6.3[(ii)], both $\widehat{f'} \leq \hat{f}$ and $\widehat{g'} \leq \hat{g}$ and by [$\mathcal{O}$**dis.3**], $\widehat{f'} \,\underline{\perp}\, \widehat{g'}$ as $\hat{f} \,\underline{\perp}\, \hat{g}$. This means $f' \perp g'$.

[**Dis.4**] Symmetry of $\perp$ follows immediately from the symmetry of $\underline{\perp}$.

[**Dis.5**] Assume $f \perp g$, i.e., $\overline{f} \,\underline{\perp}\, \overline{g}$ and $\hat{f} \,\underline{\perp}\, \hat{g}$. By [$\mathcal{O}$**dis.5**], $\overline{hf} \,\underline{\perp}\, \overline{hg}$. By Lemma 3.6.3[(i)], $\widehat{hf} \leq \hat{f}$ and $\widehat{hg} \leq \hat{g}$. Therefore $\widehat{hf} \,\underline{\perp}\, \widehat{hg}$ by [$\mathcal{O}$**dis.3**] and therefore $hf \perp hg$.

[**Dis.6**] Assume $f \,{}_{A}{\perp}_{B}\, g$ which gives both $\overline{f} \,\underline{\perp}\, \overline{g}$ and $\hat{f} \,\underline{\perp}\, \hat{g}$. By Lemma 3.6.2 for any map h, $\hat{\overline{h}} = \overline{h}$ and by Lemma 3.1.3 we have $\overline{\overline{h}} = \overline{h}$. Thus, we have both $\overline{\overline{f}} \,\underline{\perp}\, \overline{\overline{g}}$ and $\hat{\overline{f}} \,\underline{\perp}\, \hat{\overline{g}}$ and therefore $\overline{f} \,{}_{A}{\perp}_{A}\, \overline{g}$. Similarly for any map h, [**RR.1**] gives $\overline{\hat{h}} = \hat{h}$ and Lemma 3.6.2 gives $\hat{\hat{h}} = \hat{h}$. This means $\overline{\hat{f}} \,\underline{\perp}\, \overline{\hat{g}}$ and $\hat{\hat{f}} \,\underline{\perp}\, \hat{\hat{g}}$ which gives $\hat{f} \,{}_{B}{\perp}_{B}\, \hat{g}$.

[**Dis.7**] Start with the assumption $\overline{f} \perp \overline{g}$ and $\hat{h} \perp \hat{k}$, which gives $\overline{f} \,\underline{\perp}\, \overline{g}$ and $\hat{h} \,\underline{\perp}\, \hat{k}$. By Lemma 3.2.2[(ii)], both $\overline{fh} \leq \overline{f}$ and $\overline{gk} \leq \overline{g}$. Therefore, $\overline{fh} \,\underline{\perp}\, \overline{gk}$ by [$\mathcal{O}$**dis.3**]. By Lemma 3.6.3[(i)], $\widehat{fh} \leq \hat{h}$ and $\widehat{gk} \leq \hat{k}$, giving us $\widehat{fh} \,\underline{\perp}\, \widehat{gk}$ also by [$\mathcal{O}$**dis.3**]. This means $fh \perp gk$.

$\square$

Theorem 6.2.7. *To give a disjointness relation $\perp$ on $\mathbb{X}$ is to give an open disjointness relation $\underline{\perp}$ on $\mathbb{X}$.*

Proof. This requires showing there is a bijection between disjointness relations and open disjointness relations. That is, give a disjointness relation $\perp$, it generates an open disjointness relation, $\underline{\perp}$. We then need to show that the disjointness relation generated from $\underline{\perp}$ is in fact $\perp$.

Starting with the disjointness relation $\perp$, by Proposition 6.2.5, this is an open disjointness $\underline{\perp} = \perp \cap (\cup_{A \in \mathbb{X}} \mathcal{O}(A) \times \mathcal{O}(A))$.

By Proposition 6.2.6, ${}_{A}{\perp}_{B}$ defined from $\underline{\perp}$ is a disjointness relation on $\mathbb{X}$.

Assume $f \perp g$. By [**Dis.6**] and Proposition 6.2.5, $\overline{f} \underline{\perp} \overline{g}$ and $\hat{f} \underline{\perp} \hat{g}$. Then by its definition, we have $f\,{}_A\!\perp_B g$.

Assume $f\,{}_A\!\perp_B g$, which required that both $\overline{f} \underline{\perp} \overline{g}$ and $\hat{f} \underline{\perp} \hat{g}$. Therefore $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$. By Proposition 6.2.3, $f \perp g$.

We have shown $f \perp g \iff f\,{}_A\!\perp_B g$. We may also conclude that if we started with an open disjointness relation $\underline{\perp}$ and used it to construct ${}_A\!\perp_B$, then the relation ${}_A\!\perp_B \cap (\cup_{A \in \mathbb{X}} \mathcal{O}(A) \times \mathcal{O}(A))$ would again be $\underline{\perp}$.

Hence there is a bijection between disjointness relations and open disjointness relations on an inverse category $\mathbb{X}$. $\square$

Disjointness is additional structure on a restriction category, i.e., it is possible to have more than one disjointness relation on the category. To see this, consider the trivial disjointness relation, where $f \perp_0 g$ if and only if $f = 0$ or $g = 0$. As $\overline{0} = 0 = \hat{0}$, $f \leq 0 \iff f = 0$ and $h0 = 0$ for any map h, axioms [**Dis.1**] through [**Dis.7**] are immediately satisfied. $\perp_0$ is definable on any inverse category with zero maps, hence is definable on PINJ.

However, PINJ does have a more useful disjointness relation:

Example 6.2.8 (PINJ has a disjointness relation)**.** Consider the inverse category PINJ, introduced in Example 3.1.8 and Example 4.3.2. Note the restriction zero is the empty set, $\emptyset$. Recalling that a map $f : A \to B$ in PINJ is given by the set $\{(a, b) | a \in A, b \in B\}$ (see Example 2.6.5), we see the initial map $?_A : \emptyset \to A$ must be $\{\}$, i.e., the empty set. Similarly, $!_A : A \to \emptyset$ must be the partial map also given by $\emptyset$ and therefore $\mathbf{0}_{A,B} = \emptyset$.

Define the disjointness relation $\perp$ by $f \perp g$ if and only if $\overline{f} \cap \overline{g} = \emptyset$ and $\hat{f} \cap \hat{g} = \emptyset$. It is then reasonably straightforward to verify [**Dis.1**] through [**Dis.7**]. For example, take [**Dis.7**]:

Proof. We are given $\overline{f} \perp \overline{g}$ and $\hat{h} \perp \hat{k}$. This means

$$\overline{f} \cap \overline{g} = \emptyset \text{ and } \hat{h} \cap \hat{k} = \emptyset.$$

As discussed in Example 3.4.4, both $\overline{mn} \subseteq \overline{m}$ and $\widehat{mn} \subseteq \hat{n}$. Hence,

$$\overline{fh} \cap \overline{gk} \subseteq \overline{f} \cap \overline{g} = \emptyset$$
$$\widehat{fh} \cap \widehat{gk} \subseteq \hat{h} \cap \hat{k} = \emptyset.$$

Therefore, $fh \perp gk$. □

The open disjointness relation, $\perp\!\!\!\perp$, on the idempotents is given by $e \perp\!\!\!\perp e' \iff e \cap e' = \emptyset$.

Although disjointness is additional structure, one can use the disjointness structure of the base categories to define a disjointness structure on the product category.

Lemma 6.2.9. *If $\mathbb{X}$ and $\mathbb{Y}$ are inverse categories with restriction zeros and respective disjointness relations $\perp$ and $\perp'$, then there is a disjointness relation $\perp_\times$ on $\mathbb{X} \times \mathbb{Y}$.*

Proof. The restriction, the inverse and the restriction zero on the product category are defined pointwise.

- If (f, g) is a map in $\mathbb{X} \times \mathbb{Y}$, then $(f, g)^{(-1)} = (f^{(-1)}, g^{(-1)})$;
- If (f, g) is a map in $\mathbb{X} \times \mathbb{Y}$, then $\overline{(f, g)} = (\overline{f}, \overline{g})$;
- The map $(\mathbf{0}_X, \mathbf{0}_Y)$ is the restriction zero in $\mathbb{X} \times \mathbb{Y}$.

Following this pattern, for (f, g) and (h, k) maps in $\mathbb{X} \times \mathbb{Y}$, $(f, g) \perp_\times (h, k)$ iff $f \perp h$ and $g \perp' k$.

Verifying the disjointness axioms is straightforward, we show axioms 2 and 5. Proofs of the others are similar.

[**Dis.2**] : Given $(f, g) \perp_\times (h, k)$, $\overline{(f, g)}(h, k) = (\overline{f}, \overline{g})(h, k) = (\overline{f}h, \overline{g}k) = (\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

[**Dis.5**] : The assumption is $(f, g) \perp_\times (h, k)$. For the map $z = (x, y)$ in $\mathbb{X} \times \mathbb{Y}$, [**Dis.5**] in the base categories gives $xf \perp xh$ and $yg \perp yk$. Thus

$$z(f, g) = (xf, yg) \perp_\times (xh, yk) = z(h, k).$$

□

6.3 Disjoint joins

This section will now consider additional structure on the inverse category, the disjoint join, dependent upon the disjointness relation. Although we have only considered binary disjointness up to this point, extending disjointness to sets of maps is done by considering disjointness pairwise: $\perp \{f_1, f_2, \ldots, f_n\} := f_i \perp f_k$ whenever $i \neq j$.

Definition 6.3.1. An inverse category $\mathbb{X}$ with a restriction 0 and a disjointness relation $\perp$ has *disjoint joins* when there is a binary operator on disjoint parallel maps:

$$\frac{f : A \to B,\ g : A \to B,\ f \perp g}{f \sqcup g : A \to B}$$

such that the following hold:

- [**DJ.1**] $f \leq f \sqcup g$ and $g \leq f \sqcup g$;
- [**DJ.2**] $f \leq h,\ g \leq h$ and $f \perp g$ implies $f \sqcup g \leq h$;
- [**DJ.3**] $h(f \sqcup g) = hf \sqcup hg$. (Stable)
- [**DJ.4**] $\perp \{f, g, h\}$ if and only if $f \perp (g \sqcup h)$.

The binary operator, $\sqcup$, is called the *disjoint join*.

Given a specific disjointness relation on a category, there is only one disjoint join:

Lemma 6.3.2. *Suppose $\mathbb{X}$ in an inverse category with a disjointness relation $\perp$, then if $\sqcup$ and $\Box$ are disjoint joins for $\perp$ then $\sqcup = \Box$.*

Proof. [**DJ.1**] gives us:

$$f, g \leq f \sqcup g \text{ and } f, g \leq f \Box g.$$

By [**DJ.2**], $f \sqcup g \leq f \Box g$ and $f \Box g \leq f \sqcup g$, hence $f \sqcup g = f \Box g$. $\Box$

Lemma 6.3.3. *In an inverse category with disjoint joins, the disjoint join is a partial associative and commutative operation, with identity* $\mathbf{0}$*. Additionally, it respects the restriction and is universal. That is, the following hold:*

(i) $f \sqcup \mathbf{0} = f$*;*

(ii) $f \perp g,\ g \perp h,\ f \perp h$ *implies that* $(f \sqcup g) \sqcup h = f \sqcup (g \sqcup h)$*;*

(iii) $f \perp g$ *implies* $f \sqcup g = g \sqcup f$*;*

(iv) $\overline{f \sqcup g} = \overline{f} \sqcup \overline{g}$*;*

(v) $(f \sqcup g)k = fk \sqcup gk$ *(*$\sqcup$ *is universal).*

Proof.

(i) *Identity*: By [**DJ.1**], $f \leq f \sqcup \mathbf{0}$. As $\mathbf{0} \leq f$ and $f \leq f$, by [**DJ.2**], $f \sqcup \mathbf{0} \leq f$ and therefore $f = f \sqcup \mathbf{0}$.

(ii) *Associativity*: Note that [**DJ.4**] shows that both sides of the equation exist. From [**DJ.1**] $f \sqcup g, h \leq (f \sqcup g) \sqcup h$, which also means $f, g \leq (f \sqcup g) \sqcup h$. Similarly, $g \sqcup h \leq (f \sqcup g) \sqcup h$ and then $f \sqcup (g \sqcup h) \leq (f \sqcup g) \sqcup h$. Conversely, $f, g, h \leq f \sqcup (g \sqcup h)$ and therefore $(f \sqcup g) \sqcup h \leq f \sqcup (g \sqcup h)$ and both sides are equal.

(iii) *Commutativity*: Note first that both f and g are less than or equal to each of $f \sqcup g$ and $g \sqcup f$, by [**DJ.1**]. By [**DJ.2**], $f \sqcup g \leq g \sqcup f$ and $g \sqcup f \leq f \sqcup g$ and thus $f \sqcup g = g \sqcup f$.

(iv) As $\overline{f}, \overline{g} \leq \overline{f \sqcup g}$, [**DJ.2**] gives $\overline{f} \sqcup \overline{g} \leq \overline{f \sqcup g}$. To show the other direction, consider

$$\begin{aligned}
\overline{f}(\overline{f} \sqcup \overline{g})(f \sqcup g) &= (\overline{f}\,\overline{f} \sqcup \overline{f}\overline{g})(f \sqcup g) && [\mathbf{DJ.3}] \\
&= (\overline{f}\,\overline{f} \sqcup 0)(f \sqcup g) && [\mathbf{Dis.2}] \\
&= \overline{f}(f \sqcup g) && \text{(i), Lemma 3.1.3} \\
&= f.
\end{aligned}$$

Hence, $f \leq (\overline{f} \sqcup \overline{g})(f \sqcup g)$ and similarly, so is g. By [**DJ.2**] and $\overline{f} \sqcup \overline{g}$ being a restriction idempotent,

$$f \sqcup g \leq (\overline{f} \sqcup \overline{g})(f \sqcup g) \leq f \sqcup g$$

and therefore $f \sqcup g = (\overline{f} \sqcup \overline{g})(f \sqcup g)$. By Lemma 3.2.2, $\overline{f \sqcup g} \leq \overline{f} \sqcup \overline{g}$ and so $\overline{f \sqcup g} = \overline{f} \sqcup \overline{g}$.

(v) First consider when f, g and k are restriction idempotents, say e_0, e_1 and e_2. Then, $(e_0 \sqcup e_1)e_2 = e_2(e_0 \sqcup e_1) = e_2e_0 \sqcup e_2e_1 = e_0e_2 \sqcup e_1e_2$. Next, note that for general f, g, h, by [**DJ.2**] $fk \sqcup gk \leq (f \sqcup g)k$ as both $fk, gk \leq (f \sqcup g)k$. By Lemma 3.2.2, we need only show that their restrictions are equal:

$$\begin{aligned}
\overline{(f \sqcup g)k} &= \overline{\overline{f \sqcup g}(f \sqcup g)k} && [\mathbf{R.1}] \\
&= \overline{f \sqcup g}\,\overline{(f \sqcup g)k} && [\mathbf{R.3}] \\
&= (\overline{f} \sqcup \overline{g})\overline{(f \sqcup g)k} && \text{previous item} \\
&= \overline{f}\,\overline{(f \sqcup g)k} \sqcup \overline{g}\,\overline{(f \sqcup g)k} && \text{idempotent universal} \\
&= \overline{\overline{f}(f \sqcup g)k} \sqcup \overline{\overline{g}(f \sqcup g)k} && [\mathbf{R.3}] \\
&= \overline{fk} \sqcup \overline{gk} && [\mathbf{DJ.3}] \\
&= \overline{fk \sqcup gk}.
\end{aligned}$$

Therefore, as the restrictions are equal, $(f \sqcup g)k = fk \sqcup gk$.

$\square$

Note that the previous lemma and proof of associativity allows a simple inductive argument which shows that having binary disjoint joins extends to disjoint joins of an arbitrary finite collection of disjoint maps.

Recalling our notation for disjointness of a set of maps, $\sqcup\{f_i\}$ will mean the disjoint join of all maps f_i, i.e., $f_1 \sqcup f_2 \sqcup \cdots \sqcup f_n$.

Lemma 6.3.4. *In an inverse category $\mathbb{X}$ with disjoint joins,*

(i) $\perp \{f_i\}$ *if and only if* $\sqcup\{f_i\}$ *is defined,*

(ii) if $f_i, g_j : A \to B$ *and* $\perp_{i\in I} \{f_i\}$ *and* $\perp_{j\in J} \{g_j\}$, *then* $\sqcup_{i\in I}\{f_i\} \perp \sqcup_{j\in J}\{g_j\}$ *if and only* $f_i \perp g_j$ *for all* $i \in I$ *and* $j \in J$.

Proof. For (i), using [**Dj.4**], proceed as in the proof of Lemma 6.3.3[(ii)], inducting on n.

To show (ii), first assume $\sqcup\{f_i\} \perp \sqcup\{g_j\}$. By [**Dj.4**] and associativity, $\sqcup\{f_i\} \perp g_j$ for each j. Using the symmetry of $\perp$, [**Dj.4**] and associativity, $f_i \perp g_j$ for each i and j.

Next, assume $f_i \perp g_j$ for each i and j. Then by [**Dj.4**] and associativity, $f_i \perp \sqcup\{g_j\}$ for each i. Applying [**Dj.4**] again, $\sqcup\{f_i\} \perp \sqcup\{g_j\}$. $\square$

Clearly the product of two inverse categories with disjoint joins has a disjoint join:

Lemma 6.3.5. *Given $\mathbb{X}, \mathbb{Y}$ are inverse categories with disjoint joins, $\sqcup$ and $\sqcup'$ respectively, then the category $\mathbb{X} \times \mathbb{Y}$ is an inverse category with disjoint joins.*

Proof. From Lemma 6.2.9, $\mathbb{X}\times\mathbb{Y}$ has a disjointness relation that is defined point-wise. Define $\sqcup_\times$ the disjoint join on $\mathbb{X} \times \mathbb{Y}$ by

$$(f, g) \sqcup_\times (h, k) = (f \sqcup h, g \sqcup' k). \tag{6.1}$$

It remains to prove each of the axioms in Definition 6.3.1 hold.

[**DJ.1**] From Equation (6.1), since $f, h \le f \sqcup h$ and $g, k \le g \sqcup' k$, both $(f, g) \le (f, g) \sqcup_\times (h, k)$ and $(h, k) \le (f, g) \sqcup_\times (h, k)$.

[**DJ.2**] Suppose $(f, g) \leq (x, y)$, $(h, k) \leq (x, y)$ and $(f, g) \perp_\times (h, k)$. Then regarding it point-wise, $(f, g) \sqcup_\times (h, k) = (f \sqcup h, g \sqcup' k) \leq (x, y)$.

[**DJ.3**] Calculating,

$$(x, y)\,((f, g) \sqcup_\times (h, k)) = (x(f \sqcup h), y(g \sqcup' k)) =$$
$$(xf \sqcup xh, yg \sqcup' yk) = (xf, yg) \sqcup_\times (xh, yk) =$$
$$((x, y)(f, g)) \sqcup_\times ((x, y)(h, k)).$$

[**DJ.4**] Given $\perp_\times[(f, g), (h, k), (x, y)]$, both $f \perp (h \sqcup x)$ and $g \perp' (k \sqcup' y)$. Hence, $(f, g) \perp_\times ((h, k) \sqcup_\times (x, y))$. The opposite direction is similar.

$\square$

Example 6.3.6 (Pinj has a disjoint join)**.** Continuing from Example 6.2.8, we show that Pinj has disjoint joins. If $f = \{(a, b)\}$ and $g = \{(a', b')\}$ are disjoint parallel maps in Pinj from A to B, define $f \sqcup g := \{(a'', b'') | (a'', b'') \in f \text{ or } (a'', b'') \in g\}$, i.e., the union of f and g.

This is still a partial injective map, due to the requirement of disjointness. Recall that $f \perp g$ means that $\overline{f} \cap \overline{g} = \emptyset$ and $\hat{f} \cap \hat{g} = \emptyset$ and that the respective meets will also be $\emptyset$. The empty meet of the restrictions means that $f \sqcup g$ is still a partial function, as each a'' will appear only once. The empty meet of the ranges gives us that $f \sqcup g$ is injective, because each b'' is unique.

The axioms for disjoint joins all hold:

[**DJ.1**] By construction, both f and g are less than $f \sqcup g$.

[**DJ.2**] $f \leq h,\ g \leq h$ means that h must contain all of the $(a, b) \in f$ and $(a', b') \in g$ and therefore $f \sqcup g \leq h$.

[**DJ.3**] Suppose $h : C \to A = \{(c, \dot{a})\}$. Then

$$\begin{aligned} h(f \sqcup g) &= \{(c, \dot{b}) | (\exists a, \dot{a}.\dot{a} = a, (a, \dot{b}) \in f, (c, \dot{a}) \in h) \\ &\qquad \text{or } (\exists a', \dot{a}.\dot{a} = a', (a', \dot{b}) \in g, (c, \dot{a}) \in h)\} \\ &= \{(c, \dot{b}) | \exists a, \dot{a}.\dot{a} = a, (a, \dot{b}) \in f, (c, \dot{a}) \in h\} \bigcup \\ &\qquad \{(c, \dot{b}) | \exists a', \dot{a}.\dot{a} = a', (a', \dot{b}) \in g, (c, \dot{a}) \in h\} \\ &= hf \sqcup hg. \end{aligned}$$

[**DJ.4**] Suppose $\perp [f, f', f'']$, $f = \{(a, b)\}, f' = \{(a', b')\}, f'' = \{(a'', b'')\}$. Then the set $\{a\}$ intersects neither $\{a'\}$ nor $\{a''\}$ and similarly for the sets $\{b\}, \{b'\}$ and $\{b''\}$. Thus we have $f \perp (g \sqcup h)$. The reverse direction is argued similarly.

Chapter 7

Disjoint sums

This chapter introduces the disjoint sum and shows how it may be used to define a symmetric monoidal tensor in an inverse category. Conversely, it goes on to then show how a tensor, denoted as $\oplus$, may be used to create a disjointness relation and a disjoint join in an inverse category — at which point we can prove that $A \oplus B$ is a disjoint sum with respect to this structure.

7.1 Disjoint sums

This chapter explores what conditions will allow the definition of a tensor in an inverse category which already has a disjoint join. As we shall see, it is sufficient to have "enough" *disjoint sums*, as defined below.

Definition 7.1.1. In an inverse category with disjoint joins, an object X is the *disjoint sum* of A and B when there exist maps $i_1 : A \to X$, $i_2 : B \to X$ such that:

(i) i_1 and i_2 are total (equivalently – monic);

(ii) ${i_1}^{(-1)} i_1 \perp {i_2}^{(-1)} i_2$ and ${i_1}^{(-1)} i_1 \sqcup {i_2}^{(-1)} i_2 = 1_X$.

The maps i_1 and i_2 will be referred to as the *injection* maps of the disjoint sum. This arrangement may be visualized as:

$$\begin{tikzcd} A \arrow[r, "i_1"] & X \arrow[l, bend left, "{i_1}^{(-1)}"] \arrow[r, bend right, "{i_1}^{(-1)}"'] & B \arrow[l, "i_2"'] \end{tikzcd}$$

Lemma 7.1.2. *The disjoint sum X of A and B is unique up to isomorphism.*

Proof. Assume there are two disjoint sums over A and B:

$$\begin{tikzcd} A \arrow[r, "i_1"] & X \arrow[l, bend left, "{i_1}^{(-1)}"] \arrow[r, bend right, "{i_2}^{(-1)}"'] & B \arrow[l, "i_2"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[r, "j_1"] & Y \arrow[l, bend left, "{j_1}^{(-1)}"] \arrow[r, bend right, "{j_2}^{(-1)}"'] & B \arrow[l, "j_2"'] \end{tikzcd} .$$

We will show that the map ${i_1}^{(-1)}j_1 \sqcup {i_2}^{(-1)}j_2 : X \to Y$ is an isomorphism.

Note by the fact that i_1, i_2 are monic and [**Dis.2**], $0 = \overline{{i_1}^{(-1)}i_1}{i_2}^{(-1)} = \overline{{i_1}^{(-1)}\overline{i_1}}{i_2}^{(-1)} = \overline{{i_1}^{(-1)}}{i_2}^{(-1)} = \widehat{i_1}{i_2}^{(-1)}$. But then $i_1{i_2}^{(-1)} = i_1\widehat{i_1}{i_2}^{(-1)} = i_1 0 = 0$. Similarly, $i_2{i_1}^{(-1)} = 0$, $j_1{j_2}^{(-1)} = 0$ and $j_2{j_1}^{(-1)} = 0$.

Next, by Lemma 6.2.3, $\overline{{i_1}^{(-1)}} \perp \overline{{i_2}^{(-1)}}$ as both i_1 and i_2 are monic. By the same lemma, $\hat{j_1} \perp \hat{j_2}$ as ${j_1}^{(-1)}, {j_2}^{(-1)}$ are the inverses of monic maps. Then, from [**Dis.7**], ${i_1}^{(-1)}j_1 \perp {i_2}^{(-1)}j_2$, hence we may form ${i_1}^{(-1)}j_1 \sqcup {i_2}^{(-1)}j_2 : X \to Y$.

Similarly, we may form the map ${j_1}^{(-1)}i_1 \sqcup {j_2}^{(-1)}i_2 : Y \to X$. Computing their composition:

$$
\begin{aligned}
&({i_1}^{(-1)}j_1 \sqcup {i_2}^{(-1)}j_2)({j_1}^{(-1)}i_1 \sqcup {j_2}^{(-1)}i_2) \\
&\qquad = ({i_1}^{(-1)}j_1({j_1}^{(-1)}i_1 \sqcup {j_2}^{(-1)}i_2)) \sqcup ({i_2}^{(-1)}j_2({j_1}^{(-1)}i_1 \sqcup {j_2}^{(-1)}i_2)) \\
&\qquad = {i_1}^{(-1)}j_1{j_1}^{(-1)}i_1 \sqcup {i_1}^{(-1)}j_1{j_2}^{(-1)}i_2 \sqcup {i_2}^{(-1)}j_2{j_1}^{(-1)}i_1 \sqcup {i_2}^{(-1)}j_2{j_2}^{(-1)}i_2 \\
&\qquad = {i_1}^{(-1)}\,1\,i_1 \sqcup {i_1}^{(-1)}\,0\,i_2 \sqcup {i_2}^{(-1)}\,0\,i_1 \sqcup {i_2}^{(-1)}\,1\,i_2 \\
&\qquad = {i_1}^{(-1)}i_1 \sqcup {i_2}^{(-1)}i_2 = 1.
\end{aligned}
$$

Computing the other direction,

$$
\begin{aligned}
&({j_1}^{(-1)}i_1 \sqcup {j_2}^{(-1)}i_2)({i_1}^{(-1)}j_1 \sqcup {i_2}^{(-1)}j_2) \\
&\qquad = ({j_1}^{(-1)}i_1({i_1}^{(-1)}j_1 \sqcup {i_2}^{(-1)}j_2)) \sqcup ({j_2}^{(-1)}i_2({i_1}^{(-1)}j_1 \sqcup {i_2}^{(-1)}j_2)) \\
&\qquad = {j_1}^{(-1)}i_1{i_1}^{(-1)}j_1 \sqcup {j_1}^{(-1)}i_1{i_2}^{(-1)}j_2 \sqcup {j_2}^{(-1)}i_2{i_1}^{(-1)}j_1 \sqcup {j_2}^{(-1)}i_2{i_2}^{(-1)}j_2 \\
&\qquad = {j_1}^{(-1)}\,1\,j_1 \sqcup {j_1}^{(-1)}\,0\,j_2 \sqcup {j_2}^{(-1)}\,0\,j_1 \sqcup {j_2}^{(-1)}\,1\,j_2 \\
&\qquad = {j_1}^{(-1)}j_1 \sqcup {j_2}^{(-1)}j_2 = 1.
\end{aligned}
$$

This shows that the map between any two disjoint sums over the same two objects is an isomorphism. □

When there are "enough" disjoint sums in a category, this gives rise to tensor. In anticipation of the result, henceforth we will write $A \oplus B$ for the disjoint sum of A and B.

First, the next lemma shows how to construct maps between the disjoint sums:

Lemma 7.1.3. *Given $\mathbb{X}$ an inverse category with all disjoint sums and suppose $f : A \to C$ and $g : B \to D$ in $\mathbb{X}$. Then ${i_1}^{(-1)} f i_1 \perp {i_2}^{(-1)} g i_2 : A \oplus B \to C \oplus D$ and therefore ${i_1}^{(-1)} f i_1 \sqcup {i_2}^{(-1)} g i_2 : A \oplus B \to C \oplus D$.*

Proof. Note that $\overline{{i_1}^{(-1)} f i_1} = \overline{{i_1}^{(-1)} f} \le \overline{{i_1}^{(-1)}}$ and similarly $\overline{{i_2}^{(-1)} g i_2} \le \overline{{i_2}^{(-1)}}$. Then, by [**Dis.3**], we have $\overline{{i_1}^{(-1)} f i_1} \perp \overline{{i_2}^{(-1)} g i_2}$. Similarly, as $\widehat{{i_1}^{(-1)} f i_1} \le \widehat{i_1}$ and $\widehat{{i_2}^{(-1)} g i_2} \le \widehat{i_2}$, this gives $\widehat{{i_1}^{(-1)} f i_1} \perp \widehat{{i_2}^{(-1)} g i_2}$ and by Lemma 6.2.3, the conclusion is ${i_1}^{(-1)} f i_1 \perp {i_2}^{(-1)} g i_2$ and therefore the disjoint join ${i_1}^{(-1)} f i_1 \sqcup {i_2}^{(-1)} g i_2$ is a map between the two disjoint sums. $\square$

The argument in the above lemma is one we will use from time to time. In particular, this means $\overline{{i_1}^{(-1)} g} \perp \overline{{i_2}^{(-1)} h}$ for arbitrary g, h and that $\widehat{h i_1} \perp \widehat{k i_2}$.

Proposition 7.1.4. *Given $\mathbb{X}$ is an inverse category where every pair of objects has a disjoint sum, then the tensor $\oplus : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ where $A \oplus B$ is a disjoint sum of A, B is a symmetric monoidal tensor.*

Proof. It is necessary to show that $\oplus$ is a bi-functor and that the required structural maps exist.

First note that $f \oplus g$ is given by ${i_1}^{(-1)} f i_1 \sqcup {i_2}^{(-1)} g i_2$ as shown in Lemma 7.1.3. By the definition of the disjoint sum, identity maps are taken to identity maps and by the stability and universality of the disjoint join, composition is preserved and therefore $\oplus$ is a bi-functor as required.

We must now show the restriction zero is the unit of $\oplus$ and then give the structural maps.

Considering the disjoint sum $A \oplus 0$, we see that $i_2 = \mathbf{0}$, the zero map. This gives $1_{A \oplus \mathbf{0}} = {i_1}^{(-1)} i_1 \sqcup {i_2}^{(-1)} i_2 = {i_1}^{(-1)} i_1 \sqcup 0 = {i_1}^{(-1)} i_1 = \overline{{i_1}^{(-1)}}$, meaning ${i_1}^{(-1)}$ is total. Thus, ${i_1}^{(-1)}$ is an isomorphism and is $u_\oplus^r$. Similarly, ${i_2}^{(-1)} = u_\oplus^l : \mathbf{0} \oplus A \to A$ is an isomorphism.

For the symmetry map, note $c_\oplus = {i_1}^{(-1)} i_2 \sqcup {i_2}^{(-1)} i_1 : A \oplus B \to B \oplus A$ and that

$$
\begin{aligned}
({i_1}^{(-1)} i_2 \sqcup {i_2}^{(-1)} i_1)&({i_1}^{(-1)} i_2 \sqcup {i_2}^{(-1)} i_1) \\
&= {i_1}^{(-1)} i_2 ({i_1}^{(-1)} i_2 \sqcup {i_2}^{(-1)} i_1) \sqcup {i_2}^{(-1)} i_1 ({i_1}^{(-1)} i_2 \sqcup {i_2}^{(-1)} i_1) \\
&= {i_1}^{(-1)} i_2 {i_1}^{(-1)} i_2 \sqcup {i_1}^{(-1)} i_2 {i_2}^{(-1)} i_1 \sqcup {i_2}^{(-1)} i_1 {i_1}^{(-1)} i_2 \sqcup {i_2}^{(-1)} i_1 {i_2}^{(-1)} i_1 \\
&= 0 \sqcup 1 \sqcup 1 \sqcup 0 \\
&= 1.
\end{aligned}
$$

Thus, $c_\oplus c_\oplus = 1$.

For associativity, set $a_\oplus = {i_1}^{(-1)} i_1 i_1 \sqcup {i_2}^{(-1)} {i_1}^{(-1)} i_2 i_1 \sqcup {i_2}^{(-1)} {i_2}^{(-1)} i_2 : A \oplus (B \oplus C) \to (A \oplus B) \oplus C$. To visualize this,

$$
\begin{tikzcd}
B \oplus C \arrow[r, "{i_1}^{(-1)}"] \arrow[dr, "{i_2}^{(-1)}"'] & B \arrow[r, "i_2"] & A \oplus B \arrow[dd, "i_1"] \\
 & C \arrow[dr, "i_2"] & \\
A \oplus (B \oplus C) \arrow[uu, "{i_2}^{(-1)}"] \arrow[rr, dotted, "a_\oplus"] \arrow[d, "{i_1}^{(-1)}"'] & & (A \oplus B) \oplus C \\
A \arrow[rr, "i_1"'] & & A \oplus B. \arrow[u, "i_1"']
\end{tikzcd}
$$

The inverse of the $a_\oplus$ is obtained by taking the inverses of the arrows in the above diagram, yielding ${i_1}^{(-1)} {i_1}^{(-1)} i_1 \sqcup {i_1}^{(-1)} {i_2}^{(-1)} i_1 i_1 \sqcup {i_2}^{(-1)} i_2 i_2$.

Thus, $\oplus$ is a symmetric monoidal tensor on $\mathbb{X}$. $\square$

7.2 Disjointness via a tensor

The next objective is to characterize when a symmetric monoidal tensor provides a disjoint sum. The first step will be to show how a disjointness tensor, defined below, may be used to create a disjointness relation in an inverse category. This is done by first determining when it is possible for maps to work separately on the components of the tensor. This ability to separate the action of the maps will allow us to define when the restriction and range of functions are disjoint, and therefore when the maps are disjoint.

Suppose $\mathbb{X}$ is an inverse category with a restriction zero, and $\oplus$ is a tensor product on $\mathbb{X}$. As in Definition 2.8.1, we are assuming the following naming for the standard monoidal tensor isomorphisms:

$$u_\oplus^l : 0 \oplus A \to A \qquad\qquad u_\oplus^r : A \oplus 0 \to A$$

$$a_\oplus : (A \oplus B) \oplus C \to A \oplus (B \oplus C) \qquad\qquad c_\oplus : A \oplus B \to B \oplus A.$$

Definition 7.2.1. Given an inverse category $\mathbb{X}$ with restriction zero and $\oplus$ a symmetric monoidal tensor, then $\oplus$ is a *disjointness tensor* when:

(i) It is a restriction functor — i.e., $\_ \oplus \_ : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$.

(ii) The unit is the restriction zero. ($0 : \mathbf{1} \to \mathbb{X}$ picks out the restriction zero in $\mathbb{X}$).

(iii) Define $\amalg_1 = {u_\oplus^r}^{(-1)}(1 \oplus 0) : A \to A \oplus B$ and $\amalg_2 = {u_\oplus^l}^{(-1)}(0 \oplus 1) : A \to B \oplus A$. The maps $\amalg_1$ and $\amalg_2$ must be jointly epic. That is, if $\amalg_1 f = \amalg_1 g$ and $\amalg_2 f = \amalg_2 g$, then $f = g$.

(iv) Define $\amalg_1^* := (1 \oplus 0)u_\oplus^r : A \oplus B \to A$ and $\amalg_2^* := (0 \oplus 1)u_\oplus^l : A \oplus B \to B$. $\amalg_1^*$ and $\amalg_2^*$ must be jointly monic. That is, whenever $f\amalg_1^* = g\amalg_1^*$ and $f\amalg_2^* = g\amalg_2^*$ then $f = g$. (Note this is derivable by taking the inverses of maps in (iii), as will be shown below).

Example 7.2.2 (PINJ has a disjointness tensor)**.** In PINJ, the disjoint union, $\uplus$, is a disjointness tensor. Designate elements of the disjoint union as a pair (x, n) where x is an element of the sets in the disjoint union and n is the order of the disjoint union. That is, when

$$A = \{a\}, B = \{b\}, \text{ then } A \uplus B = \{(x, n) | n \in \{1, 2\}, n = 1 \implies x \in A, n = 2 \implies x \in B\}.$$

Setting $\oplus$ as $\uplus$, the identity for the tensor is $\emptyset$. The action of the tensor on maps $f : A \to C = \{(a,c)\}$, $g : B \to D = \{(b,d)\}$ is given by:

$$f \oplus g : A \oplus B \to C \oplus D = \{((x,n),(v,m)) | (x,v) \in f \text{ or } (x,v) \in g\}.$$

From our definitions above, define our tensor structure maps:

$$\begin{aligned}
u^l_\oplus &: 0 \oplus A \to A & (a,2) &\mapsto a, \\
u^r_\oplus &: A \oplus 0 \to A & (a,1) &\mapsto a, \\
a_\oplus &: (A \oplus B) \oplus C \to A \oplus (B \oplus C) & ((a,1),1) &\mapsto (a,1) \\
& & ((b,2),1) &\mapsto ((b,1),2) \\
& & (c,2) &\mapsto ((c,1),2), \\
c_\oplus &: A \oplus B \to B \oplus A & (a,1) &\mapsto (a,2) \\
& & (a,2) &\mapsto (a,1).
\end{aligned}$$

The map $\amalg_1 = {u^r_\oplus}^{(-1)}(1 \oplus 0)$ is given by the mapping $a \in A \mapsto (a,1) \in A \oplus B$. Similarly, $\amalg_2 = {u^l_\oplus}^{(-1)}(0 \oplus 1)$ is given by the mapping $a \in A \mapsto (a,2) \in B \oplus A$. From their definitions, $\amalg_1$ and $\amalg_2$ are jointly epic. Similarly, ${\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)}$ are jointly monic.

Lemma 7.2.3. *Given an inverse category $\mathbb{X}$ with restriction zero and $+$ a disjointness tensor, then the map $\mathbf{0} \oplus \mathbf{0} : A \oplus B \to C \oplus D$ is the map $\mathbf{0} : A \oplus B \to C \oplus D$.*

Proof. Recall the zero map factors through the restriction zero, i.e. $\mathbf{0} : A \to B$ is the same as saying $A \xrightarrow{!} \mathbf{0} \xrightarrow{?} B$. Additionally, as objects, $\mathbf{0} \oplus \mathbf{0} \cong \mathbf{0}$ — the restriction zero.

Therefore the map $\mathbf{0} \oplus \mathbf{0} : A \oplus B \to C \oplus D$ is writable as

$$A \oplus B \xrightarrow{!\oplus!} \mathbf{0} \oplus \mathbf{0} \xrightarrow{?\oplus?} C \oplus D,$$

which may then be rewritten as

$$A \oplus B \xrightarrow{!\oplus!} \mathbf{0} \oplus \mathbf{0} \xrightarrow{u^l_\oplus} \mathbf{0} \xrightarrow{{u^l_\oplus}^{(-1)}} \mathbf{0} \oplus \mathbf{0} \xrightarrow{?\oplus?} C \oplus D.$$

But by the properties of the restriction zero, $(!\oplus!)u_\oplus^l = !$ and ${u_\oplus^l}^{(-1)}(? \oplus ?) = ?$ and therefore the map $\mathbf{0} \oplus \mathbf{0} : A \oplus B \to C \oplus D$ is the same as the map $\mathbf{0} : A \oplus B \to C \oplus D$. $\square$

Lemma 7.2.4. *Given an inverse category $\mathbb{X}$ with restriction zero and $\oplus$ a disjointness tensor, then $\amalg_1^* = {\amalg_1}^{(-1)}$ and $\amalg_2^* = {\amalg_2}^{(-1)}$ and the following identities hold:*

(i) $\amalg_i^* \amalg_i = \overline{\amalg_i^*}$ *and* $\amalg_i \amalg_i^* = \overline{\amalg_i} = 1$*;*

(ii) $\overline{\amalg_1^*}\amalg_2^* = \mathbf{0}$ *and* $\overline{\amalg_2^*}\amalg_1^* = \mathbf{0}$*;*

(iii) $\amalg_2 \amalg_1^* = \mathbf{0}$*,* $\amalg_2 \overline{\amalg_1^*} = \mathbf{0}$*,* $\amalg_1 \amalg_2^* = \mathbf{0}$ *and* $\amalg_1 \overline{\amalg_2^*} = \mathbf{0}$*;*

(iv) the maps $\amalg_1$ *and* $\amalg_2$ *are monic.*

Proof.

(i) Recalling that the restriction zero is its own partial inverse,

$$ {\amalg_1}^{(-1)} = \left({u_\oplus^r}^{(-1)}(1 \oplus 0)\right)^{(-1)} = (1 \oplus 0)^{(-1)} u_\oplus^r = (1 \oplus 0) u_\oplus^r = \amalg_1^*. $$

Similarly,

$$ {\amalg_2}^{(-1)} = \left({u_\oplus^l}^{(-1)}(0 \oplus 1)\right)^{(-1)} = (0 \oplus 1) u_\oplus^l = \amalg_2^*. $$

Hence, calculating the restriction of $\amalg_1$,

$$ \amalg_1 \amalg_1^* = {u_\oplus^r}^{(-1)}(1 \oplus 0)(1 \oplus 0)u_\oplus^r = ({u_\oplus^r}^{(-1)}(1 \oplus 0))u_\oplus^r = 1{u_\oplus^r}^{(-1)}u_\oplus^r = 1. $$

The calculation for $\amalg_2^*$ and $\amalg_2$ is analogous.

(ii) To show $\overline{\amalg_1^*}\amalg_2^* = 0$ and $\overline{\amalg_2^*}\amalg_1^* = 0$,

$$ \overline{\amalg_1^*}\amalg_2^* = \overline{(1 \oplus 0)u_\oplus^r}(0 \oplus 1)u_\oplus^l = \overline{1 \oplus 0}(0 \oplus 1)u_\oplus^l = (1 \oplus 0)(0 \oplus 1)u_\oplus^l = (0 \oplus 0)u_\oplus^l = 0, $$

and

$$ \overline{\amalg_2^*}\amalg_1^* = \overline{(0 \oplus 1)u_\oplus^l}(1 \oplus 0)u_\oplus^r = (0 \oplus 1)(1 \oplus 0)u_\oplus^r = (0 \oplus 0)u_\oplus^r = 0. $$

(iii) $\amalg_i \amalg_j^* = 0$, $\amalg_i \overline{\amalg_j^*} = 0$ when $i \neq j$,

$$\amalg_1 \amalg_2^* = ({u_\oplus^r}^{(-1)}(1 \oplus 0))(0 \oplus 1)u_\oplus^l = {u_\oplus^r}^{(-1)}(0 \oplus 0)u_\oplus^l = 0$$

and

$$\amalg_2 \amalg_1^* = ({u_\oplus^l}^{(-1)}(0 \oplus 1))(1 \oplus 0)u_\oplus^r = {u_\oplus^l}^{(-1)}(0 \oplus 0)u_\oplus^r = 0.$$

As $\overline{\amalg_1^*} = 1 \oplus 0$ and $\overline{\amalg_2^*} = 0 \oplus 1$, we see the other two identities hold as well.

(iv) The first requirement is to show $\amalg_1$ is monic. Suppose $f\amalg_1 = g\amalg_1$. Therefore we must have

$$f = f(\amalg_1 {\amalg_1}^{(-1)}) = (f\amalg_1){\amalg_1}^{(-1)} = (g\amalg_1){\amalg_1}^{(-1)} = g(\amalg_1 {\amalg_1}^{(-1)}) = g.$$

The proof that $\amalg_2$ is monic follows via a similar argument.

□

As the above proof showed $\amalg_i^* = {\amalg_i}^{(-1)}$, the explicit notation of ${\amalg_i}^{(-1)}$ will be used for the remainder of this thesis.

Corollary 7.2.5. *In an inverse category $\mathbb{X}$ with a restriction zero and $\oplus$ a disjointness tensor, the following identities hold:*

(i) $\amalg_1(f \oplus g){\amalg_1}^{(-1)} = f$;

(ii) $\amalg_1(f \oplus g){\amalg_2}^{(-1)} = 0$;

(iii) $\amalg_2(f \oplus g){\amalg_1}^{(-1)} = 0$;

(iv) $\amalg_2(f \oplus g){\amalg_2}^{(-1)} = g$.

Additionally, if $t : A \oplus B \to C \oplus D$ is a map such that there are maps $t_1 : A \to C$ and $t_2 : B \to D$ with:

$$\amalg_i t {\amalg_j}^{(-1)} = \begin{cases} t_i & : \quad i \neq j \\ 0 & : \quad i = j, \end{cases}$$

then $t = t_1 \oplus t_2$.

Proof. The calculations for $f \oplus g$ follow from Lemma 7.2.4. For example, $\amalg_1(f \oplus g){\amalg_1}^{(-1)} = f\,\amalg_1\,{\amalg_1}^{(-1)} = f$.

For the second claim, note that $\amalg_1(t{\amalg_1}^{(-1)}) = t_1 = \amalg_1(t_1 \oplus t_2){\amalg_1}^{(-1)}$ and $\amalg_2(t{\amalg_1}^{(-1)}) = 0 = \amalg_2(t_1 \oplus t_2){\amalg_1}^{(-1)}$, hence $t{\amalg_1}^{(-1)} = (t_1 \oplus t_2){\amalg_1}^{(-1)}$. Similarly, $t{\amalg_2}^{(-1)} = (t_1 \oplus t_2){\amalg_2}^{(-1)}$ and therefore $t = t_1 \oplus t_2$. $\square$

Definition 7.2.6. In an inverse category $\mathbb{X}$ with a restriction zero and $\oplus$ a disjointness tensor, define a partial pairing and a partial copairing operation on arrows in $\mathbb{X}$. First, for arrows $f : A \to B$ and $g : A \to C$, define $f \triangledown g$ as being the map that makes Diagram (7.1) below commute, when it exists.

$$
\begin{tikzcd}
 & A \arrow[dl, "f"'] \arrow[d, dotted, "f \triangledown g"] \arrow[dr, "g"] & \\
B & B \oplus C \arrow[l, "{\amalg_1}^{(-1)}"] \arrow[r, "{\amalg_2}^{(-1)}"'] & C.
\end{tikzcd}
\tag{7.1}
$$

Then for $h : B \to A$, $k : C \to A$, define $h \vartriangle k$ as the map that makes Diagram (7.2) commute, if it exists.

$$
\begin{tikzcd}
B \arrow[r, "\amalg_1"] \arrow[dr, "h"'] & B \oplus C \arrow[d, dotted, "h \vartriangle k"] & C \arrow[l, "\amalg_2"'] \arrow[dl, "k"] \\
 & A. &
\end{tikzcd}
\tag{7.2}
$$

Due to ${\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)}$ being jointly monic, $f \triangledown g$ is unique when it exists. Similarly, as $\amalg_1$ and $\amalg_2$ are jointly epic, $f \vartriangle g$ is unique when it exists.

Example 7.2.7 (Pinj)**.** Continuing from Example 6.3.6, $f \triangledown g$ can only exist when $\overline{f} \cap \overline{g} = 0$, as it must be a set function, i.e., $f \triangledown g$ of some element a must be either $(b, 1)$ when $f(a) = b \in B$ or $(c, 2)$ when $g(a) = c \in C$.

Similarly, $h \vartriangle k$ can only exist when $\hat{h} \cap \hat{k} = 0$, otherwise $h \vartriangle k$ would not be injective.

The partial pairing and copairing of Definition 7.2.6 will provide our mechanism for defining disjointness and eventually the disjoint join of maps. As we are operating in an inverse category, the existence of the pairing map $f \triangledown g$ ensures the restrictions of f and g are disjoint, while the copairing map $f \vartriangle d$ exists only when the ranges of f and g are disjoint.

To arrive at the disjointness relation we first give the following lemma detailing properties of the two operations $\triangledown$ and $\vartriangle$:

Lemma 7.2.8. *Given $\mathbb{X}$ is an inverse category with a restriction zero and $\oplus$ a disjointness tensor then the following relations hold for $\triangledown$ and $\vartriangle$:*

(i) *If $f \triangledown g$ exists, then $g \triangledown f$ exists. If $f \vartriangle g$ exists, then $g \vartriangle f$ exists.*

(ii) *$f \triangledown 0$ and $f \vartriangle 0$ always exist.*

(iii) *When $f \triangledown g$ exists, $\overline{f}(f \triangledown g) = f \triangledown 0$, $\overline{f}g = 0$, $\overline{g}(f \triangledown g) = 0 \triangledown g$ and $\overline{g}f = 0$.*

(iv) *Dually to (iii), when $f \vartriangle g$ exists, $(f \vartriangle g)\hat{f} = f \vartriangle 0$, $g\hat{f} = 0$, $(f \vartriangle g)\hat{g} = 0 \vartriangle g$ and $f\hat{g} = 0$.*

(v) *When $f \triangledown g$ exists, $f \triangledown g(h \oplus k) = fh \triangledown gk$.*

(vi) *Dually to (v), when $f \vartriangle g$ exists, $(h \oplus k)f \vartriangle g = hf \vartriangle kg$.*

(vii) *When $f \triangledown g$ exists, then $h(f \triangledown g) = hf \triangledown hg$ and when $f \vartriangle g$ exists, $(f \vartriangle g)h = fh \vartriangle gh$.*

(viii) *If $\overline{f} \triangledown \overline{g}$ exists, then $\overline{f} \vartriangle \overline{g}$ exists and is the partial inverse of $\overline{f} \triangledown \overline{g}$.*

(ix) *If $f \triangledown g$ exists and $f' \leq f$, $g' \leq g$, then $f' \triangledown g'$ exists.*

(x) *When $f \vartriangle g$ exists, $(f \vartriangle g)(f \vartriangle g)^{(-1)} = \overline{f} \oplus \overline{g}$.*

(xi) *Given $f \triangledown g$ and $h \triangledown k$ exist, then $(f \oplus h) \triangledown (g \oplus k) = (f \triangledown g) \oplus (h \triangledown k)$. Dually, the existence of $f \vartriangle g$ and $h \vartriangle k$ implies $(f \oplus h) \vartriangle (g \oplus k) = (f \vartriangle g) \oplus (h \vartriangle k)$.*

Proof.

(i) $g \triangledown f = (f \triangledown g)c_\oplus$ and $g \vartriangle f = c_\oplus(f \vartriangle g)$.

(ii) Consider $f\amalg_1$. Then $f\,\amalg_1\,{\amalg_1}^{(-1)} = f$ and $f\,\amalg_1\,{\amalg_2}^{(-1)} = f0 = 0$. Hence, $f\amalg_1 = f \triangledown 0$.

Consider ${\amalg_1}^{(-1)}f$. Then $\amalg_1{\amalg_1}^{(-1)}f = f$ and $\amalg_2{\amalg_1}^{(-1)}f = 0f = 0$ and therefore ${\amalg_1}^{(-1)}f = (f \vartriangle 0)$.

(iii) Using Lemma 7.2.4

$$\overline{f}g = \overline{(f \triangledown g){\amalg_1}^{(-1)}}(f \triangledown g){\amalg_2}^{(-1)} = (f \triangledown g)\overline{{\amalg_1}^{(-1)}}{\amalg_2}^{(-1)} = 0.$$

Similarly, $\overline{g}f = f \triangledown g\overline{{\amalg_2}^{(-1)}}{\amalg_1}^{(-1)} = 0$.

Recall that ${\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)}$ are jointly monic. Then, $\overline{f}(f \triangledown g){\amalg_1}^{(-1)} = \overline{f}f = f = (f \triangledown 0){\amalg_1}^{(-1)}$ and $\overline{f}(f \triangledown g){\amalg_2}^{(-1)} = \overline{f}g = 0 = (f \triangledown 0){\amalg_2}^{(-1)}$. Therefore, $\overline{f}(f \triangledown g) = f \triangledown 0$. Similarly, $\overline{g}(f \triangledown g) = 0 \triangledown g$.

(iv) Using Lemma 7.2.4

$$\begin{aligned} g\hat{f} = \amalg_2(f \vartriangle g)(\widehat{\amalg_1(f \vartriangle g)}) &= \amalg_2(f \vartriangle g)\overline{(f \vartriangle g)^{(-1)}{\amalg_1}^{(-1)}} = \\ \amalg_2\,(f \vartriangle g)\overline{(f \vartriangle g)^{(-1)}\overline{{\amalg_1}^{(-1)}}} &= \overline{\amalg_2\overline{(f \vartriangle g)}\,\overline{{\amalg_1}^{(-1)}}}\,\amalg_2\,(f \vartriangle g) = \\ \overline{\amalg_2\overline{{\amalg_1}^{(-1)}}\,\overline{(f \vartriangle g)}}\,\amalg_2\,(f \vartriangle g) &= \overline{0}\,\amalg_2\,(f \vartriangle g) = 0. \end{aligned}$$

Similarly, $f\hat{g} = 0$.

Recall that $\amalg_1$ and $\amalg_2$ are jointly epic. Then $\amalg_1(f \vartriangle g)\hat{f} = f\hat{f} = f = \amalg_1(f \vartriangle 0)$ and $\amalg_2(f \vartriangle g)\hat{f} = g\hat{f} = 0 = \amalg_2(f \vartriangle 0)$. Therefore, $(f \vartriangle g)\hat{f} = f \vartriangle 0$. Similarly, $(f \vartriangle g)\hat{g} = 0 \vartriangle g$.

(v) Calculating,

$$f \triangledown g(h \oplus k){\amalg_1}^{(-1)} = f \triangledown g{\amalg_1}^{(-1)}h = fh$$

and

$$f \triangledown g(h \oplus k){\amalg_2}^{(-1)} = f \triangledown g{\amalg_2}^{(-1)}k = gk,$$

which means that $f \triangledown g(h \oplus k) = fh \triangledown gk$ by the joint monic property of ${\amalg_1}^{(-1)}$, ${\amalg_2}^{(-1)}$.

(vi) The proof for this is dual to (v), and depends on the joint epic property of $\amalg_1$ and $\amalg_2$.

(vii) The assumption is $f \triangledown g$ exists, thus $f = (f \triangledown g)\amalg_1{}^{(-1)}$ and $g = (f \triangledown g)\amalg_2{}^{(-1)}$. But this means $hf = h(f \triangledown g)\amalg_1{}^{(-1)}$ and $hg = h(f \triangledown g)\amalg_2{}^{(-1)}$, from which we may conclude $hf \triangledown hg = h(f \triangledown g)$ by the fact that $\amalg_1{}^{(-1)}$ and $\amalg_2{}^{(-1)}$ are jointly monic. The proof of $(f \vartriangle g)h = fh \vartriangle gh$ is similar.

(viii) The assumption is $\overline{f} = \overline{f} \triangledown \overline{g}\amalg_1{}^{(-1)}$. Therefore,

$$\overline{f} = \overline{f}^{(-1)} = \amalg_1{}^{(-1)(-1)}(\overline{f} \triangledown \overline{g})^{(-1)} = \amalg_1(\overline{f} \triangledown \overline{g})^{(-1)}.$$

Similarly, $\overline{g} = \amalg_2(\overline{f} \triangledown \overline{g})^{(-1)}$. But this means $(\overline{f} \triangledown \overline{g})^{(-1)} = \overline{f} \vartriangle \overline{g}$.

(ix) Note that (v) implies $f \triangledown g = \overline{f} \triangledown \overline{g}(f \oplus g)$. By assumption, $f' \leq f$ and $g' \leq g$. This gives $\overline{f'}f = f'$, $\overline{g'}g = g'$, $\overline{f'}\,\overline{f} = \overline{f'}$ and $\overline{g'}\,\overline{g} = \overline{g'}$. Consider the map $\overline{f} \triangledown \overline{g}(\overline{f'} \oplus \overline{g'})(f \oplus g)$. Calculating,

$$\begin{aligned}
\overline{f} \triangledown \overline{g}(\overline{f'} \oplus \overline{g'})(f \oplus g) &= \overline{f} \triangledown \overline{g}(\overline{f'} \oplus \overline{g'})(\overline{f'} \oplus \overline{g'})(f \oplus g) \\
&= \overline{f} \triangledown \overline{g}(\overline{f'} \oplus \overline{g'})(f' \oplus g') \\
&= \overline{f}\,\overline{f'} \triangledown \overline{g}\overline{g'}(f' \oplus g') \\
&= \overline{f'}\,\overline{f} \triangledown \overline{g'}\overline{g}(f' \oplus g') \\
&= \overline{f'} \triangledown \overline{g'}(f' \oplus g') \\
&= f' \triangledown g'.
\end{aligned}$$

(x) From the diagram for $\vartriangle$, we know:

$$\begin{aligned}
f^{(-1)} &= (f \vartriangle g)^{(-1)}\amalg_1{}^{(-1)} \text{ and} \\
g^{(-1)} &= (f \vartriangle g)^{(-1)}\amalg_2{}^{(-1)}.
\end{aligned}$$

As well, from the diagram, $\amalg_1(f \vartriangle g) = f$ and $\amalg_1(f \vartriangle g) = g$. Therefore:

$$\amalg_1(f \vartriangle g)(f \vartriangle g)^{(-1)}\amalg_1{}^{(-1)} = \overline{f} \text{ and } \amalg_2\,(f \vartriangle g)(f \vartriangle g)^{(-1)}\amalg_2{}^{(-1)} = \overline{g}.$$

As $f \perp_{\oplus} g$, by [**Dis.2**] $fg^{(-1)} = f\hat{g}g^{(-1)} = 0g^{(-1)} = 0$ and therefore,

$$\amalg_1(f \vartriangle g)(f \vartriangle g)^{(-1)}\amalg_2{}^{(-1)} = 0 \text{ and } \amalg_2\,(f \vartriangle g)(f \vartriangle g)^{(-1)}\amalg_1{}^{(-1)} = 0.$$

By Corollary 7.2.5 this means $(f \vartriangle g)(f \vartriangle g)^{(-1)} = \overline{f} \oplus \overline{g}$.

(xi) The fact that $(f \triangledown g) \oplus (h \triangledown k)\amalg_1{}^{(-1)} = (f \triangledown g)$ and $(f \triangledown g) \oplus (h \triangledown k)\amalg_2{}^{(-1)} = (h \triangledown k)$, implies that $(f \triangledown g) \oplus (h \triangledown k)$ satisfies the diagram for $(f \oplus h) \triangledown (g \oplus k)$. Dually, as $\amalg_1(f \vartriangle g) \oplus (h \vartriangle k) = (f \vartriangle g)$ and $\amalg_2(f \vartriangle g) \oplus (h \vartriangle k) = (h \vartriangle k)$, $(f \vartriangle g) \oplus (h \vartriangle k)$ satisfies the diagram for $(f \oplus h) \vartriangle (g \oplus k)$.

□

We are now set up to prove that it is possible to create a disjointness relation based on the existence of our pairing and copairing maps:

Lemma 7.2.9. *Define $f \perp_{\oplus} g$ (f is tensor disjoint to g) when $f, g : A \to B$ and both $f \triangledown g$ and $f \vartriangle g$ exist. If $\mathbb{X}$ is an inverse category with a restriction zero and $\oplus$ a disjointness tensor then the relation $\perp_{\oplus}$ is a disjointness relation.*

Proof. The proof requires us to show that $\perp_{\oplus}$ satisfies the disjointness axioms. We will use [**Dis.6'**] in place of [**Dis.6**] and [**Dis.7**] as discussed in Lemma 6.2.2.

[**Dis.1**] The requirement is $f \perp_{\oplus} 0$. This follows immediately from Lemma 7.2.8, item (ii).

[**Dis.2**] Show $f \perp_{\oplus} g$ implies $\overline{f}g = 0$. This is a direct consequence of Lemma 7.2.8, item (iii).

[**Dis.3**] This requires that $f \perp_{\oplus} g$, $f' \leq f$, $g' \leq g$ implies $f' \perp_{\oplus} g'$. From Lemma 7.2.8, item (ix), $f' \triangledown g'$ exists. Using a similar argument to the proof of this item, $f' \vartriangle g'$ exists and hence $f' \perp_{\oplus} g'$.

[**Dis.4**] Commutativity of $\perp_{\oplus}$ follows from the symmetry of the two required diagrams, see Lemma 7.2.8, item (i).

[**Dis.5**] Show that if $f \perp_{\oplus} g$ then $hf \perp_{\oplus} hg$ for any map h. By Lemma 7.2.8(vii), when $f \triangledown g$ exists, $hf \triangledown hg$ exists. Assuming $f \vartriangle g$, by (vi) of the same lemma, $(hf) \vartriangle (hg)$ exists and equals $(h \oplus h)(f \vartriangle g)$ and therefore $hf \perp_{\oplus} hg$.

[**Dis.6'**] This requires showing $f \perp_{\oplus} g$ if and only if $\overline{f} \perp_{\oplus} \overline{g}$ and $\hat{f} \perp_{\oplus} \hat{g}$. This follows directly from Lemma 7.2.8(v) and (vi), which give us $f \triangledown g = \overline{f} \triangledown \overline{g}(f \oplus g)$ and $f \vartriangle g = (f \oplus g)\hat{f} \vartriangle \hat{g}$, where the equalities hold if either side of the equation exists.

□

Example 7.2.10 ($\perp_{\oplus}$ in PINJ)**.** Referring to Example 7.2.7, $f \triangledown g$ exists when $\overline{f} \cap \overline{g} = 0$ and that $f \vartriangle g$ exists when $\hat{f} \cap \hat{g} = 0$. But this agrees with the initial definition of disjointness ($\perp$) in PINJ from Example 6.2.8 and hence $\perp_{\oplus}$ is the same relation as $\perp$ in PINJ.

Note that this is due to the actual choices of the tensor and $\perp$. If we had initially chosen a different $\perp$, such as $\perp_0$, where $f \perp_0 g$ iff f or g are $\mathbf{0}$, then $\perp_{\oplus} \neq \perp_0$.

7.3 Disjoint joins via a tensor

The operations $\triangledown$ and $\vartriangle$ are sufficient to define a disjointness relation on an inverse category. However, when attempting to extend this to a disjoint join, the two relations are insufficient to prove [**DJ.4**], That is, requiring that $\perp_{\oplus}[f, g, h]$ implies $f \perp_{\oplus} (g \sqcup_{\oplus} h)$.

Therefore, we must add one more assumption regarding our tensor in order to define disjointness.

Definition 7.3.1. Let $\mathbb{X}$ be an inverse category with $\oplus$ a disjointness tensor and a restriction zero. Consider the commutative diagrams (7.3) and (7.4).

$$\begin{tikzcd}
A \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& X \oplus Y \oplus Z \arrow[r, "\amalg_{1,3}{}^{(-1)}"'] \arrow[d, "\amalg_{1,2}{}^{(-1)}"] & X \oplus Z \arrow[d, "\amalg_1{}^{(-1)}"] \\
& X \oplus Y \arrow[r, "\amalg_1{}^{(-1)}"'] & X,
\end{tikzcd} \tag{7.3}$$

$$\begin{tikzcd}
& & & A \\
X \oplus Y \arrow[r, "\amalg_{1,2}"'] \arrow[urrr, bend left, "h"] & X \oplus Y \oplus Z \arrow[urr, dotted, "\beta"] & & \\
X \arrow[r, "\amalg_1"'] \arrow[u, "\amalg_1"] & X \oplus Z. \arrow[u, "\amalg_{1,3}"'] \arrow[uurr, bend right, "k"'] & &
\end{tikzcd} \tag{7.4}$$

Then $\oplus$ is a *disjoint sum tensor* when the following two conditions hold:

- α exists if and only if $f\amalg_2{}^{(-1)} \triangledown g\amalg_2{}^{(-1)}$ exists;
- β exists if and only if $\amalg_2 h \vartriangle \amalg_2 k$ exists.

Example 7.3.2 (In Pinj, $\oplus$ is a disjoint sum tensor)**.** In Pinj, Diagram (7.3) means that f and g must agree on those elements of A that map to $(x, 1)$ in either $X \oplus Y$ or $X \oplus Z$. The statement that $f\amalg_2{}^{(-1)} \triangledown g\amalg_2{}^{(-1)}$ exists means that if $f(a) = (y, 2)$, then $g(a)$ must be undefined and vice versa. In such a case α exists and is defined as:

$$\alpha(a) = \begin{cases} (x,1) & f(a) = (x,1) \in X \oplus Y \text{ and } g(a) = (x,1) \in X \oplus Z \\ (y,2) & f(a) = (y,2) \in X \oplus Y \text{ and } g(a)\uparrow \\ (z,3) & g(a) = (z,2) \in X \oplus Z \text{ and } f(a)\uparrow . \end{cases}$$

For the converse, assume α exists, meaning $\alpha(a)$ must be one of $(x, 1), (y, 2)$ or $(z, 3)$. As $f\amalg_2{}^{(-1)} = \alpha\amalg_{1,2}{}^{(-1)}\amalg_2{}^{(-1)}$ and $g\amalg_2{}^{(-1)} = \alpha\amalg_{1,3}{}^{(-1)}\amalg_2{}^{(-1)}$, this requires that $\overline{f\amalg_2{}^{(-1)}} \cap \overline{g\amalg_2{}^{(-1)}} = 0$ and therefore $f\amalg_2{}^{(-1)} \triangledown g\amalg_2{}^{(-1)}$ exists.

The reasoning for Diagram (7.4) is similar.

Lemma 7.3.3. *Let $\mathbb{X}$ be an inverse category with a disjoint sum tensor as in Definition 7.3.1 where $f, g, h : A \to B$ with $\perp_{\oplus}[f, g, h]$. Then both $f \triangledown (g \triangledown h)$ and $f \vartriangle (g \vartriangle h)$ exist.*

Proof. As all the maps are disjoint, we know the maps $\triangledown$ and $\vartriangle$ exist for each pair. Consider the diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[drr, bend left=10, "g \triangledown f"] \arrow[dr, dotted, "\alpha"] \arrow[ddr, bend right, "g \triangledown h"'] & & B \\
& B \oplus B \oplus B \arrow[r, "\amalg_{1,3}{}^{(-1)}"'] \arrow[d, "\amalg_{1,2}{}^{(-1)}"] & B \oplus B \arrow[u, "\amalg_2{}^{(-1)}"'] \arrow[d, "\amalg_1{}^{(-1)}"] \\
& B \oplus B \arrow[r, "\amalg_1{}^{(-1)}"'] & B
\end{tikzcd}$$

where we claim $\alpha = (g \triangledown h) \triangledown f$.

The lower part of the diagram commutes as it fulfills the conditions of Definition 7.3.1. The upper rightmost triangle of the diagram commutes by the definition of $g \triangledown f$. Noting that $\amalg_{0,1}{}^{(-1)} : B \oplus B \oplus B \to B \oplus B$ is the same map as $\amalg_1{}^{(-1)} : (B \oplus B) \oplus B \to (B \oplus B)$ and $\amalg_{0,2}{}^{(-1)} \amalg_2{}^{(-1)} : B \oplus B \oplus B \to B \oplus B \to B$ is the same map as $\amalg_2{}^{(-1)} : (B \oplus B) \oplus B \to B$, therefore α does make the $\triangledown$ diagram for $g \triangledown h$ and f commute. Therefore by Lemma 7.2.8, $f \triangledown (g \triangledown h)$ exists and is equal to $\alpha c_{\oplus\{01,2\}}$.

A dual diagram and corresponding reasoning shows $f \vartriangle (g \vartriangle h)$ exists. $\square$

Lemma 7.3.4. *In an inverse category $\mathbb{X}$ with a disjoint sum tensor, when $\perp_{\oplus}[f, g, h]$, then:*

(i) $f \triangledown (g \triangledown h) = ((f \triangledown g) \triangledown h) a_{\oplus}$ *and both exist,*

(ii) $f \vartriangle (g \vartriangle h) = ((f \vartriangle g) \vartriangle h) a_{\oplus}$ *and both exist.*

Proof. Consider the diagram

$$
\begin{tikzcd}
A \arrow[rr, "h"] \arrow[drr, bend left=10, "f \triangledown h"] \arrow[dr, dotted, "\alpha"] \arrow[ddr, bend right, "f \triangledown g"'] & & B \\
& B \oplus B \oplus B \arrow[r, "\amalg_{1,3}{}^{(-1)}"'] \arrow[d, "\amalg_{1,2}{}^{(-1)}"] & B \oplus B \arrow[u, "\amalg_2{}^{(-1)}"'] \arrow[d, "\amalg_1{}^{(-1)}"] \\
& B \oplus B \arrow[r, "\amalg_1{}^{(-1)}"'] & B
\end{tikzcd}
\tag{7.5}
$$

which gives us $\alpha = (f \triangledown g) \triangledown h : A \to (B \oplus B) \oplus B$ and $\alpha a_\oplus : A \to B \oplus (B \oplus B)$. Next consider the diagram

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[drr, bend left=10, "g \triangledown f"] \arrow[dr, dotted, "\gamma"] \arrow[ddr, bend right, "g \triangledown h"'] & & B \\
& B \oplus B \oplus B \arrow[r, "\amalg_{1,3}{}^{(-1)}"'] \arrow[d, "\amalg_{1,2}{}^{(-1)}"] & B \oplus B \arrow[u, "\amalg_2{}^{(-1)}"'] \arrow[d, "\amalg_1{}^{(-1)}"] \\
& B \oplus B \arrow[r, "\amalg_1{}^{(-1)}"'] & B
\end{tikzcd}
\tag{7.6}
$$

which gives us $\gamma c_\oplus = f \triangledown (g \triangledown h) : A \to B \oplus (B \oplus B)$.

From Diagrams (7.5) and (7.6):

$$
\gamma c_\oplus \amalg_0{}^{(-1)} = f = \alpha a_\oplus \amalg_1{}^{(-1)}
$$

$$
\gamma c_\oplus \amalg_1{}^{(-1)} \amalg_0{}^{(-1)} = g = \alpha a_\oplus \amalg_2{}^{(-1)} \amalg_1{}^{(-1)}
$$

$$
\gamma c_\oplus \amalg_1{}^{(-1)} \amalg_2{}^{(-1)} = h = \alpha a_\oplus \amalg_2{}^{(-1)} \amalg_2{}^{(-1)}.
$$

By the assumption that $\amalg_1{}^{(-1)}, \amalg_2{}^{(-1)}$ are jointly monic, $\alpha = \gamma c_\oplus a_\oplus$. Therefore $f \triangledown (g \triangledown h) = (f \triangledown g) \triangledown h$, up to the associativity isomorphism. $\square$

We may now state the main result of this section:

Proposition 7.3.5. *An inverse category with a restriction zero and a disjoint sum tensor has disjoint joins.*

Given two maps f, g with $f \perp_{\oplus} g$, the candidate for disjoint join is the map $f \sqcup_{\oplus} g = \overline{f} \triangledown \overline{g}(f \oplus g)\hat{f} \vartriangle \hat{g}$. The proof that this is a disjoint join will follow after giving an example and a lemma detailing the properties of $\sqcup_{\oplus}$.

For reference, the map $f \sqcup_{\oplus} g$ may be visualized as follows:

$$\begin{tikzcd}
 & A & B \arrow[d, "\amalg_1"'] \arrow[dr, "\hat{f}"] & \\
A \arrow[ur, "\overline{f}"] \arrow[r, dotted, "\overline{f} \triangledown \overline{g}"] \arrow[dr, "\overline{g}"'] & A \oplus A \arrow[u, "{\amalg_1}^{(-1)}"'] \arrow[d, "{\amalg_2}^{(-1)}"'] \arrow[r, "f \oplus g"] & B \oplus B \arrow[r, dotted, "\hat{f} \vartriangle \hat{g}"] & B. \\
 & A & B \arrow[u, "\amalg_2"] \arrow[ur, "\hat{g}"'] &
\end{tikzcd} \tag{7.7}$$

Using Lemma 7.2.8, this may be rewritten in a variety of equivalent ways:

$$f \sqcup_{\oplus} g = (\overline{f} \triangledown \overline{g})(f \oplus g)(\hat{f} \vartriangle \hat{g}) \tag{7.8}$$

$$= (f \triangledown g)\,(\hat{f} \vartriangle \hat{g}) \tag{7.9}$$

$$= (\overline{f} \triangledown \overline{g})\,(f \vartriangle g) \tag{7.10}$$

$$= (f \triangledown g)(f^{(-1)} \oplus g^{(-1)})(f \vartriangle g). \tag{7.11}$$

In particular, note that $\overline{f} \sqcup_{\oplus} \overline{g} = (\overline{f} \triangledown \overline{g})(\overline{f} \vartriangle \overline{g})$ as $\hat{\overline{g}} = \overline{g}$.

Example 7.3.6 (Disjoint sum tensor in $\textsc{Pinj}$)**.** Using Equation (7.10) for $f \sqcup_{\oplus} g$, calculate:

$$\overline{f} \triangledown \overline{g}(a) = \begin{cases} (a, 1) & \overline{f}(a) = a, \overline{g}\uparrow \\ (a, 2) & \overline{g}(a) = a, \overline{f}\uparrow \end{cases} \tag{7.12}$$

and

$$f \vartriangle g((a, n)) = \begin{cases} f(a) & n = 1 \\ g(a) & n = 2. \end{cases} \tag{7.13}$$

Combining Equation (7.12) with Equation (7.13) then gives the same definition as that of $\sqcup$ as given in Example 6.3.6.

Lemma 7.3.7. *Let $\mathbb{X}$ be an inverse category with a disjoint sum tensor and restriction zero. Let $\mathbb{X}$ have the maps $f, g : A \to B$ with $f \perp_{\oplus} g$. Then $\sqcup_{\oplus}$ has the following properties.*

(i) For all maps $h : A \to B$, $\overline{f}h \sqcup_{\oplus} \overline{g}h = (\overline{f} \sqcup_{\oplus} \overline{g})h$.

(ii) $\overline{f} \sqcup_{\oplus} \overline{g} = \overline{f \sqcup_{\oplus} g}$.

Proof.

(i) By Lemma 6.2.3 (ii), $\overline{f}h \perp_{\oplus} \overline{g}h$, hence the disjoint join exists: $\overline{f}h \sqcup_{\oplus} \overline{g}h$. Also, noting that

$$h\widehat{\overline{f}h} = h\overline{h^{(-1)}\overline{f}} = \overline{hh^{(-1)}\overline{f}}h = \overline{\overline{h}\overline{f}}h = \overline{\overline{f}h}h = \overline{f}h,$$

we may then calculate from the left hand side as follows:

$$\begin{aligned}
\overline{f}h \sqcup_{\oplus} \overline{g}h &= (\overline{f}h \triangledown \overline{g}h)(\widehat{\overline{f}h} \vartriangle \widehat{\overline{g}h}) \\
&= (\overline{f} \triangledown \overline{g})(h\widehat{\overline{f}h} \vartriangle h\widehat{\overline{g}h}) \\
&= (\overline{f} \triangledown \overline{g})(\overline{f}h \vartriangle \overline{g}h) \\
&= (\overline{f} \triangledown \overline{g})(\overline{f} \vartriangle \overline{g})h \\
&= (\overline{f} \sqcup_{\oplus} \overline{g})h.
\end{aligned}$$

(ii) Using Lemma 7.2.8 (x):

$$\begin{aligned}
\overline{f \sqcup_{\oplus} g} &= f \sqcup_{\oplus} g(f \sqcup_{\oplus} g)^{(-1)} \\
&= \left((\overline{f} \triangledown \overline{g})(f \vartriangle g)\right)\left((f \triangledown g)^{(-1)}(\overline{f} \triangledown \overline{g})^{(-1)}\right) \\
&= \overline{f} \triangledown \overline{g}(f \triangledown g)(f \triangledown g)^{(-1)}\overline{f} \vartriangle \overline{g} \\
&= \overline{f} \triangledown \overline{g}(\overline{f} \oplus \overline{g})\overline{f} \vartriangle \overline{g} \\
&= \overline{f} \triangledown \overline{g}\,\overline{f} \vartriangle \overline{g} \\
&= \overline{f} \sqcup_{\oplus} \overline{g}.
\end{aligned}$$

$\square$

We may now complete the proof of Proposition 7.3.5:

Proof. [**DJ.1**] This requires $f, g \leq f \sqcup_\oplus g$.

$$\begin{aligned}
\overline{f}\,(\overline{f} \triangledown \overline{g}) f \vartriangle g &= (\overline{f} \triangledown \overline{g}) {\amalg_1}^{(-1)} (\overline{f} \triangledown \overline{g}) f \vartriangle g \\
&= \overline{(\overline{f} \triangledown \overline{g}) {\amalg_1}^{(-1)}} (\overline{f} \triangledown \overline{g}) f \vartriangle g \\
&= (\overline{f} \triangledown \overline{g}) \overline{{\amalg_1}^{(-1)}} f \vartriangle g \\
&= (\overline{f} \triangledown \overline{g}) {\amalg_1}^{(-1)}\, \amalg_1\, f \vartriangle g \\
&= ((\overline{f} \triangledown \overline{g}) {\amalg_1}^{(-1)}) (\amalg_1 (f \vartriangle g)) \\
&= \overline{f} f \\
&= f.
\end{aligned}$$

Thus, $f \leq f \sqcup_\oplus g$. Showing $g \leq f \sqcup_\oplus g$ proceeds in the same manner.

[**DJ.2**] The requirement is that $f \leq h,\ g \leq h$ and $f \perp_\oplus g$ implies $f \sqcup_\oplus g \leq h$.

$$\begin{aligned}
\overline{f \sqcup_\oplus g}\, h &= \overline{\overline{f} h \sqcup_\oplus \overline{g} h}\, h \\
&= \overline{(\overline{f} \sqcup_\oplus \overline{g}) h}\, h \\
&= \overline{\overline{(\overline{f} \sqcup_\oplus \overline{g})} h}\, h \\
&= \overline{\overline{(\overline{f} \sqcup_\oplus \overline{g})} h}\, (\overline{f} \sqcup_\oplus \overline{g}) h \\
&= \overline{(\overline{f} \sqcup_\oplus \overline{g}) h}\, (\overline{f} \sqcup_\oplus \overline{g}) h \\
&= (\overline{f} \sqcup_\oplus \overline{g}) h \\
&= (\overline{f} h \sqcup_\oplus \overline{g} h) \\
&= (f \sqcup_\oplus g).
\end{aligned}$$

[**DJ.3**] This is the stability of $\sqcup_\oplus$, i.e., that $h(f \sqcup_\oplus g) = hf \sqcup_\oplus hg$.

$$
\begin{aligned}
h(f \sqcup_{\oplus} g) &= h((\overline{f} \triangledown \overline{g})(f \vartriangle g)) \\
&= (h\overline{f} \triangledown h\overline{g})(f \vartriangle g) \\
&= (\overline{hf}h \triangledown \overline{hg}h)(f \vartriangle g) \\
&= (\overline{hf} \triangledown \overline{hg})(h \oplus h)(f \vartriangle g) \\
&= (\overline{hf} \triangledown \overline{hg})(hf \vartriangle hg) \\
&= hf \sqcup_{\oplus} hg.
\end{aligned}
$$

[**DJ.4**] This requires $\perp_{\oplus}[f, g, h]$ if and only if $f \perp_{\oplus} (g \sqcup_{\oplus} h)$. For the right to left implication, note that the existence of $g \sqcup_{\oplus} h$ implies $g \perp_{\oplus} h$. By [**DJ.1**] $g, h \leq g \sqcup_{\oplus} h$, as shown in this proof. This gives that $f \perp_{\oplus} g$ and $f \perp_{\oplus} h$, hence $\perp_{\oplus}[f, g, h]$.

For the left to right implication, use Lemma 7.3.3. As $\perp_{\oplus}[f, g, h]$, this means $f \triangledown (g \triangledown h)$ and $f \vartriangle (g \vartriangle h)$ exist.

Recall that $g \sqcup_{\oplus} h = (g \triangledown h)(\hat{g} \vartriangle \hat{h})$. Then the map

$$A \xrightarrow{f \triangledown (g \triangledown h)} B \oplus B \oplus B \xrightarrow{1 \oplus (\hat{g} \vartriangle \hat{h})} B \oplus B$$

makes the diagram for $f \triangledown (g \sqcup_{\oplus} h)$ commute.

Recalling that $g \sqcup_{\oplus} h = (\overline{g} \triangledown \overline{h})(g \vartriangle h)$, this gives

$$A \oplus A \xrightarrow{1 \oplus (\overline{g} \triangledown \overline{h})} A \oplus A \oplus A \xrightarrow{f \vartriangle (g \vartriangle h)} B$$

provides the witness map for $f \vartriangle (g \sqcup_{\oplus} h)$ and hence $f \perp_{\oplus} (g \sqcup_{\oplus} h)$.

$\square$

Next, we show that the disjoint sum tensor is universal with respect to the disjoint join.

Lemma 7.3.8. *Given an inverse category $\mathbb{X}$ with $\oplus$ a disjoint sum tensor, then $\oplus$ preserves the disjoint join. That is,*

$$f \perp_{\oplus} g,\ h \perp_{\oplus} k \text{ implies } f \oplus h \perp_{\oplus} g \oplus k, \tag{7.14}$$

$$f \perp_{\oplus} g,\ h \perp_{\oplus} k \text{ implies } (f \sqcup_{\oplus} g) \oplus (h \sqcup_{\oplus} k) = (f \oplus h) \sqcup_{\oplus} (g \oplus k). \tag{7.15}$$

Proof. For Condition (7.14), suppose $f \perp_{\oplus} g$ and $h \perp_{\oplus} k$. From Lemma 7.2.8(xi), both $(f \oplus h) \triangledown (g \oplus k)$ and $(f \oplus h) \vartriangle (g \oplus k)$ exist, hence $(f \oplus h) \perp_{\oplus} (g \oplus k)$.

For Condition (7.15), compute from the right hand side:

$$\begin{aligned}
(f \oplus h) \sqcup_{\oplus} (g \oplus k) &= (f \oplus h) \triangledown (g \oplus k) (\widehat{f \oplus h}) \vartriangle (\widehat{g \oplus k}) \\
&= ((f \triangledown g) \oplus (h \triangledown k)) \left((\hat{f} \oplus \hat{h}) \vartriangle (\hat{g} \oplus \hat{k}) \right) \\
&= ((f \triangledown g) \oplus (h \triangledown k)) \left((\hat{f} \vartriangle \hat{g}) \oplus (\hat{h} \vartriangle \hat{k}) \right) \\
&= \left((f \triangledown g)(\hat{f} \vartriangle \hat{g}) \right) \oplus \left((h \triangledown k)(\hat{h} \vartriangle \hat{k}) \right) \\
&= (f \sqcup_{\oplus} g) \oplus (h \sqcup_{\oplus} k).
\end{aligned}$$

The second and third lines above again use Lemma 7.2.8(xi).

□

7.4 Disjoint sums via a disjoint sum tensor

The significant amount of technical machinery of Section 7.2 and Section 7.3 now allow us to show that a disjoint sum tensor will produce disjoint sums in a category and conversely, having all disjoint sums in a category produces a disjoint sum tensor.

Proposition 7.4.1. *A disjoint sum tensor in an inverse category $\mathbb{X}$ gives disjoint sums, i.e., for each pair of objects A, B, $A \oplus B$ is a disjoint sum.*

Proof. Setting $i_i = \amalg_i$ and $x_i = {\amalg_i}^{(-1)}$ and setting $X = A \oplus B$ produces a disjoint sum in $\mathbb{X}$. We show this satisfies the four conditions of Definition 7.1.1.

(i) From Lemma 7.2.4, both $\amalg_1$ and $\amalg_2$ are monic maps.

(ii) $\amalg_1 : A \to A \oplus B$, $\amalg_2 : B \to A \oplus B$, ${\amalg_1}^{(-1)} : A \oplus B \to A$ and ${\amalg_2}^{(-1)} : A \oplus B \to B$.

(iii) ${\amalg_1}^{(-1)} = {\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)} = {\amalg_2}^{(-1)}$.

(iv) ${i_1}^{(-1)} i_1 = 1 \oplus 0 \perp_{\oplus} 0 \oplus 1 = {i_2}^{(-1)} i_2$ as

$$1 \oplus 0 \triangledown 0 \oplus 1 = ({u^r_{\oplus}}^{(-1)} \oplus {u^l_{\oplus}}^{(-1)}) \text{ and } 1 \oplus 0 \vartriangle 0 \oplus 1 = ({\amalg_1}^{(-1)} \oplus {\amalg_2}^{(-1)}).$$

For their join,

$$(1 \oplus 0) \sqcup_{\oplus} (0 \oplus 1) = ({u^r_{\oplus}}^{(-1)} \oplus {u^l_{\oplus}}^{(-1)})({\amalg_1}^{(-1)} \oplus {\amalg_2}^{(-1)}) = \\ {u^r_{\oplus}}^{(-1)} {\amalg_1}^{(-1)} \oplus {u^l_{\oplus}}^{(-1)} {\amalg_2}^{(-1)} = 1 \oplus 1 = 1.$$

$\square$

Finally, to complete the cycle:

Proposition 7.4.2. *Given $\mathbb{X}$ is an inverse category where every pair of objects has a disjoint sum, then the tensor derived from the disjoint sum of A, B is a disjoint sum tensor.*

Proof. Proposition 7.1.4 showed that when $\oplus$ is derived from the disjoint sum, it is a symmetric monoidal tensor. Therefore, it only remains to show that it is a disjointness tensor and that it satisfies Definition 7.3.1, i.e., is a disjoint sum tensor as well.

First note that we immediately have that it is a disjointness tensor as:

(i) Proposition 7.1.4 shows it is a restriction functor.

(ii) Proposition 7.1.4 shows that the unit is the restriction zero.

(iii) From the above, $\amalg_1 = i_1$ and $\amalg_2 = i_2$. Assume $\amalg_1 f = \amalg_1 g$ and $\amalg_2 f = \amalg_2 g$. As ${\amalg_1}^{(-1)} \amalg_1 \sqcup {\amalg_2}^{(-1)} \amalg_2 = 1$,

$$f = ({\amalg_1}^{(-1)} \amalg_1 \sqcup {\amalg_2}^{(-1)} \amalg_2) f = {\amalg_1}^{(-1)} \amalg_1 f \sqcup {\amalg_2}^{(-1)} \amalg_2 f = \\ {\amalg_1}^{(-1)} \amalg_1 g \sqcup {\amalg_2}^{(-1)} \amalg_2 g = ({\amalg_1}^{(-1)} \amalg_1 \sqcup {\amalg_2}^{(-1)} \amalg_2) g = g$$

and therefore $\amalg_1$ and $\amalg_2$ are jointly monic.

(iv) ${\amalg_1}^{(-1)}$ and ${\amalg_2}^{(-1)}$ are jointly epic by a similar argument as (iii).

Note that we may also immediately conclude that the original disjointness relation on $\mathbb{X}$ is the same as the one generated by $\oplus$. This is because $f \triangledown g$ will be equal to $f\,\amalg_1 \sqcup g\amalg_2$ and $f \vartriangle g$ will be equal to $\amalg_1{}^{(-1)} f \sqcup \amalg_2{}^{(-1)} g$, therefore, $f \triangledown g$ and $f \vartriangle g$ will exist if and only if $f \perp g$.

To show it is a disjoint sum tensor, we must show the existence of the maps α and β from Definition 7.3.1.

For Diagram (7.3), assuming $f\amalg_2{}^{(-1)} \triangledown g\amalg_2{}^{(-1)}$ exists, then

$$\alpha = (f\amalg_1{}^{(-1)})\,\amalg_1 \sqcup ((f\amalg_2{}^{(-1)} \triangledown g\amalg_2{}^{(-1)})\amalg_2)$$

satisfies the diagram. If α does exist, then $\alpha\amalg_2{}^{(-1)}$ satisfies the diagram for $f\amalg_2{}^{(-1)} \triangledown g\amalg_2{}^{(-1)}$. The argument for satisfying Diagram (7.4) is analogous.

Therefore, $\oplus$ is a disjoint sum tensor. $\square$

Chapter 8

Matrix categories

Inverse categories with all disjoint sums are, in fact, Unique Decomposition Categories [35]. However, what about inverse categories with disjoint joins? The previous chapter showed that in the presence of a disjoint sum tensor, an inverse category $\mathbb{X}$ with disjoint joins also has disjoint sums. This chapter will show how to add a disjoint sum to such an arbitrary $\mathbb{X}$.

8.1 Matrices

In this section, we will show that when given an inverse category $\mathbb{X}$ with a disjoint joins, one can define a matrix category based on $\mathbb{X}$, called $\textsc{iMat}(\mathbb{X})$. Furthermore, we will show that $\textsc{iMat}(\mathbb{X})$ is an inverse category with disjoint sums and that $\mathbb{X}$ embeds within this category.

The types of matrices allowed in the matrix category are subject to certain constraints:

Definition 8.1.1. Suppose $\mathbb{X}$ is an inverse category with disjoint joins, then a *disjoint sum matrix* in $\mathbb{X}$ is a matrix of maps $[f_{ij}]$ where $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ with $f_{ij} : A_i \to B_j$ which satisfy the two conditions:

$$\text{Rows: For each } i,\ \perp \{\overline{f_{ij}}\}_{j=1,\dots,m}. \tag{8.1}$$

$$\text{Columns: For each } j,\ \perp \{\widehat{f_{ij}}\}_{i=1,\dots,n}. \tag{8.2}$$

In other words, for each column, the ranges of the functions in that column are disjoint and for each row the restrictions of the functions in that row are disjoint.

We will show that this type of matrix corresponds to maps in the category $\textsc{iMat}(\mathbb{X})$. In $\textsc{iMat}(\mathbb{X})$ the composition is given by "matrix multiplication", with the operations of multiplication and addition replaced with composition in $\mathbb{X}$ and the disjoint join respectively.

Definition 8.1.2. Given an inverse category $\mathbb{X}$ with disjoint joins, we define the *inverse matrix category* of $\mathbb{X}$, $\text{iMat}(\mathbb{X})$, as follows:

Objects: Lists of the objects of $\mathbb{X}$;

Maps: Disjoint sum matrices $[f_{ij}] : [A_i] \to [B_j]$. In such a matrix each individual map $f_{ij} : A_i \to B_j$ is a map in $\mathbb{X}$;

Identity: The disjoint sum matrix I — A diagonal matrix with $1 : A_i \to A_i$ at the i, i position and zero maps elsewhere;

Composition: Given $[f_{ij}] : [A_i] \to [B_j]$ and $[g_{jk}] : [B_j] \to [C_k]$, then $[h_{ik}] = [f_{ij}][g_{jk}] : [A_i] \to [C_k]$ is defined as $h_{ik} = \bigsqcup_j f_{ij} g_{jk}$;

Restriction: We set $\overline{[f_{ij}]}$ to be $[f'_{ij}]$ where $f'_{ij} = 0$ when $i \neq j$ and $f'_{ii} = \sqcup_j \overline{f_{ij}}$.

In the following, we will use the notation $\text{diag}[d_j]$ for diagonal matrices where the j, j entry is d_j.

Lemma 8.1.3. *When $\mathbb{X}$ is an inverse category with disjoint joins, $\text{iMat}(\mathbb{X})$ is an inverse category.*

Proof. We need to show the following:

- Composition is well defined and associative.
- The restriction is well defined.
- Each map must have a partial inverse.

Composition is well defined: Consider $[h_{ik}] = [f_{ij}][g_{jk}]$ where $[f_{ij}] : [A_1, \ldots, A_n] \to [B_1, \ldots, B_m]$ and $[g_{jk}] : [B_1, \ldots, B_m] \to [C_1, \ldots, C_\ell]$. As each of $[f_{ij}]$ and $[g_{jk}]$ are disjoint sum matrices, by [**Dis.7**] we know that $\perp \{f_{ij} g_{jk}\}$ for each choice of i and k. Hence, we know the composition $h_{ik} = \bigsqcup_j f_{ij} g_{jk}$ is defined and $h_{ik} : A_i \to C_k$. We must now show that the h_{ik} satisfy Equation (8.1) and Equation (8.2). Calculating the restriction of row elements,

$$\overline{h_{ik}} = \overline{\bigsqcup_j f_{ij}g_{jk}} = \bigsqcup_j \overline{f_{ij}g_{jk}}.$$

By [**DJ.4**], these are disjoint for the i's when each component is disjoint. But each component is of the form $\overline{f_{ij}g_{jk}}$ and by Lemma 3.2.2, it is less than or equal to $\overline{f_{ij}}$ each of which are disjoint by assumption. Therefore, by [**Dis.3**] the restriction of each of the entries in a row of $[h_{ik}]$ are disjoint. By a similar argument, the range of each of the entries in a column of $[h_{ik}]$ are disjoint.

Thus the matrix $[h_{ik}]$ is a disjoint sum matrix and is in the category. Therefore composition is well-defined.

Associativity of composition. We have

$$\begin{aligned}
([f_{ij}][g_{jk}])[h_{k\ell}] &= \left[(\bigsqcup_j f_{ij}g_{jk}) \right] [h_{k\ell}] \\
&= \left[\bigsqcup_k (\bigsqcup_j f_{ij}g_{jk}) h_{k\ell} \right] \\
&= \left[\bigsqcup_j f_{ij} (\bigsqcup_k g_{jk}h_{k\ell}) \right] \\
&= [f_{ij}]([g_{jk}][h_{k\ell}]).
\end{aligned}$$

The restriction axioms.

$$[\mathbf{R.1}] \quad \overline{[f_{ij}]}[f_{ij}] = \begin{bmatrix} (\sqcup_j \overline{f_{1j}})f_{11} & \cdots & (\sqcup_j \overline{f_{1j}})f_{1n} \\ & \vdots & \\ (\sqcup_j \overline{f_{mj}})f_{m1} & \cdots & (\sqcup_j \overline{f_{mj}})f_{mn} \end{bmatrix} = [f_{ij}].$$

[**R.2**] $\overline{[f_{ij}]}\overline{g_{ij}} = \overline{g_{ij}}\overline{[f_{ij}]}$ as diagonal matrices commute and $\sqcup$ is commutative.

$$[\mathbf{R.3}] \quad \overline{\overline{[f_{ij}]}[g_{jk}]} = \overline{\mathrm{diag}[\sqcup_j \overline{f_{1j}}, \ldots, \sqcup_j \overline{f_{nj}}][g_{jk}]}$$

$$= \overline{\begin{bmatrix} \sqcup_j \overline{f_{1j}} g_{11} & \ldots & \sqcup_j \overline{f_{1j}} g_{1k} \\ & \vdots & \\ \sqcup_j \overline{f_{nj}} g_{n1} & \ldots & \sqcup_j \overline{f_{nj}} g_{nk} \end{bmatrix}}$$

$$= \mathrm{diag}[\sqcup_k \overline{(\sqcup_j (\overline{f_{1j}} g_{1k}))}, \ldots, \sqcup_k \overline{(\sqcup_j (\overline{f_{nj}}) g_{nk})}]$$

$$= \mathrm{diag}[\sqcup_k (\sqcup_j (\overline{f_{1j}}) \overline{g_{1k}}), \ldots, \sqcup_k (\sqcup_j (\overline{f_{nj}}) \overline{g_{nk}})]$$

$$= \mathrm{diag}[(\sqcup_j (\overline{f_{1j}}) \sqcup_k \overline{g_{1k}}), \ldots, (\sqcup_j (\overline{f_{nj}}) \sqcup_k \overline{g_{nk}})] = \overline{[f_{ij}]}\,\overline{[g_{jk}]}.$$

$$[\mathbf{R.4}] \quad [f_{ij}]\overline{[g_{jk}]} = [f_{ij}]\, \mathrm{diag}_j [\sqcup_k \overline{g_{jk}}]$$

$$= \begin{bmatrix} f_{11} \sqcup_k \overline{g_{1k}} & \ldots & f_{1n} \sqcup_k \overline{g_{nk}} \\ & \vdots & \\ f_{m1} \sqcup_k \overline{g_{1k}} & \ldots & f_{mn} \sqcup_k \overline{g_{nk}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqcup_k f_{11} \overline{g_{1k}} & \ldots & \sqcup_k f_{1n} \overline{g_{nk}} \\ & \vdots & \\ \sqcup_k f_{m1} \overline{g_{1k}} & \ldots & \sqcup_k f_{mn} \overline{g_{nk}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqcup_k \overline{f_{11} g_{1k}} f_{11} & \ldots & \sqcup_k \overline{f_{1n} g_{nk}} f_{1n} \\ & \vdots & \\ \sqcup_k \overline{f_{m1} g_{1k}} f_{m1} & \ldots & \sqcup_k \overline{f_{mn} g_{nk}} f_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} \sqcup_j \sqcup_k \overline{f_{1j} g_{jk}} f_{11} & \ldots & \sqcup_j \sqcup_k \overline{f_{1j} g_{jk}} f_{1n} \\ & \vdots & \\ \sqcup_j \sqcup_k \overline{f_{mj} g_{jk}} f_{m1} & \ldots & \sqcup_j \sqcup_k \overline{f_{mj} g_{jk}} f_{mn} \end{bmatrix}$$

$$= \overline{[f_{ij}][g_{jk}]}[f_{ij}].$$

Thus, $\text{IMAT}(\mathbb{X})$ is a restriction category.

Existence of partial inverses. The inverse of the map $f = [f_{ij}]$ is the map $f^{(-1)} := [{f_{ji}}^{(-1)}]$.

For the off diagonal elements of $ff^{(-1)}$, they are a disjoint join of $\mathbb{X}$ maps of the form $f_{ij}{f_{\ell j}}^{(-1)}$. By the definition of a disjoint sum matrix $\widehat{f_{ij}} \perp \widehat{f_{\ell j}}$, thus by Lemma 6.2.1 (iii), $0 = \widehat{\widehat{f_{ij}}\widehat{f_{\ell j}}} = \widehat{f_{ij}}\widehat{f_{\ell j}}$. Then,

$$f_{ij}{f_{\ell j}}^{(-1)} = f_{ij}\widehat{f_{ij}}\overline{{f_{\ell j}}^{(-1)}}{f_{\ell j}}^{(-1)} = f_{ij}\widehat{f_{ij}}\widehat{f_{\ell j}}{f_{\ell j}}^{(-1)} = f_{ij}0{f_{\ell j}}^{(-1)} = 0.$$

Therefore all off-diagonal elements are 0. The j^{th} diagonal element is $\sqcup_k f_{jk}{f_{j_k}}^{(-1)} = \sqcup\overline{f_{jk}}$ and thus $ff^{(-1)} = {}_{\text{diag}_j}[\sqcup_k \overline{f_{jk}}] = \overline{f}$. $\square$

Furthermore, $\text{iMat}(\mathbb{X})$ is actually a disjoint sum category:

Theorem 8.1.4. *Given $\mathbb{X}$ an inverse category with disjoint joins and restriction zero, $\text{iMat}(\mathbb{X})$ is an inverse category with disjoint sums.*

Lemma 8.1.3 shows that $\text{iMat}(\mathbb{X})$ is an inverse category. We will prove $\text{iMat}(\mathbb{X})$ has disjoint sums in a series of lemmas.

Lemma 8.1.5. *Given $\mathbb{X}$ is an inverse restriction category with a restriction zero and a disjoint join, then $\text{iMat}(\mathbb{X})$ has a restriction zero.*

Proof. The restriction zero in $\text{iMat}(\mathbb{X})$ is the list $[0]$ where 0 is the restriction zero in $\mathbb{X}$.

For the object $A = [A_1, \ldots, A_n]$, the 0 map is given by the $n \times 1$ matrix $[0, \ldots, 0]$. The map from 0 is given by the $1 \times n$ matrix $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

$\square$

Lemma 8.1.6. *Given $\mathbb{X}$ is an inverse restriction category with a restriction zero, 0, and a disjoint join, then the monoid $\oplus$ defined by list catenation of objects is a disjointness tensor.*

Proof. We first note the monoidal isomorphisms:

$$u^l_\oplus : [0, A_1, A_2, \dots, A_n] \to [A_1, A_2, \dots, A_n] \qquad u^l_\oplus := \begin{bmatrix} 0 & \cdots & 0 \\ & I_{n\times n} & \end{bmatrix}$$

$$u^r_\oplus : [A_1, A_2, \dots, A_n, 0] \to [A_1, A_2, \dots, A_n] \qquad u^r_\oplus := \begin{bmatrix} & I_{n\times n} & \\ 0 & \cdots & 0 \end{bmatrix}$$

$$a_\oplus : (A \oplus B) \oplus C \to A \oplus (B \oplus C) \qquad a_\oplus := id$$

$$c_\oplus : [A_1, \dots, A_n, B_1, \dots, B_m] \to [B_1, \dots, B_m, A_1, \dots, A_n] \qquad c_\oplus := \begin{bmatrix} 0_{m\times n} & I_{n\times n} \\ I_{m\times m} & 0_{n\times m} \end{bmatrix}.$$

The action of $\oplus$ on maps is given by:

$$[f_{ij}] \oplus [g_{\ell k}] = \begin{bmatrix} [f_{ij}] & 0 \\ 0 & [g_{\ell k}] \end{bmatrix}.$$

With this definition, we see that $\oplus$ is a restriction functor:

$$1_X \oplus 1_Y = 1_{X\oplus Y},$$

$$f_1g_1 \oplus f_2g_2 = \begin{bmatrix} f_1g_1 & 0 \\ 0 & f_2g_2 \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}\begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} = (f_1 \oplus f_2)(g_1 \oplus g_2).$$

Following Definition 7.2.1, we note $\amalg_1{}^{(-1)} = (1 \oplus 0)u^r_\oplus = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and similarly $\amalg_2{}^{(-1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose we have $f = [f_{ij}]$ and $g = [g_{ij}]$ where $i \in \{1, \dots, n\}$ and $j \in \{1, 2\}$. Further suppose $f\amalg_1{}^{(-1)} = g\amalg_1{}^{(-1)}$ and $f\amalg_1{}^{(-1)} = g\amalg_1{}^{(-1)}$. Therefore, $f\amalg_1{}^{(-1)} = [f_{i1}] = [g_{i1}] = g\amalg_1{}^{(-1)}$ and $f\amalg_2{}^{(-1)} = [f_{i2}] = [g_{i2}] = g\amalg_2{}^{(-1)}$, but this means that $f = g$ and we may conclude $\amalg_1{}^{(-1)}$ and $\amalg_2{}^{(-1)}$ are jointly monic. Similarly, $\amalg_1 = [1\ 0]$ and $\amalg_2 = [0\ 1]$ are jointly epic. $\square$

Lemma 8.1.7. *Given $\mathbb{X}$ is an inverse category with a disjoint join and restriction zero, then* $\text{iMat}(\mathbb{X})$ *has a disjoint sum tensor.*

Proof. By Lemma 8.1.6, we know that the tensor defined by list catenation is a disjointness tensor. To show that it is a disjoint sum tensor, we must show the diagrams and conditions of Definition 7.3.1 hold.

For the diagram below we show that α exists if and only if $f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$:

$$
\begin{tikzcd}
{[A]} \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& {[X,Y,Z]} \arrow[r, "{\amalg_{1,3}}^{(-1)}"'] \arrow[d, "{\amalg_{1,2}}^{(-1)}"] & {[X,Z]} \arrow[d, "{\amalg_1}^{(-1)}"] \\
& {[X,Y]} \arrow[r, "{\amalg_1}^{(-1)}"'] & {[X].}
\end{tikzcd}
$$

The existence of $f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$ means there is an $h = [h_1, h_2] : [A] \to [Y, Z]$ such that $h{\amalg_1}^{(-1)} = f{\amalg_2}^{(-1)}$ and $h{\amalg_2}^{(-1)} = g{\amalg_2}^{(-1)}$. From the diagram, given that $f = [f_1, f_2]$ and $g = [g_1, g_2]$, we know that $f_1 = f{\amalg_1}^{(-1)} = g{\amalg_1}^{(-1)} = g_1$. We also have $h_1 = f_2$ and $h_2 = g_2$. If we set α to the matrix $[f_1, f_2, g_2]$, the diagram above commutes. We need only show that α is a map in $\textsc{iMat}(\mathbb{X})$. As f, g and h are maps in $\textsc{iMat}(\mathbb{X})$, we know that:

$$
\begin{aligned}
f_1\amalg_1 &\perp f_2\amalg_2 \\
(f_1\amalg_1 =)g_1\amalg_1 &\perp g_2\amalg_2 \\
(f_2\amalg_2 =)h_1\amalg_1 &\perp h_2\,\amalg_2\,(= g_2\amalg_2).
\end{aligned}
$$

From this, we can conclude $\perp [f_1\amalg_1, f_2\amalg_2, g_2\amalg_3]$.

Conversely, suppose we have an $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ that makes the above diagram commute. Then $h := [\alpha_2, \alpha_3]$ is a map in $\mathbb{X}$. Since $[\alpha_1, \alpha_3] = g$ and $[\alpha_1, \alpha_2] = f$, we have $h{\amalg_1}^{(-1)} = f{\amalg_2}^{(-1)}$ and $h{\amalg_2}^{(-1)} = g{\amalg_2}^{(-1)}$, hence $h = f{\amalg_2}^{(-1)} \triangledown g{\amalg_2}^{(-1)}$.

The proof that β in the diagram below exists if and only if $\amalg_2 h \bigtriangleup \amalg_2 k$ is similar:

$$\begin{tikzcd}
 & & {[A]} \\
{[X,Y]} \arrow[r, "\amalg_{1,2}"'] \arrow[urr, bend left, "h"] & {[X,Y,Z]} \arrow[ur, dotted, "\beta"] & \\
{[X]} \arrow[u, "\amalg_1"] \arrow[r, "\amalg_1"'] & {[X,Z].} \arrow[u, "\amalg_{1,3}"] \arrow[uur, bend right, "k"'] &
\end{tikzcd}$$

□

We are now ready to prove Theorem 8.1.4, that $\text{iMat}(\mathbb{X})$ has disjoint sums.

Proof. By Lemma 8.1.7, we know $\text{iMat}(\mathbb{X})$ has a disjoint sum tensor and therefore by Proposition 7.3.5, it has a disjoint join. By Proposition 7.4.1 we know that $[A, B] = A \oplus B$ is a disjoint sum of A and B for any two objects in $\text{iMat}(\mathbb{X})$, and hence, $\text{iMat}(\mathbb{X})$ has disjoint sums. □

8.2 Equivalence between a disjoint sum category and its matrix category

This section starts by giving a functor from an inverse category with disjoint joins to its matrix category, followed by exhibiting a reflection between inverse categories with disjoint sums (Disjoint Sum Cats) and categories with a disjoint join (Disjoint Join Cats). That is,

$$\begin{tikzcd}
\text{Disjoint Sum Cats} \arrow[r, hook] \arrow[r, phantom, shift left=3, "\top"] & \text{Disjoint Join Cats.} \arrow[l, bend right=40, "\text{Matrix}"']
\end{tikzcd} \tag{8.3}$$

Then, we will provide a restriction functor from the matrix category of an inverse category with a disjoint sum to itself, i.e., $\text{iMat}(\mathbb{X}) \to \mathbb{X}$. Furthermore, we will show in the case where $\mathbb{X}$ is an inverse category with a disjoint sum, $\mathbb{X} \cong \text{iMat}(\mathbb{X})$.

Definition 8.2.1. Given $\mathbb{X}$ has disjoint joins and restriction zero, define $M : \mathbb{X} \to \text{iMat}(\mathbb{X})$ by:

$$\text{On objects: } M(A) := [A].$$

$$\text{On maps: } M(f) := [f] \text{ – The } 1 \times 1 \text{ matrix with entry } f.$$

Lemma 8.2.2. *The map M from Definition 8.2.1 is a restriction functor.*

Proof. From the definition of $\text{iMat}(\mathbb{X})$, we have

$$f : A \to B \text{ if and only if } M(f) : M(A) \to M(B) \text{ (i.e., } [f] : [A] \to [B]),$$

$$M(id_A) = [id_A] = id_M(A),$$

$$M(fg) = [fg] = [f][g] = M(f)M(g), \text{ and}$$

$$M(\overline{f}) = [\overline{f}] = \overline{[f]} = \overline{M(f)}.$$

□

Let us represent the category of inverse categories with disjoint sums as DSum and the category of inverse categories with disjoint joins as DJoin.

Note that any inverse category with disjoint sums is an inverse category with disjoint joins. Hence, we have the obvious forgetful functor U : DSum $\to$ DJoin. From above, we also have a functor Mat : DJoin $\to$ DSum given by $Mat : \mathbb{X} \mapsto \text{iMat}(\mathbb{X})$. From the definition of $\text{iMat}(\mathbb{X})$, we see that we have the correspondence

$$\frac{\text{DSum}(\text{iMat}(\mathbb{X}), \mathbb{Y})}{\text{DJoin}(\mathbb{X}, \mathbb{Y})}$$

meaning that we have an adjunction, $Mat \vdash U$: DJoin $\to$ DSum, as noted at the beginning of this section.

Let us now consider when we apply the matrix construction to an inverse category which already has a disjoint sum:

Definition 8.2.3. Given $\mathbb{X}$ has disjoint sums with restriction zero 0, and disjoint sum tensor $\oplus$, define $S : \text{IMAT}(\mathbb{X}) \to \mathbb{X}$ by:

$$\text{Objects: } S([A_1, A_2, \ldots, A_n]) := A_1 \oplus A_2 \oplus \cdots \oplus A_n$$
$$\text{Maps: } S([f_{ij}]) := \bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j).$$

Lemma 8.2.4. *The map S from Definition 8.2.3 is a restriction functor.*

Proof. From the definition of $\text{IMAT}(\mathbb{X})$, where $A = [A_1, A_2, \ldots, A_n]$, $B = [B_1, B_2, \ldots, B_M]$, and $f = [f_{ij}]$ we have

$$S(id_A) = S([id_{A_i}]) = \bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j \amalg_j) = id_{S(A)}$$
$$f : A \to B \iff S(f) : S(A) \to S(B) \iff$$
$$\bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j) : A_1 \oplus \cdots \oplus A_n \to B_1 \oplus \cdots \oplus B_m$$
$$M(\overline{f}) = [\overline{f}] = \overline{[f]} = \overline{M(f)}.$$

For composition, we have

$$\begin{aligned}
S(f)S(g) &= (\bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j))(\bigsqcup_{j'} \amalg_{j'}{}^{(-1)}(\sqcup_k g_{jk} \amalg_k)) \\
&= \bigsqcup_i \amalg_i{}^{(-1)} \bigsqcup_j \bigsqcup_{j'} f_{ij}\ \amalg_j\ \amalg_{j'}{}^{(-1)}(\sqcup_k g_{j'k} \amalg_k) \\
&= \bigsqcup_i \amalg_i{}^{(-1)} \bigsqcup_j f_{ij}(\sqcup_k g_{jk} \amalg_k) \\
&= \bigsqcup_i \amalg_i{}^{(-1)} \bigsqcup_k (\sqcup_j f_{ij} g_{jk} \amalg_k) \\
&= S([\sqcup_j f_{ij} g_{jk}]) \\
&= S(fg).
\end{aligned}$$

□

The functors S and M provide an equivalence:

Proposition 8.2.5. *Given an inverse category $\mathbb{X}$ with $\oplus$ a disjoint sum tensor and a restriction zero, then the categories $\mathbb{X}$ and* $\textsc{iMat}(\mathbb{X})$ *are equivalent.*

Proof. The functors of the equivalence are S from Definition 8.2.3 and M from Definition 8.2.1.

First, we see that $MS : \mathbb{X} \to \mathbb{X}$ is the identity functor as

$$\text{Objects: } S(M(A)) = S([A]) = A,$$

$$\text{Maps: } S(M(f)) = S([f]) = f.$$

Next, we need to show that there is a natural transformation and isomorphism ρ such that $\rho(SM) = I_{\textsc{iMat}(\mathbb{X})}$. For each object $A = [A_1, A_2, \ldots, A_n]$, set $\rho_A = \begin{bmatrix} \amalg_1{}^{(-1)} & \cdots & \amalg_n{}^{(-1)} \end{bmatrix}$.

Note that the functor SM has the following effect:

$$\text{On objects:} \quad M(S([A_1, \ldots, A_n])) = M(A_1 \oplus \cdots \oplus A_n) = [A_1 \oplus \cdots \oplus A_n].$$

$$\text{On maps:} \quad M(S([f_{ij}]) = M(\bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j)) = [\bigsqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j)].$$

We can now draw the commuting naturality square for $f = [f_{ij}] : [A_i] \to [B_j]$:

$$\begin{tikzcd}
SM([A_i]) \arrow[r, equal] & {[\oplus_i A_i]} \arrow[rr, "{\begin{bmatrix} \amalg_1{}^{(-1)} & \cdots & \amalg_n{}^{(-1)} \end{bmatrix}}"] \arrow[d, "SM(f)"'] & & {[A_i]} \arrow[d, "f"] \\
SM([B_j]) \arrow[r, equal] & {[\oplus_j B_j]} \arrow[rr, "{\begin{bmatrix} \amalg_1{}^{(-1)} & \cdots & \amalg_m{}^{(-1)} \end{bmatrix}}"'] & & {[B_j]}.
\end{tikzcd}$$

Following the square by the top–right path from $[\oplus_i A_i]$ to $[B_j]$, by the definition of the maps in the category $\textsc{iMat}(\mathbb{X})$, we see each $B_j = \sqcup_i \amalg_i{}^{(-1)} f_{ij}(\oplus_i A_i)$. Following the left–bottom path, composing $SM(f)$ with $\begin{bmatrix} \amalg_1{}^{(-1)} & \cdots & \amalg_m{}^{(-1)} \end{bmatrix}$ gives us the map

$$\begin{bmatrix} \sqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j)\amalg_1{}^{(-1)} & \cdots & \sqcup_i \amalg_i{}^{(-1)}(\sqcup_j f_{ij} \amalg_j)\amalg_m{}^{(-1)} \end{bmatrix} =$$

$$\begin{bmatrix} \sqcup_i \amalg_i{}^{(-1)} f_{i1} & \cdots & \sqcup_i \amalg_i{}^{(-1)} f_{im} \end{bmatrix}.$$

Applying this to $[\oplus_i A_i]$, we see each $B_j = \sqcup_i {\amalg_i}^{(-1)} f_{ij}(\oplus_i A_i)$ and the two directions are equal.

Finally, we know that ${\rho_{A_i}}^{(-1)} = \begin{bmatrix} \amalg_1 \\ \vdots \\ \amalg_n \end{bmatrix}$ and defines an isomorphism between any object of the form $[\oplus_i A_i]$ and the object $[A_1, \ldots, A_n]$. $\square$

Note that as we have an equivalence, we have the adjoint equivalence between hom-sets of

$$\frac{\mathbb{X}(S(X), Y)}{\textsc{iMat}(\mathbb{X})(X, M(Y))}$$

as S is the left adjoint of M.

Example 8.2.6. We may obtain a matrix representative of any map $f : A \oplus B \to C \oplus D$ by applying the construction of Definition 8.2.3 in reverse.

Then given a function $f : A \oplus B \to C \oplus D$ define

$$f_M = \begin{bmatrix} \amalg_1 f {\amalg_1}^{(-1)} & \amalg_1 f {\amalg_2}^{(-1)} \\ \amalg_2 f {\amalg_1}^{(-1)} & \amalg_2 f {\amalg_2}^{(-1)} \end{bmatrix}.$$

Thus, applying the functor S from Definition 8.2.4, we have

$$\begin{aligned} S(f_M) &= {\amalg_1}^{(-1)}(\amalg_1 f {\amalg_1}^{(-1)}\, \amalg_1 \sqcup \amalg_1\, f {\amalg_2}^{(-1)} \amalg_2) \sqcup {\amalg_2}^{(-1)}(\amalg_2 f {\amalg_1}^{(-1)}\, \amalg_1 \sqcup \amalg_2\, f {\amalg_2}^{(-1)} \amalg_2) \\ &= {\amalg_1}^{(-1)}\, \amalg_1\, f({\amalg_1}^{(-1)}\, \amalg_1 \sqcup {\amalg_2}^{(-1)} \amalg_2) \sqcup {\amalg_2}^{(-1)}\, \amalg_2\, f({\amalg_1}^{(-1)}\, \amalg_1 \sqcup {\amalg_2}^{(-1)} \amalg_2) \\ &= {\amalg_1}^{(-1)}\, \amalg_1\, f \sqcup {\amalg_2}^{(-1)}\, \amalg_2\, f \\ &= ({\amalg_1}^{(-1)}\, \amalg_1 \sqcup {\amalg_2}^{(-1)} \amalg_2) f \\ &= f. \end{aligned}$$

In particular, we note that we may represent $f : A \to B$ by the matrix

$$\begin{bmatrix} 1f1 & 1f0 \\ 0f1 & 0f0 \end{bmatrix} = \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$$

as $A \cong A \oplus 0$ and $B \cong B \oplus 0$. Equivalently, this is the matrix $\begin{bmatrix} f \end{bmatrix}$.

Definition 8.2.7. A unique decomposition category [35] is a category where any

$$h : A \oplus B \to C \oplus D$$

is uniquely determined by the four maps:

$$\amalg_1 h \amalg_1{}^{(-1)} : A \to C, \quad \amalg_1 h \amalg_2{}^{(-1)} : A \to D,$$
$$\amalg_2 h \amalg_1{}^{(-1)} : B \to C, \quad \amalg_2 h \amalg_2{}^{(-1)} : B \to D.$$

That is, it is writable as the matrix:

$$\begin{bmatrix} \amalg_1 h \amalg_1^{(-1)} & \amalg_1 h \amalg_2^{(-1)} \\ \amalg_2 h \amalg_1^{(-1)} & \amalg_2 h \amalg_2^{(-1)} \end{bmatrix} : A \oplus B \to C \oplus D.$$

The map from 0 in the category corresponds to the 0-dimensional matrix,

$$0 \xrightarrow{[]} B.$$

Corollary 8.2.8. *If $\mathbb{X}$ is an inverse category with disjoint sums, then it is a unique decomposition category.*

The fact that we get a unique decomposition category is important as Proposition 4.0.11 of [35] gives a formula for computing a trace, when it exists. In a unique decomposition category, for a function $f : X \oplus U \to Y \oplus U$, which may be represented by the matrix

$$f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix},$$

the unique inductive trace is given by the formula:

$$Tr^U_{X,Y}(f) = f_{11} + \sum_{n=0}^{\infty} f_{12} f_{22}^n f_{21}, \tag{8.4}$$

whenever this sum exists. Therefore, for example, an inverse category with countable disjoint sums will have a trace.

Chapter 9

Distributive inverse categories

We now consider inverse categories with both a disjoint sum and inverse product, where the inverse product distributes over the disjoint sum in a specific way. This chapter will show that the Cartesian Completion of such a category is a distributive restriction category where the product distributes over the coproduct.

9.1 Distributive restriction categories

Definition 9.1.1. A Cartesian category $\mathbb{B}$ with coproduct $+$ and is called a *distributive* [14] category when

$$(A \times B) + (A \times C) \cong A \times (B + C).$$

Definition 9.1.2. A Cartesian restriction category $\mathbb{D}$ (as in Definition 3.9.3), with a restriction zero and coproducts is called a *distributive restriction category* [23] when there is an isomorphism ρ such that

$$A \times (B + C) \xrightarrow{\rho} (A \times B) + (A \times C).$$

If $\mathbb{D}$ is a distributive restriction category, then $\text{Total}(\mathbb{D})$ is a distributive category as in Definition 9.1.1.

9.2 Distributive inverse categories

Definition 9.2.1. A *distributive inverse category* $\mathbb{D}$ consists of the following:

- $\mathbb{D}$ is an inverse category;
- $\mathbb{D}$ has an inverse product with tensor $\otimes$, per Definition 4.3.1;

- $\mathbb{D}$ has a disjoint sum tensor, $\oplus$, per Definition 7.3.1 and
- There is a family of isomorphisms, d, such that

$$\begin{tikzcd} & & A \otimes B \\ A \otimes (B \oplus C) \arrow[urr, "1\otimes\amalg_1{}^{(-1)}"] \arrow[rr, "d"] \arrow[drr, "1\otimes\amalg_2{}^{(-1)}"'] & & (A \otimes B) \oplus (A \otimes C) \arrow[u, "\amalg_1{}^{(-1)}"'] \arrow[d, "\amalg_2{}^{(-1)}"] \\ & & A \otimes C, \end{tikzcd} \tag{9.1}$$

commutes in $\mathbb{D}$ for any choices of objects A, B, C.

Example 9.2.2 (Pinj is a distributive inverse category)**.** The following defines the isomorphism d of Diagram (9.1):

$$d((a,(x,n)) = \begin{cases} ((a,x),1) & n = 1 \\ ((a,x),2) & n = 2. \end{cases} \tag{9.2}$$

Note that as we are operating in an inverse category, we also have the inverse of diagram (9.1) available to us. That is,

$$\begin{tikzcd} A \otimes B \arrow[d, "\amalg_1"'] \arrow[drr, "1\otimes\amalg_1"] & & \\ (A \otimes B) \oplus (A \otimes C) \arrow[rr, "d^{(-1)}"] & & A \otimes (B \oplus C) \\ A \otimes C \arrow[u, "\amalg_2"] \arrow[urr, "1\otimes\amalg_2"'] & & \end{tikzcd} \tag{9.3}$$

is also a commuting diagram in $\mathbb{D}$.

Definition 9.2.3. Suppose $\mathbb{X}$ is an inverse category with $\oplus$ a disjoint sum tensor and a restriction zero. Then for maps $f : A \to B$ and $g : A \to C$ with $\overline{f} \perp \overline{g}$, define the map $[f, g] : A \to B \oplus C$ as $(f\amalg_1) \sqcup (g\amalg_2)$. This is well defined as $\widehat{\amalg_1} \perp \widehat{\amalg_2}$ and therefore by [**Dis.7**], $f\amalg_1 \perp g\amalg_2$.

Lemma 9.2.4. *Suppose $\mathbb{X}$ is an inverse category $\mathbb{X}$ with:*

- *$\oplus$ a disjoint sum tensor,*
- *a restriction zero, and*
- *an inverse product $\otimes$ which distributes over disjoint joins, (that is, $f \otimes (g \sqcup h) = (f \otimes g) \sqcup (f \otimes h)$).*

Then, $\mathbb{X}$ is an distributive inverse category.

Proof. By assumption, we have the first three items of Definition 9.2.1. Therefore, we need to construct an isomorphism d such that diagram (9.1) commutes. We claim that the map $d = [1 \otimes \amalg_1{}^{(-1)}, 1 \otimes \amalg_2{}^{(-1)}]$ does this.

First, note that the typing of d is correct. By Definition 9.2.3,

$$d = ((1 \otimes \amalg_1{}^{(-1)})\amalg_1) \sqcup ((1 \otimes \amalg_2{}^{(-1)})\amalg_2) : A \otimes (B \oplus C) \to (A \otimes B) \oplus (A \otimes C)$$

as

$$A \otimes (B \oplus C) \xrightarrow{(1 \otimes \amalg_1{}^{(-1)})} A \otimes B \xrightarrow{\amalg_1{}^{(-1)}} (A \otimes B) \oplus (A \otimes C),$$
$$A \otimes (B \oplus C) \xrightarrow{(1 \otimes \amalg_2{}^{(-1)})} A \otimes C \xrightarrow{\amalg_2{}^{(-1)}} (A \otimes B) \oplus (A \otimes C).$$

Next, we need to show d is an isomorphism. We will do this by showing both $\overline{d} = 1$ and $\overline{d^{(-1)}} = 1$. As a consequence of Lemma 6.3.3, we know the inverse of d is

$$((1 \otimes \amalg_1{}^{(-1)})\amalg_1)^{(-1)} \sqcup ((1 \otimes \amalg_2{}^{(-1)})\amalg_2)^{(-1)} = (\amalg_1{}^{(-1)}(1 \otimes \amalg_1)) \sqcup (\amalg_2{}^{(-1)}(1 \otimes \amalg_2)).$$

Having $\otimes$ distribute over the disjoint sum means that for any maps f, h, k with $h \perp k$, we have $f \otimes (h \sqcup k) = (f \otimes h) \sqcup (f \otimes k)$. We use this in the calculation of the restriction of d:

$$\begin{aligned}
\overline{((1 \otimes \amalg_1{}^{(-1)})\amalg_1) \sqcup ((1 \otimes \amalg_2{}^{(-1)})\amalg_2)} &= \overline{((1 \otimes \amalg_1{}^{(-1)})\amalg_1)} \sqcup \overline{((1 \otimes \amalg_2{}^{(-1)})\amalg_2)} \\
&= (1 \otimes \overline{\amalg_1{}^{(-1)}}) \sqcup (1 \otimes \overline{\amalg_2{}^{(-1)}}) \\
&= (1 \otimes (\overline{\amalg_1{}^{(-1)}} \sqcup \overline{\amalg_2{}^{(-1)}}) \\
&= 1 \otimes ((1 \oplus 0) \sqcup (0 \oplus 1)) \\
&= 1 \otimes 1 = 1.
\end{aligned}$$

The calculation for $\overline{d^{(-1)}}$ also shows it is 1:

$$\begin{aligned}
\overline{(\amalg_1{}^{(-1)}(1 \otimes \amalg_1)) \sqcup (\amalg_2{}^{(-1)}(1 \otimes \amalg_2))} &= \overline{(\amalg_1{}^{(-1)}(1 \otimes \amalg_1))} \sqcup \overline{(\amalg_2{}^{(-1)}(1 \otimes \amalg_2))} \\
&= \overline{(\amalg_1{}^{(-1)}\overline{(1 \otimes \amalg_1)})} \sqcup \overline{(\amalg_2{}^{(-1)}\overline{(1 \otimes \amalg_2)})} \\
&= \overline{(\amalg_1{}^{(-1)})} \sqcup \overline{(\amalg_2{}^{(-1)})} \\
&= (1 \oplus 0) \sqcup (0 \oplus 1) \\
&= 1.
\end{aligned}$$

Hence, $[1\otimes\amalg_1{}^{(-1)}, 1\otimes\amalg_2{}^{(-1)}]$ is an isomorphism. Finally, we must show that diagram (9.1) commutes:

$$\begin{aligned}
d\amalg_1{}^{(-1)} &= \Big(((1 \otimes \amalg_1{}^{(-1)})\amalg_1) \sqcup ((1 \otimes \amalg_2{}^{(-1)})\amalg_2)\Big)\, \amalg_1{}^{(-1)} \\
&= \Big(((1 \otimes \amalg_1{}^{(-1)})\amalg_1)\amalg_1{}^{(-1)}\Big) \sqcup \Big(((1 \otimes \amalg_2{}^{(-1)})\amalg_2)\amalg_1{}^{(-1)}\Big) \\
&= \Big((1 \otimes \amalg_1{}^{(-1)})1\Big) \sqcup \Big((1 \otimes \amalg_2{}^{(-1)})0\Big) \\
&= (1 \otimes \amalg_1{}^{(-1)}) \sqcup 0 \\
&= 1 \otimes \amalg_1{}^{(-1)}
\end{aligned}$$

and

$$\begin{aligned}
d\amalg_2{}^{(-1)} &= \Big(((1 \otimes \amalg_1{}^{(-1)})\amalg_1) \sqcup ((1 \otimes \amalg_2{}^{(-1)})\amalg_2)\Big)\, \amalg_2{}^{(-1)} \\
&= \Big(((1 \otimes \amalg_1{}^{(-1)})\amalg_1)\amalg_2{}^{(-1)}\Big) \sqcup \Big(((1 \otimes \amalg_2{}^{(-1)})\amalg_2)\amalg_2{}^{(-1)}\Big) \\
&= 0 \sqcup (1 \otimes \amalg_2{}^{(-1)}) \\
&= 1 \otimes \amalg_2{}^{(-1)}.
\end{aligned}$$

This shows the fourth condition is satisfied and $\mathbb{X}$ is a distributive inverse category. $\square$

We have seen that a second tensor distributing over the disjoint joins implies that we have an inverse distributive category. We now show the converse is true.

Lemma 9.2.5. *Given an inverse distributive category $\mathbb{X}$, then $h \otimes (f \triangledown g) = (h \otimes f) \triangledown (h \otimes g)$ whenever $f \triangledown g$ exists and $h \otimes (f \vartriangle g) = (h \otimes f) \vartriangle (h \otimes g)$ whenever $f \vartriangle g$ exists.*

Proof. Let $h : A \to B$, $f : C \to D$ and $g : C \to E$. Consider the following diagram:

$$\begin{tikzcd}
 & & B \otimes E \\
A \otimes C \arrow[urr, "h\otimes f"] \arrow[r, "h\otimes(f \triangledown g)"] \arrow[drr, "h\otimes g"'] & B \otimes (D \oplus E) \arrow[ur, "1\otimes \amalg_1{}^{(-1)}"] \arrow[r, "\cong"] \arrow[dr, "1\otimes \amalg_2{}^{(-1)}"] & (B \otimes D) \oplus (B \otimes E) \arrow[u, "\amalg_1{}^{(-1)}"'] \arrow[d, "\amalg_2{}^{(-1)}"] \\
 & & B \otimes D.
\end{tikzcd}$$

The two leftmost triangles commute by the diagram for $f \triangledown g$. The right hand triangles commute as per Definition 9.2.1. By the uniqueness of the $\triangledown$ operation we see $h \otimes (f \triangledown g) = (h \otimes f) \triangledown (h \otimes g)$,

The argument for showing $h \otimes (f \vartriangle g) = (h \otimes f) \vartriangle (h \otimes g)$ follows the same pattern. $\square$

Lemma 9.2.6. *Given an inverse distributive category $\mathbb{X}$, $\otimes$ distributes over the disjoint join.*

Proof. First recall the definition of $f \sqcup g = (\overline{f} \triangledown \overline{g})(f \vartriangle g)$. In order to show $h \otimes (f \sqcup g) = (h \otimes f) \sqcup (h \otimes g)$, we need to show that

$$h \otimes (\overline{f} \triangledown \overline{g})(f \vartriangle g) = (\overline{h \otimes f} \triangledown \overline{h \otimes g})(h \otimes f \vartriangle h \otimes g). \tag{9.4}$$

Since $h \otimes (\overline{f} \triangledown \overline{g})(f \vartriangle g) = (\overline{h} \otimes (\overline{f} \triangledown \overline{g}))(h \otimes (f \vartriangle g))$, Equation (9.4) follows directly from Lemma 9.2.5 and the fact that $\otimes$ is a restriction functor. $\square$

Corollary 9.2.7. *Suppose we have an inverse distributive category $\mathbb{X}$. Then,*

(i) if $f \perp g$, then $(h \otimes f) \perp (h \otimes g)$ for any h,

(ii) if $f \perp g : A \to B$ and $h \perp k : C \to D$, then $(f \otimes h) \perp (g \otimes k)$.

Proof.

(i) As $f \perp g$, we have $f \vartriangle g$ and $f \triangledown g$. By Lemma 9.2.5, both $h \otimes f \vartriangle h \otimes g$ and $h \otimes f \triangledown h \otimes g$ exist and therefore $h \otimes f \perp h \otimes g$.

(ii) By the previous item, we have that $((f \sqcup g) \otimes h) \perp ((f \sqcup g) \otimes k)$. Then, by [**DJ.1**] and [**Dis.3**] we have $(f \otimes h) \perp (g \otimes k)$.

$\square$

9.3 Discrete inverse categories with disjoint sums

We now consider the case where we have a discrete inverse category with inverse product $\otimes$ and a disjoint sum $\oplus$, where the $\otimes$ tensor preserves the disjoint join.

Recall that the Cartesian Completion, from Definition 5.1.1, when applied to the discrete inverse category $\mathbb{X}$ produces the discrete Cartesian restriction category $\widetilde{\mathbb{X}}$. A map in $\widetilde{\mathbb{X}}$ is related to a map in $\mathbb{X}$ in the following way:

$$\frac{A \xrightarrow{(f,C)} B \text{ in } \widetilde{\mathbb{X}}}{A \xrightarrow{f} B \otimes C \text{ in } \mathbb{X}.}$$

Our goal is to show that a disjoint sum in a distributive inverse category becomes a coproduct in $\widetilde{\mathbb{X}}$.

Lemma 9.3.1. *Given $\mathbb{X}$ is a distributive inverse category, then $\widetilde{\mathbb{X}}$ has a restriction zero.*

Proof. Recall from Theorem 5.2.6 that $\mathbb{X}$ is equivalent as a category to $\widetilde{\mathbb{X}}$ under the identity on objects functor

$$\mathbf{T} : \mathbb{X} \to \widetilde{\mathbb{X}}; \quad \begin{array}{c} A \\ \downarrow f \\ B \end{array} \mapsto \begin{array}{c} A \\ \downarrow (f u_\otimes^{r\,(-1)},1) \\ B \end{array} .$$

In $\mathbb{X}$, we know 0 is a terminal and initial object, with maps $A \xrightarrow{t_A} 0$ and $0 \xrightarrow{z_A} A$, where $\overline{0_{A,A}} = 0_{A,A} = t_A z_A$.

First we note that 0 is both initial and terminal in $\widetilde{\mathbb{X}}$, with the terminal maps being $\mathbf{T}(t_A)$ and initial maps being $\mathbf{T}(z_A)$.

As was also shown in Theorem 5.2.6, $\mathbf{T}$ is a restriction functor, so in $\widetilde{\mathbb{X}}$ we have

$$0_{A,A} = \mathbf{T}(t_A)\mathbf{T}(z_A) = \mathbf{T}(t_A z_A) = \mathbf{T}(0_{A,A}) = \mathbf{T}(\overline{0_{A,A}}) = \overline{\mathbf{T}(0_{A,A})} = \overline{0_{A,A}}.$$

Hence, $0_{A,A}$ is a restriction zero in $\widetilde{\mathbb{X}}$. $\Box$

Lemma 9.3.2. *In a distributive inverse category $\mathbb{X}$, the following hold:*

(i) Given $f : A \to Y \otimes C$ we can construct maps $f' : A \to Y \otimes (C \oplus D)$ and $f'' : A \to Y \otimes (E \otimes C)$ for some objects D, E such that $f \simeq f'$ and $f \simeq f''$.

(ii) Given $f : A \to Y \otimes C$, $g : A \to Y \otimes D$, then the $f' : A \to Y \otimes (C \oplus D)$, $g'' : B \to Y \otimes (C \oplus D)$ as constructed in (i) satisfies $\amalg_1{}^{(-1)} f' \perp \amalg_2{}^{(-1)} g''$.

Proof.

(i) Set $f' = f(1 \otimes \amalg_1)$. To show $f \simeq f'$, we must first show their restriction is the same:

$$\overline{f(1 \otimes \amalg_1)} = \overline{f\overline{(1 \otimes \amalg_1)}} = \overline{f1} = \overline{f}.$$

The mediating map between f and f' is, of course, $1 \otimes \amalg_1$:

$$\begin{tikzcd} & Y \otimes C \arrow[dd, dotted, "1\otimes\amalg_1"] \\ A \arrow[ur, "f"] \arrow[dr, "f'"'] & \\ & Y \otimes (C \oplus D). \end{tikzcd}$$

By the same reasoning we may also create $f'' : A \to Y \otimes (D \oplus C)$ by setting $f'' = f(1 \otimes \amalg_2)$.

(ii) First note we have $\amalg_1{}^{(-1)} f', \amalg_2{}^{(-1)} g'' : A \oplus B \to Y \otimes (C \oplus D)$. In order to show $\amalg_1{}^{(-1)} f' \perp \amalg_2{}^{(-1)} g''$, we will proceed by showing their restrictions and ranges are disjoint. As $\overline{\amalg_1{}^{(-1)}} \perp \overline{\amalg_2{}^{(-1)}}$ and $\overline{\amalg_1{}^{(-1)} f'} \leq \overline{\amalg_1{}^{(-1)}}$ and $\overline{\amalg_2{}^{(-1)} g'} \leq \overline{\amalg_2{}^{(-1)}}$, we immediately have $\overline{\amalg_1{}^{(-1)} f'} \perp \overline{\amalg_2{}^{(-1)} g''}$.

For the ranges, we have

$$\begin{aligned}\widehat{\amalg_1^{(-1)}} f' = \amalg_1^{(-1)}\widehat{(f(1 \otimes \amalg_1))} &= \overline{((1 \otimes \amalg_1^{(-1)})f^{(-1)})\amalg_1} \\ &= \overline{((1 \otimes \amalg_1^{(-1)})f^{(-1)}} \\ &\leq \overline{(1 \otimes \amalg_1^{(-1)})}\end{aligned}$$

and similarly

$$\widehat{\amalg_2^{(-1)}} g'' \leq \overline{(1 \otimes \amalg_2^{(-1)})}.$$

Using Lemma 6.2.3 we know that $\overline{(\amalg_1^{(-1)})} \perp \overline{(\amalg_2^{(-1)})}$. From Corollary 9.2.7 we conclude that $\overline{(1 \otimes \amalg_1^{(-1)})} \perp \overline{(1 \otimes \amalg_2^{(-1)})}$ and giving us $\widehat{\amalg_1^{(-1)}} f' \perp \widehat{\amalg_2^{(-1)}} g''$ and therefore $\amalg_1^{(-1)} f' \perp \amalg_2^{(-1)} g''$.

$\square$

Theorem 9.3.3. *Given $\mathbb{X}$ is a distributive inverse category, then the category $\widetilde{\widetilde{\mathbb{X}}}$ has coproducts.*

Proof. The tensor object $A \oplus B$ in $\mathbb{X}$ will become the coproduct of A, B in $\widetilde{\widetilde{\mathbb{X}}}$.

The injection maps of the coproduct are $i_1 = (\amalg_1 {u_\otimes^r}^{(-1)}, 1)$ and $i_2 = (\amalg_2 {u_\otimes^r}^{(-1)}, 1)$.

Consider the following diagram in $\widetilde{\widetilde{\mathbb{X}}}$:

$$\begin{tikzcd}
A \arrow[dr, "i_1"'] \arrow[drr, "{(f,C)}"] & & \\
& A \oplus B \arrow[r, dotted, "h"] & Y. \\
B \arrow[ur, "i_2"] \arrow[urr, "{(g,D)}"'] & &
\end{tikzcd}$$

In $\mathbb{X}$, this comes from the diagram:

$$\begin{tikzcd}
A \arrow[dr, "\amalg_1"'] \arrow[rr, "f"] & & Y \otimes C \\
& A \oplus B & \\
B \arrow[ur, "\amalg_2"] \arrow[rr, "g"] & & Y \otimes D
\end{tikzcd} \tag{9.5}$$

where the extraneous unit isomorphisms are removed.

This corresponds to the conditions of Lemma 9.3.2. Hence by that lemma we may revise Diagram (9.5) as

$$\begin{tikzcd}
A \arrow[dr, "\amalg_1"] \arrow[drr, "f'"] & & \\
& A \oplus B \arrow[r, "{\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g''"] & Y \otimes (C \oplus D) \\
B \arrow[ur, "\amalg_2"] \arrow[urr, "g''"'] & &
\end{tikzcd} \tag{9.6}$$

where f' and g'' are respectively equivalent to f, g.

Lifting Diagram (9.6) to $\mathbb{X}$, we see this corresponds to the desired coproduct diagram, where h in $\widetilde{\mathbb{X}}$ is the map $({\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g'', (C \oplus D))$.

By construction, in $\mathbb{X}$, we have

$$\amalg_1({\amalg_1}^{(-1)} f' \sqcup \amalg_1 {\amalg_2}^{(-1)} g'') = (\amalg_1 {\amalg_1}^{(-1)} f') \sqcup (\amalg_1 {\amalg_2}^{(-1)} g'') = f' \sqcup 0 = f'$$

and

$$\amalg_2({\amalg_1}^{(-1)} f' \sqcup \amalg_1 {\amalg_2}^{(-1)} g'') = g''.$$

Hence, in $\widetilde{\mathbb{X}}$, we have $(i_1 u^r_\otimes, 1)h = f$ and $(i_2 u^r_\otimes, 1)h = g$.

All that remains to be shown is that h is unique.

Suppose there is another (k, E) in $\widetilde{\mathbb{X}}$ such that it satisfies the coproduct properties, i.e., that $i_1(k, E) = (f', C \oplus D)$ and $i_2(k, E) = (g'', C \oplus D)$. In $\mathbb{X}$, $k : A \oplus B \to Y \otimes E$ and we have

$$\amalg_1 k \simeq f' \quad \text{and}$$

$$\amalg_2 k \simeq g'.$$

Since equivalence is a transitive relation, this means we have

$$f \overset{q_1}{\simeq} \amalg_1 k \quad \text{and}$$
$$g \overset{q_2}{\simeq} \amalg_2 k,$$

where the maps $q_1 : Y \otimes C \to Y \otimes E$ and $q_2 : Y \otimes D \to Y \otimes E$ fulfill the respective equivalence diagrams.

Explicitly for k, f and q_1, this gives us:

$$\begin{tikzcd}
 & Y \otimes C \arrow[dd, dotted, "q_1"] \\
A \arrow[ur, "f"] \arrow[dr, "\amalg_1 k"'] & \\
 & Y \otimes E.
\end{tikzcd}$$

Now, we turn out attention to showing that $k \simeq h = {\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g''$. Consider

$$\begin{tikzcd}
 & Y \otimes (C \oplus D) \arrow[dd, dotted, "t"] \\
A \arrow[ur, "{\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g''"] \arrow[dr, "k"'] & \\
 & Y \otimes E.
\end{tikzcd} \tag{9.7}$$

As $\mathbb{X}$ is an inverse category, we know there is a map t that makes this diagram commute, namely $t = ({\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g'')^{(-1)} k$. However, we must show this is in Y^{Δ}_{∇}.

Next, recalling Definition 7.2.6 consider

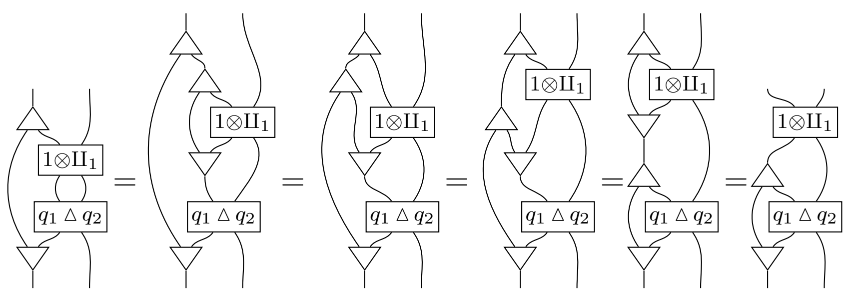
$$\begin{tikzcd}
Y \otimes C \arrow[r, "1 \otimes \amalg_1"] \arrow[dr, "q_1"'] & Y \otimes (C \oplus D) \arrow[d, dotted, "q_1 \vartriangle q_2"] & Y \otimes D \arrow[l, "1 \otimes \amalg_2"'] \arrow[dl, "q_2"] \\
 & Y \otimes E. &
\end{tikzcd}$$

The map $q_1 \vartriangle q_2$ exists iff $\widehat{q_1} \perp \widehat{q_2}$. But, the map t from above does make the diagram commute. This can be shown by

$$
\begin{aligned}
&(1 \otimes \amalg_1)({\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g'')^{(-1)} k \\
&\qquad = (1 \otimes \amalg_1)({\amalg_1}^{(-1)} f(1 \otimes \amalg_1) \sqcup {\amalg_2}^{(-1)} g(1 \otimes \amalg_2))^{(-1)} k \\
&\qquad = (1 \otimes \amalg_1)((1 \otimes {\amalg_1}^{(-1)}) f^{(-1)} \amalg_1 \sqcup (1 \otimes {\amalg_2}^{(-1)}) g^{(-1)} \amalg_2) k \\
&\qquad = ((1 \otimes 1) f^{(-1)} \amalg_1 \sqcup (1 \otimes 0) g^{(-1)} \amalg_2) k \\
&\qquad = f^{(-1)} \amalg_1 k \\
&\qquad = q_1
\end{aligned}
$$

and similarly for q_2. Therefore, $\widehat{q_1} \perp \widehat{q_2}$.

This means we may form the map $q_1 \triangle q_2 : Y \otimes (C \oplus D) \to Y \otimes E$. But then the map $(q_1 \triangle q_2)$ makes Diagram (9.7) commute. At the same time, $q_1, q_2 \in Y^{\triangle}_{\triangledown}$, which gives

$q_1 =$  $= (1 \otimes \amalg_1)(q_1 \triangle q_2)^{\triangle}_{\triangledown}.$

Similarly, $q_2 = (1 \otimes \amalg_2)(q_1 \triangle q_2)^{\triangle}_{\triangledown}$ giving us $q_1 \triangle q_2 \in Y^{\triangle}_{\triangledown}$ by the uniqueness of the $\triangle$ operation. Therefore $q_1 \triangle q_2$ provides an equivalence between k and ${\amalg_1}^{(-1)} f' \sqcup {\amalg_2}^{(-1)} g''$, meaning the coproduct is unique. $\square$

Corollary 9.3.4. *When $\mathbb{X}$ is a distributive inverse category, $\widetilde{\mathbb{X}}$ is a distributive restriction category.*

Proof. As $\widetilde{\mathbb{X}}$ has restriction products by Lemma 5.1.11, restriction coproducts by Theorem 9.3.3 and the equations for distributivity follow directly from the distributivity of the base tensors, we see $\widetilde{\mathbb{X}}$ is a distributive restriction category. $\square$

Chapter 10

Commutative Frobenius algebras and inverse categories

10.1 The category of commutative Frobenius algebras

Dagger categories generalize the category of Hilbert spaces which is often used to model quantum computation. These were introduced in [3] as *strongly compact closed categories*, an additional structure on compact closed categories.

Before introducing dagger categories, we define compact closed categories.

Definition 10.1.1. A *compact closed category* $\mathbb{D}$ is a symmetric monoidal category with tensor $\otimes$ where each object A has a dual A^*. Additionally, there must exist families of maps $\eta_A : I \to A^* \otimes A$ (the *unit*) and $\epsilon_A : A \otimes A^* \to I$ (the *counit*) such that

$$
\begin{tikzcd}
A \arrow[r, "u_A"] \arrow[d, equal] & A \otimes I \arrow[r, "1\otimes\eta_A"] & A \otimes (A^* \otimes A) \arrow[d, "a_{A,A^*,A}"] \\
A & I \otimes A \arrow[l, "u_A^{-1}"] & (A \otimes A^*) \otimes A \arrow[l, "\epsilon_A\otimes 1"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A^* \arrow[r, "u_{A^*}"] \arrow[d, equal] & I \otimes A^* \arrow[r, "\eta_A\otimes 1"] & (A^* \otimes A) \otimes A^* \arrow[d, "a^{-1}_{A^*,A,A^*}"] \\
A^* & A^* \otimes I \arrow[l, "u_{A^*}^{-1}"] & A^* \otimes (A \otimes A^*) \arrow[l, "1\otimes\epsilon_A"]
\end{tikzcd}
$$

commute.

Given a map $f : A \to B$ in a compact closed category, define the map $f^* : B^* \to A^*$ as

$$
\begin{tikzcd}
B^* \arrow[r, "u_{B^*}"] \arrow[d, "f^*"'] & I \otimes B^* \arrow[r, "\eta_A\otimes 1"] & A^* \otimes A \otimes B^* \arrow[d, "1\otimes f\otimes 1"] \\
A^* & A^* \otimes I \arrow[l, "u_{A^*}^{-1}"] & A^* \otimes B \otimes B^*. \arrow[l, "1\otimes\epsilon_B"]
\end{tikzcd}
$$

10.1.1 Dagger categories

Although dagger categories were introduced in the context of compact closed categories, the concept of a dagger is definable independently. This was first done in [64].

Definition 10.1.2. A *dagger* on a category D is a functor $\dagger : \mathbb{D}^{op} \to \mathbb{D}$, which is involutive, that is, $f^{\dagger\dagger} = f$ and which is the identity on objects. A *dagger category* is a category that has a dagger.

Typically, the dagger is written as a superscript on the morphism. So, if $f : A \to B$ is a map in $\mathbb{D}$, then $f^\dagger : B \to A$ is a map in $\mathbb{D}$ and is called the *adjoint* of f. A map where $f^{-1} = f^\dagger$ is called *unitary.* A map $f : A \to A$ with $f = f^\dagger$ is called *self-adjoint* or *Hermitian.*

Definition 10.1.3. A *dagger symmetric monoidal category* is a symmetric monoidal category $\mathbb{D}$ with a dagger operator such that:

(i) For all maps $f : A \to B$ and $g : C \to D$, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger : B \otimes D \to A \otimes C$;

(ii) The monoid structure isomorphisms $a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $u^l_A : I \otimes A \to A$, $u^r_A : A \otimes I \to A$ and $c_{A,B} : A \otimes B \to B \otimes A$ are unitary.

Definition 10.1.4. A *dagger compact closed category* $\mathbb{D}$ is a dagger symmetric monoidal category that is compact closed where the diagram

$$\begin{tikzcd} I \arrow[r, "\epsilon_A^\dagger"] \arrow[dr, "\eta_A"'] & A \otimes A^* \arrow[d, "c_{A,A^*}"] \\ & A^* \otimes A \end{tikzcd}$$

commutes for all objects A in $\mathbb{D}$.

Lemma 10.1.5. *If $\mathbb{D}$ is a dagger category with biproducts $\boxplus$, with injections in_1, in_2 and projections p_1, p_2, then the following are equivalent:*

(i) ${p_i}^\dagger = in_i, i = 1, 2,$

(ii) $(f \boxplus g)^\dagger = f^\dagger \boxplus g^\dagger$ *and* $\Delta^\dagger = \nabla$,

(iii) $\langle f, g\rangle^\dagger = [f^\dagger, g^\dagger]$,

(iv) The map $[{p_1}^\dagger, {p_2}^\dagger] : A^\dagger \boxplus B^\dagger \to (A \boxplus B)^\dagger$ *is the identity map.*

Proof. **(i)** $\implies$ **(ii)** To show $\Delta^\dagger = \nabla$, draw the product cone for Δ,

$$\begin{tikzcd} & A \arrow[dl, "id"'] \arrow[d, "\Delta"] \arrow[dr, "id"] & \\ A & A \boxplus A \arrow[l, "p_1"] \arrow[r, "p_2"'] & A \end{tikzcd}$$

and apply the dagger functor to it. As ${p_i}^\dagger = in_i$, and $\dagger$ is identity on objects, this is now a coproduct diagram and therefore $\Delta^\dagger = \nabla$.

For $(f \boxplus g)^\dagger = f^\dagger \boxplus g^\dagger$, start with the diagram defining $f \boxplus g$ as a product of the arrows:

$$\begin{tikzcd} A \arrow[d, "f"'] & A \boxplus B \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "f \boxplus g"] & A \arrow[d, "g"] \\ C & C \boxplus D \arrow[l, "p_1"] \arrow[r, "p_2"'] & D. \end{tikzcd}$$

Then, apply the dagger functor to this diagram. This is now the diagram defining the coproduct of maps and therefore $(f \boxplus g)^\dagger = f^\dagger \boxplus g^\dagger$.

(ii) $\implies$ **(iii)** The calculation showing this is

$$\begin{aligned} [f^\dagger, g^\dagger] &= \nabla; (f^\dagger \boxplus g^\dagger) \\ &= \Delta^\dagger; (f^\dagger \boxplus g^\dagger) \\ &= \Delta^\dagger; (f \boxplus g)^\dagger \\ &= ((f \boxplus g); \Delta)^\dagger \\ &= \langle f, g \rangle^\dagger. \end{aligned}$$

(iii) $\implies$ **(iv)** Under the assumption,

$$[{p_1}^\dagger, {p_2}^\dagger] = \langle p_1, p_2 \rangle^\dagger = id^\dagger = id.$$

(iv) $\implies$ **(i)** As $[in_1, in_2] : A^\dagger \boxplus B^\dagger \to A^\dagger \boxplus B^\dagger = id = [{p_1}^\dagger, {p_2}^\dagger]$, we immediately have ${p_1}^\dagger = in_1$ and ${p_2}^\dagger = in_2$.

$\square$

Definition 10.1.6. A *biproduct dagger compact closed category* is a dagger compact closed category with biproducts where the conditions of Lemma 10.1.5 hold.

10.1.2 Examples of dagger categories

Example 10.1.7 (FdHilb)**.** The category of finite dimensional Hilbert spaces is the motivating example for the creation of the dagger and is, in fact, a biproduct dagger compact closed category. The biproduct is the direct sum of Hilbert spaces and the tensor for compact closure is the standard tensor of Hilbert spaces. The dual H^* of a space H is the space of all continuous linear functions from H to the base field. The dagger is defined via the adjoint as being the unique map $f^\dagger : B \to A$ such that $\langle fa|b\rangle = \langle a|f^\dagger b\rangle$ for all $a \in A, b \in B$.

Example 10.1.8 (Rel)**.** The category Rel of sets and relations has the tensor $S \otimes T := S \times T$ and the biproduct $S \boxplus T := S \uplus T$. This is compact closed under $A^* := A$ and the dagger is the relational converse. That is, if the relation $R = \{(s,t)|s \in S, t \in T\} : S \to T$, then $R^\dagger = R^* = \{(t,s)|(s,t) \in R\}$.

Example 10.1.9 (Inverse categories)**.** An inverse category $\mathbb{X}$ is also a dagger category when the dagger is defined as the partial inverse. The unitary maps are the total maps. When the inverse category $\mathbb{X}$ is also a symmetric monoidal category where the monoid $\otimes$ is actually a restriction bi-functor, then $\mathbb{X}$ is a dagger symmetric monoidal category.

Requirement (i) of Definition 10.1.3 is fulfilled, as

$$(f \otimes g)(f \otimes g)^{(-1)} = \overline{f \otimes g} = \overline{f} \otimes \overline{g} = ff^{(-1)} \otimes gg^{(-1)} = (f \otimes g)(f^{(-1)} \otimes g^{(-1)})$$

and since the partial inverse of $f \otimes g$ is unique, $(f \otimes g)^{(-1)} = f^{(-1)} \otimes g^{(-1)}$. Requirement (ii) is that the structure isomorphisms are unitary. This is, of course, true as each of them are isomorphisms, hence total and therefore unitary.

10.1.3 Frobenius algebras

Frobenius algebras were originally defined as a finite dimensional algebra over a field together with a non-degenerate pairing operation. Here we present the general categorical definition:

Definition 10.1.10. Given a symmetric monoidal category $\mathbb{D}$, a *Frobenius algebra* is an object X of $\mathbb{D}$ and four maps, $\nabla : X \otimes X \to X$, $\eta : I \to X$, $\Delta : X \to X \otimes X$ and $\epsilon : X \to I$, with the conditions that (X, ∇, η) forms a commutative monoid, (X, Δ, ϵ) forms a commutative comonoid and the diagrams

$$
\begin{tikzcd}
X\otimes X \arrow[rr, "1\otimes\Delta"] \arrow[dd, "\Delta\otimes 1"'] \arrow[dr, "\nabla"] & & X\otimes X\otimes X \arrow[dd, "\nabla\otimes 1"] \\
& X \arrow[dr, "\Delta"] & \\
X\otimes X\otimes X \arrow[rr, "1\otimes\nabla"'] & & X\otimes X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[ddr, equal] & X\otimes X \arrow[d, "\epsilon\otimes 1"] \\
& I\otimes X \arrow[d, "u^l_\otimes"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "u^{r\,(-1)}_\otimes"] \arrow[ddr, equal] & X\otimes I \arrow[d, "1\otimes\eta"] \\
& X\otimes X \arrow[d, "\nabla"] \\
& X
\end{tikzcd}
$$

all commute. The Frobenius algebra is *special* when $\Delta\nabla = 1_X$ and *commutative* when $\Delta c_{X,X} = \Delta$. Note that special is sometimes referred to as *separable*.

Definition 10.1.11. A Frobenius algebra in a dagger symmetric monoidal category where $\Delta = \nabla^\dagger$ and $\epsilon = \eta^\dagger$ is a $\dagger$-*Frobenius algebra*.

For an example of a †-Frobenius algebra, consider a finite dimensional Hilbert space H with an orthonormal basis $\{|\phi_i\rangle\}$ and define $\Delta : H \to H \otimes H : |\phi_i\rangle \mapsto |\phi_i\rangle \otimes |\phi_i\rangle$ and $\epsilon : H \to \mathbb{C} : |\phi_i\rangle \mapsto 1$. Then $(H, \nabla = \Delta^\dagger, \eta = \epsilon^\dagger, \Delta, \epsilon)$ forms a commutative special †-Frobenius algebra.

In [27], Coecke, Pavlović and Vicary give a correspondence between Frobenius algebras and orthogonal bases in finite dimensional Hilbert spaces. An orthonormal basis for such a space determines, as above, a special commutative †-Frobenius algebra. To show the other direction, given a commutative †-Frobenius algebra, (H, ∇, η), for each element $\alpha \in H$ define the right action of α as $R_\alpha := (id \otimes \alpha)\,\nabla : H \to H$. Note the use of the fact that elements $\alpha \in H$ can be considered as linear maps $\alpha : \mathbb{C} \to H : 1 \mapsto |\alpha\rangle$. The dagger of a right action is also a right action, ${R_\alpha}^\dagger = R_{\alpha'}$ where $\alpha' = \eta\,\nabla\,(id \otimes \alpha^\dagger)$, which is a consequence of the Frobenius identities.

The $(\_)'$ construction is actually an involution:

$$
\begin{aligned}
(\alpha')' &= \eta\nabla(id \otimes \alpha'^{\dagger}) \\
&= u\nabla(id \otimes (\eta\nabla(id \otimes \alpha^{\dagger}))^{\dagger} \\
&= u\nabla(id \otimes ((id \otimes \alpha)\Delta\epsilon)) \\
&= (u \otimes \alpha)(\nabla \otimes id)(id \otimes \Delta)(id \otimes \epsilon) \\
&= (u \otimes \alpha)(id \otimes \Delta)(\nabla \otimes id)(id \otimes \epsilon) \\
&= (u \otimes \alpha)(id \otimes \epsilon) \\
&= \alpha.
\end{aligned}
$$

Lemma 10.1.12 (Coecke, Pavolvić, Vicary [27]). *Any †-Frobenius algebra in* FdHilb *is a C^*-algebra.*

Proof. The endomorphism monoid of FdHilb (H,H) is a C^*-algebra. From the proceeding, we have

$$H \cong \text{FdHilb}(\mathbb{C}, H) \cong R_{[\text{FdHilb}(\mathbb{C},H)]} \subseteq \text{FdHilb}(H, H).$$

This inherits the algebra structure from FdHilb (H,H). Furthermore, since any finite dimensional involution-closed sub-algebra of a C^*-algebra is also a C^*-algebra, this shows the †-Frobenius algebra is a C^*-algebra. □

Using the fact that the involution preserving homomorphisms from a finite dimensional commutative C^*-algebra to $\mathbb{C}$ form a basis for the dual of the underlying vector space, write these homomorphisms as ${\phi_i}^{\dagger} : H \to \mathbb{C}$. Then their adjoints, $\phi_i : \mathbb{C} \to H$ will form a basis for the space H. These are the copyable elements in H.

This, together with continued applications of the Frobenius rules and linear algebra allow Coecke, Pavlović and Vicary to prove the following Theorem.

Theorem 10.1.13 (Coecke, Pavolvić, Vicary [27]). *Every commutative †-Frobenius algebra in* FdHilb *determines an orthogonal basis consisting of its copyable elements. Conversely,* *every orthogonal basis* $\{|\phi_i\rangle\}_i$ *determines a commutative* †*-Frobenius algebra via*

$$\Delta : H \to H \otimes H : |\phi_i\rangle \mapsto |\phi_i\rangle \otimes |\phi_i\rangle \qquad \epsilon : H \to \mathbb{C} : |\phi_i\rangle \mapsto 1$$

and these constructions are inverse to each other.

An interesting aspect of Theorem 10.1.13 is that it uses only algebraic structures to determine a basis as the "copyable elements". In a quantum computation, the choice of basis determines the copyable elements.

Remark 10.1.14. Vicary, in [67], further explores †-Frobenius algebras to define *involution monoids* on †-monoidal categories with duals. This led to an alternate proof of Lemma 10.1.12 based on monoids. From this, Vicary shows that the category of commutative †-monoids in FdHilb, with monoid morphisms as maps, is equivalent to FinSets^{op} – the dual of the category of finite sets. Of course, this is another way to move to the classical world from the quantum world. Vicary goes on in [50] to define a finite quantum Boolean topos and finite Boolean topos and show that the category of classical structures in a finite quantum Boolean topos is equivalent to a finite Boolean topos and in fact, every finite Boolean topos arises in this way.

10.1.4 CFrob($\mathbb{X}$) is an inverse category

Example 10.1.15 (Commutative separable Frobenius algebras [43])**.** Let $\mathbb{X}$ be a symmetric monoidal category and form CFrob($\mathbb{X}$) as follows:

Objects: Commutative separable Frobenius algebras: These are quintuples $(A, \nabla, \eta, \Delta, \epsilon)$ where A is an object of $\mathbb{X}$ with the following maps: $\nabla : A \otimes A \to A$, $\eta : I \to A$, $\Delta : A \to A \otimes A$, $\epsilon : A \to I$ which are natural maps in $\mathbb{X}$, with (A, ∇, η) a monoid and (A, Δ, ϵ) a comonoid.

Additionally, these satisfy

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\Delta\otimes 1"] \arrow[dd, "1\otimes\Delta"'] \arrow[dr, "\nabla"] & & A \otimes (A \otimes A) \arrow[dd, "1\otimes\nabla"] \\
& A \arrow[dr, "\Delta"] & \\
(A \otimes A) \otimes A \arrow[rr, "\nabla\otimes 1"'] & & A \otimes A
\end{tikzcd}$$

Frobenius

together with the additional property that $\Delta\nabla = 1$ (separable).

Maps: The maps of $\mathbb{X}$ between the objects of $\mathbb{X}$ which preserve multiplication (∇) and comultiplication (Δ) but do not necessarily preserve the units. This means a map f must satisfy the following commuting diagrams:

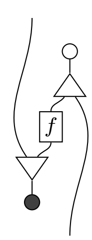
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\Delta"'] & B \arrow[d, "\Delta"] \\
A \otimes A \arrow[r, "f\otimes f"'] & B \otimes B
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \otimes A \arrow[r, "f\otimes f"] \arrow[d, "\nabla"'] & B \otimes B \arrow[d, "\nabla"] \\
A \arrow[r, "f"'] & B.
\end{tikzcd}$$

Lemma 10.1.16. *When $\mathbb{X}$ is a symmetric monoidal category,* $\mathrm{CF\textsc{rob}}(\mathbb{X})$ *is an inverse category.*

Proof. We need to show that $\mathrm{CF\textsc{rob}}(\mathbb{X})$ has restrictions and that each map has a partial inverse. We do this by exhibiting the partial inverse of a map. For $f : X \to Y$, define $f^{(-1)}$ as

$$Y \xrightarrow{1\otimes\eta} Y \otimes X \xrightarrow{1\otimes\Delta} Y \otimes X \otimes X \xrightarrow{1\otimes f\otimes 1} Y \otimes Y \otimes X \xrightarrow{\nabla\otimes 1} Y \otimes X \xrightarrow{\epsilon\otimes 1} X.$$

As a string diagram, this looks like:



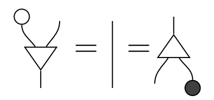
.

In the following proofs, we also use the following two identities from [43]:

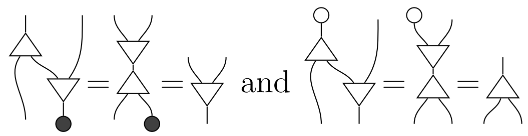
$$(1 \otimes \eta)\nabla = 1, \tag{10.1}$$

$$\Delta(1 \otimes \epsilon) = 1. \tag{10.2}$$

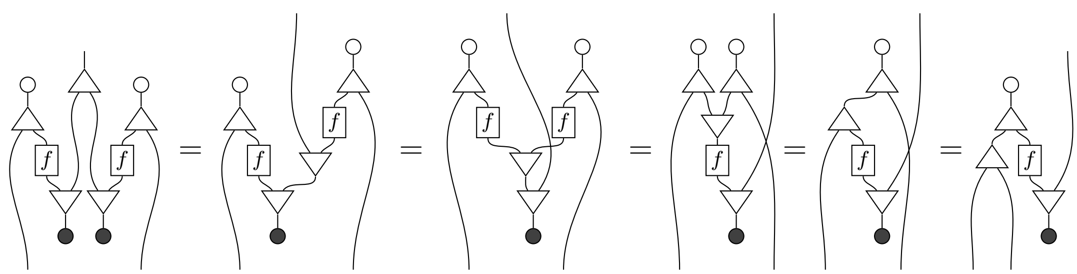
Diagrammatically, this is:



Note that when combined with the Frobenius identities, this allows transforms of the following types:



First, we must show that $f^{(-1)}$ is a map in the category, i.e., that $\Delta(f^{(-1)} \otimes f^{(-1)}) = f^{(-1)}\Delta$ and $(f^{(-1)} \otimes f^{(-1)})\nabla = \nabla f^{(-1)}$. We show this for Δ using string diagrams, starting from $\Delta(f^{(-1)} \otimes f^{(-1)})$. The proof for the preservation of ∇ proceeds in a similar manner.



$= f^{(-1)}\Delta.$

Thus, $f^{(-1)}$ is a map in the category whenever f is.

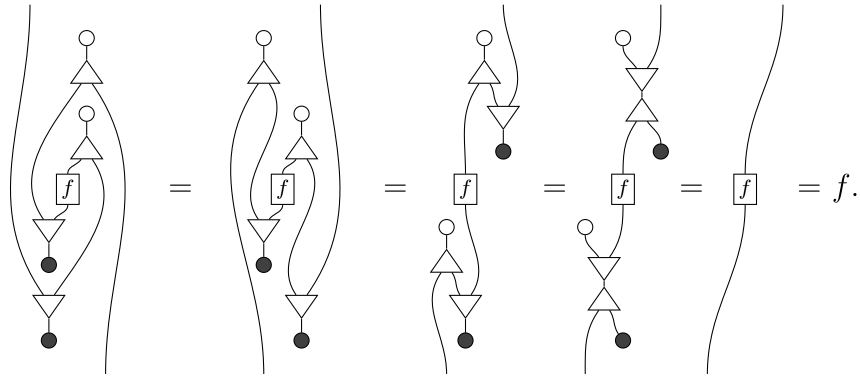
If $f^{(-1)}$ is truly a partial inverse, we may then define $\overline{f} = ff^{(-1)}$. Using Theorem 2.20 from [21], we need only show:

$$\left(f^{(-1)}\right)^{(-1)} = f \tag{10.3}$$

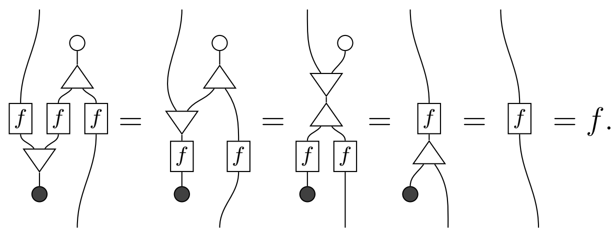
$$ff^{(-1)}f = f \tag{10.4}$$

$$ff^{(-1)}gg^{(-1)} = gg^{(-1)}ff^{(-1)}. \tag{10.5}$$

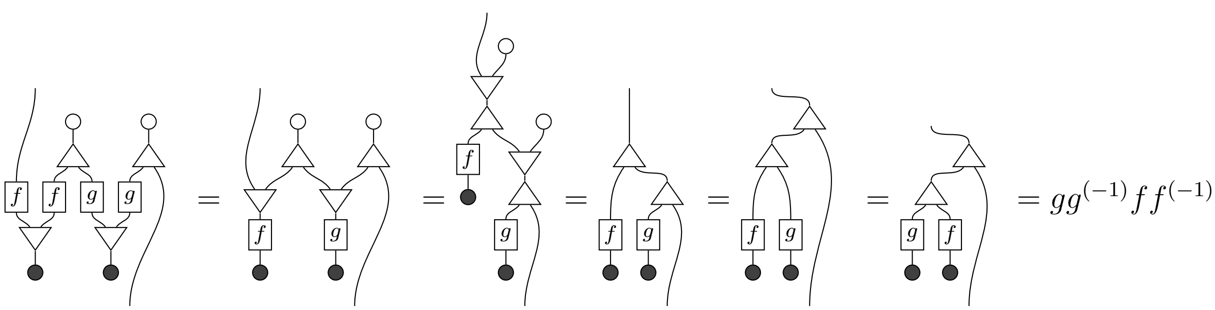
Proof of Equation (10.3): $(f^{(-1)})^{(-1)} =$



Proof of Equation (10.4): $ff^{(-1)}f =$



Proof of Equation (10.5): $ff^{(-1)}gg^{(-1)} =$



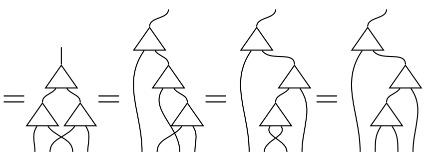
where the last step is accomplished by reversing all the previous diagrammatic steps. Hence, CFROB($\mathbb{X}$) is an inverse category. $\square$

Theorem 10.1.17. *When $\mathbb{X}$ is a symmetric monoidal category,* CFROB($\mathbb{X}$) *is a discrete inverse category.*

Proof. Lemma 10.1.16 shows CFrob($\mathbb{X}$) is an inverse category. We need to show the conditions of Definition 4.3.1 are met.

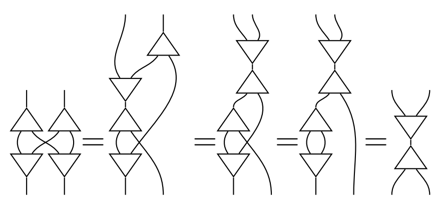
First, we see that the tensor of $\mathbb{X}$ is a tensor in CFrob($\mathbb{X}$). $A \otimes B$ is an object in CFrob($\mathbb{X}$) with $\Delta_{A\otimes B} = (\Delta_A \otimes \Delta_B)(1 \otimes c_\otimes \otimes 1)$, $\nabla_{A\otimes B} = (1 \otimes c_\otimes \otimes 1)(\nabla_A \otimes \nabla_B)$, $\eta_{A\otimes B} = \Delta_I(\eta_A \otimes \eta_B)$, and $\epsilon_{A\otimes B} = (\epsilon_A \otimes \epsilon_B)\nabla_I$.

The map $\Delta : A \to A \otimes A$ is a map in CFrob($\mathbb{X}$). To show it preserves Δ, we need to show $\Delta_A \Delta_{A\otimes A} = \Delta_A(\Delta_A \otimes \Delta_A)$:

$\Delta_A \Delta_{A\otimes A} =$  $= \Delta_A(\Delta_A \otimes \Delta_A).$

Note that in the last step, we simply reverse the various associativity steps used previously.

To show that Δ preserves the ∇, we must show that $(\Delta_A \otimes \Delta_A)\nabla_{A\otimes A} = \nabla_A \Delta_A$. Starting with $(\Delta_A \otimes \Delta_A)\nabla_{A\otimes A} =$

 $= \nabla_A \Delta_A.$

Note that the proof uses the "special" property in a non-trivial way.

Thus, we have a Δ in CFrob($\mathbb{X}$). As $\nabla = \Delta^{(-1)}$, the Frobenius requirement for the inverse product is immediately fulfilled. Commutativity, cocommutativity, associativity, coassociativity and the exchange rule all follow from the properties of the commutative Frobenius algebras and therefore CFrob($\mathbb{X}$) is a discrete inverse category. $\square$

Note that the category CFrob($\mathbb{X}$) possesses additional structure over that of a general discrete inverse category, specifically, the existence of unit maps $\eta : I \to A$ and $\epsilon : A \to I$ for each object A.

We may also consider CFrob($\mathbb{X}$) as a bicategory [47], with the following data:

(i) Objects of the bicategory are the objects of $\mathrm{CF}\textsc{rob}(\mathbb{X})$, that is, the objects of the underlying symmetric monoidal category $\mathbb{X}$.

(ii) The hom-sets $\mathrm{CF}\textsc{rob}(\mathbb{X})(A, B)$ are each categories, with the objects being the maps between A and B (elements of the hom-set) and the maps being the partial ordering given by the restriction as shown in Lemma 3.2.2.

(iii) The composition functor is based upon the composition in $\mathbb{X}$. By Lemma 3.2.2 we have that $f \le g$, $h \le k$ gives $fh \le gk$. The identity functor $I_A : 1 \to \mathrm{CF}\textsc{rob}(\mathbb{X})(A, A)$ maps to the identity map in each hom-set.

(iv) The associativity and identity transforms are identities, i.e. $f(gh) = (fg)h$ and $fI_A = f = I_A f$, hence this is a 2-category.

With this data, we see that the Frobenius structure of $(A, \nabla, \eta, \Delta, \epsilon)$ ensure that $\mathrm{CF}\textsc{rob}(\mathbb{X})$ is actually a Cartesian bicategory as defined in Carboni and Walters [12]. Moreover, as we have $\nabla\Delta = (\Delta\otimes 1)(1\otimes\nabla)$, this satisfies the further condition that each object is *discrete* and therefore is considered a "bicategory of relations" as defined in [12], Definition 2.1. Carboni and Walters describe a number of consequences resulting when the base bicategory is locally posetal.

Frobenius algebras are not the only structure of interest when considering symmetric monoidal categories. Coecke, Paquette and Pavlović [26] model quantum and classical computations in a †-symmetric monoidal category $\mathbb{D}$. Their basic definitions include a compact structure, a quantum structure and a classical structure, the latter of which is a special Frobenius algebra:

Definition 10.1.18. A *compact structure* on an object A in the category $\mathbb{D}$ is given by the object A, an object A^* called its *dual* and the maps $\eta : I \to A^* \otimes A$, $\epsilon : A \otimes A^* \to I$ such that the diagrams

$$\begin{tikzcd} & A^* \arrow[d, "\eta\otimes A^*"'] \arrow[dr, "id"] & \\ A^* \otimes A \otimes A^* \arrow[rr, "A^*\otimes\epsilon"'] & & A^* \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "A\otimes\eta"] \arrow[dr, "id"'] & A \otimes A^* \otimes A \arrow[d, "\epsilon\otimes A"] \\ & A \end{tikzcd}$$

commute.

Definition 10.1.19. A *quantum structure* is an object A and map $\eta : I \to A \otimes A$ such that $(A, A, \eta, \eta^\dagger)$ form a compact structure.

Note that A is self-dual in definition 10.1.19.

Definition 10.1.20. A *classical structure* in $\mathbb{D}$ is an object X together with two maps, $\Delta : X \to X \otimes X$, $\epsilon : X \to I$ such that $(X, \Delta^\dagger, \epsilon^\dagger, \Delta, \epsilon)$ forms a special Frobenius algebra.

Coecke, Paquette and Pavlović then examine two categories based on these structures: The category $\mathbb{D}_q$, the category whose objects are quantum structures in $\mathbb{D}$, with

$$\mathbb{D}_q((A, \eta_A), (B, \eta_B)) := \mathbb{D}(A, B)$$

and $\mathbb{D}_c$, the category whose maps are classical structures in $\mathbb{D}$, with

$$\mathbb{D}_q((X, \Delta_X, \epsilon_X), (Y, \Delta_Y, \epsilon_Y)) := \mathbb{D}(A, B).$$

The creation of these categories and the use of the classical structure motivated the examination of $\mathrm{CF{\scriptstyle ROB}}(\mathbb{X})$ in this section.

Remark 10.1.21. For an alternate way of using inverse categories to describe quantum computation, refer to Example 4.1.12 and Hines and Braunstein's paper "The structure of partial isometries" [36]. This paper discusses how the category of partial isometries (an inverse category) may be considered a reasonable interpretation of von Neumann-Birkhoff quantum logic [8]. At the same time, they show this model is inconsistent with that of [3] in that one can not model teleportation and in fact, is not even compact closed.

10.2 Disjointness in CFROB($\mathbb{X}$)

An important example of a disjointness system arises naturally from Frobenius Algebras. Recall that we defined a zero map in Section 6.1 whenever there was a zero object. This may also be defined directly:

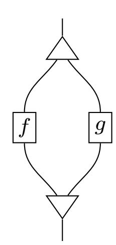
Definition 10.2.1. A family of maps in a symmetric monoidal category with monoid $\otimes$ is called a *zero map*, written as 0 whenever:

- For all f, g, $f0g = 0$;
- For all f, $f \otimes 0 = 0$ and $0 \otimes f = 0$.

Lemma 10.2.2. *If $\mathbb{X}$ is a symmetric monoidal category with a zero map, then* CFROB($\mathbb{X}$) *has a zero map.*

Proof. As $\Delta 0 = 0 = 0\Delta$ and $0\nabla = 0 = \nabla 0$, the zero maps are in CFROB($\mathbb{X}$). Since composition is inherited from $\mathbb{X}$, the first item of Definition 10.2.1 is satisfied. Finally, since the tensor in $\mathbb{X}$ is the tensor in CFROB($\mathbb{X}$), the second item is also satisfied. Hence, CFROB($\mathbb{X}$) has zero maps. $\square$

Lemma 10.2.3. *As shown in Lemma 10.1.16,* CFROB($\mathbb{X}$) *is a discrete inverse category. Suppose $\mathbb{X}$ has zero maps. Then for $f, g : A \to B$, define $f \perp g$ when*

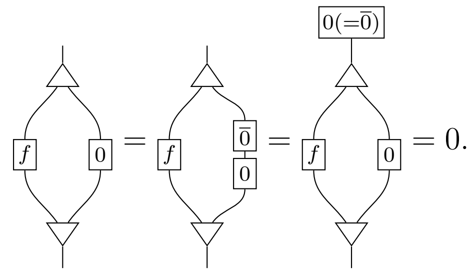


$= 0.$

Then, the relation $\perp$ is a disjointness relation.

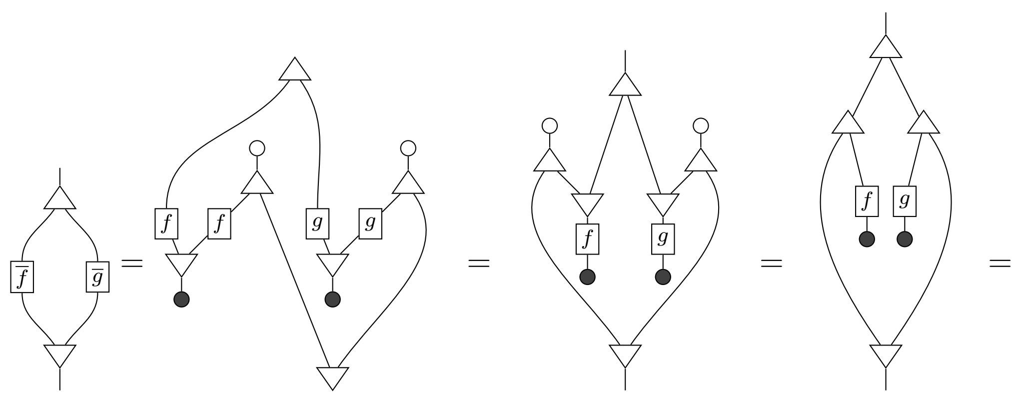
Proof. We must show the seven axioms of the disjointness relation hold. We will show [**Dis.6**] early on as its result will be used to establish some of the other axioms.

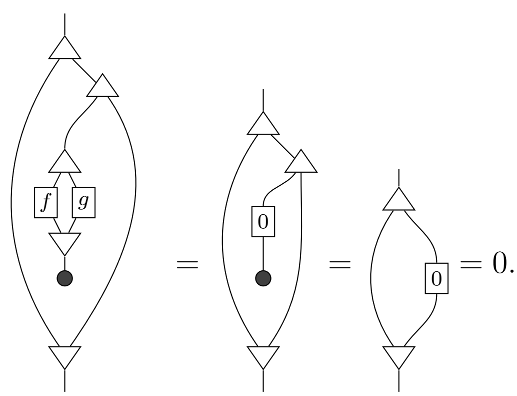
[**Dis.1**]: For all $f : A \to B,\ f \perp 0$.



[**Dis.6**]: $f \perp g$ implies $\overline{f} \perp \overline{g}$ and $\hat{f} \perp \hat{g}$.

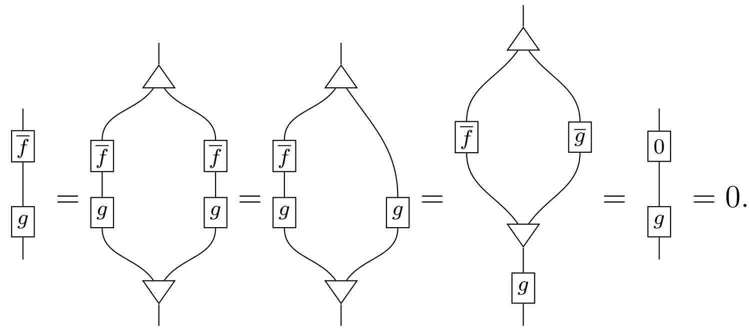
We will show the details of $\overline{f} \perp \overline{g}$, using $\overline{f} = ff^{(-1)}$ and the definition of $f^{(-1)}$ as given in Theorem 10.1.17. The proof of $(f^{(-1)}f =)\hat{f} \perp \hat{g}(= g^{(-1)}g)$ is similar.



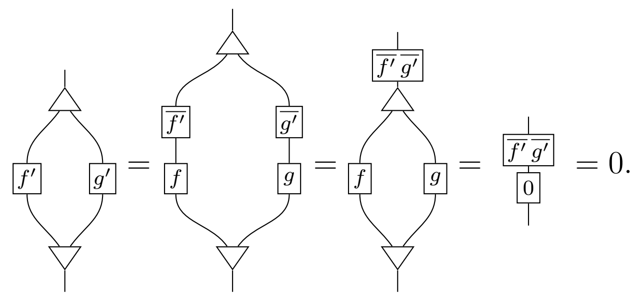


[**Dis.2**]: $f \perp g$ implies $\overline{f}g = 0$.

In this proof, we use the result of [**Dis.6**], i.e., that $\overline{f} \perp \overline{g}$.



[**Dis.3**]: $f \perp g,\ f' \leq f,\ g' \leq g$ implies $f' \perp g'$.



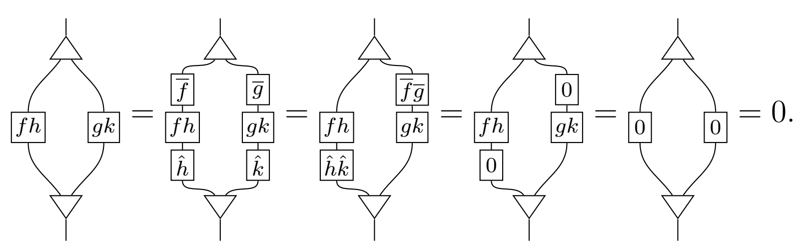
[**Dis.4**]: $f \perp g$ implies $g \perp f$.

This follows directly from the cocommutativity of Δ.

[**Dis.5**]: $f \perp g$ implies $hf \perp hg$.

This follows directly from the naturality of Δ.

[**Dis.7**]: $\overline{f} \perp \overline{g},\ \hat{h} \perp \hat{k}$ implies $fh \perp gk$.



$\square$

Note that Lemma 10.2.3 uses the meet of $\mathrm{CF\scriptsize{ROB}}(\mathbb{X})$ as defined in Proposition 4.3.6.

10.3 Disjoint joins in CFrob($\mathbb{X}$)

In the previous sections, we have shown that CFrob($\mathbb{X}$) is a discrete inverse category, with a disjointness relation whenever $\mathbb{X}$ is a symmetric monoidal category with zero maps.

We now show that if $\mathbb{X}$ has biproducts and is an additive tensor category, i.e., a symmetric monoidal category where the hom-sets are enriched in additive monoids, then CFrob($\mathbb{X}$) will have a disjoint join. Moreover, in the following section, we shall show it possesses a disjoint sum. First, we explicitly define additive tensor category:

Definition 10.3.1. Suppose $\mathbb{X}$ is a symmetric monoidal category with zero maps and the hom-sets are enriched in additive monoids. It is an *additive tensor category* when:

- $h(f+g)k = hfk + hgk$ for all $h : A \to B$, $f, g : B \to C$ and $k : C \to D$;
- $(f+g) \otimes k = f \otimes k + g \otimes k$ for all $f, g : A \to B$ and $k : C \to D$.

We begin by showing that given an additive tensor category, we may form an equivalent category which has biproducts.

Lemma 10.3.2. *Suppose $\mathbb{X}$ is an additive symmetric monoidal category. If we have the diagram*

$$\begin{tikzcd} A \arrow[r, "\sigma_1"] & X \arrow[l, bend left, "\pi_1"] \arrow[r, bend right, "\pi_2"'] & B \arrow[l, "\sigma_2"'] \end{tikzcd}$$

with

$$\sigma_1\pi_1 = 1_A, \quad \sigma_2\pi_2 = 1_B, \qquad \sigma_1\pi_2 = 0 = \sigma_2\pi_2$$

and $\pi_1\sigma_1 + \pi_2\sigma_2 = 1$, then X is a biproduct of A and B.

Corollary 10.3.3. *If $F : \mathbb{X} \to \mathbb{Y}$ is an additively enriched functor, then F preserves all biproducts.*

Corollary 10.3.4. *The functor $A \otimes \_ : \mathbb{X} \to \mathbb{X}$ preserves biproducts.*

Thus, if $\mathbb{X}$ is additively enriched, we may add biproducts by moving to the matrix category of $\mathbb{X}$, defined as:

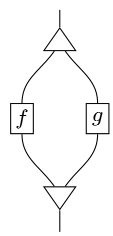
Objects: Lists of the objects $[A_i]$ of $\mathbb{X}$;

Maps: Matrices of maps in $\mathbb{X}$, $[f_{i,j}] : [A_i] \to [B_j]$;

Identity: The diagonal matrix I ($f_{i,i} = 1_{A_i}$ and $f_{i,j} = 0, i \neq j$);

Composition: Matrix multiplication.

Now, let us consider $\mathrm{CF\scriptsize{ROB}}(\mathbb{X})$ where $\mathbb{X}$ is an additive tensor category with biproducts. We know from Lemma 10.2.3 that

$f \perp g \iff$  $= 0.$

We will now show that the biproduct is the disjoint join of any two disjoint maps. To do so, we must show the biproduct of two disjoint maps is in the category $\mathrm{CF\scriptsize{ROB}}(\mathbb{X})$. We first give a lemma about the biproduct of disjoint maps.

Lemma 10.3.5. *Given $\mathbb{X}$ is an additive tensor category, when f, g are maps in $\mathrm{CF\scriptsize{ROB}}(\mathbb{X})$ with $f \perp g$, then $\Delta(f \otimes g) = 0$ and $(f \otimes g)\nabla = 0$.*

Proof. From Lemma 4.3.5 (ii) we have that $e\Delta(f \otimes g) = \Delta(ef \otimes g)$ and $(f \otimes g)\Delta^{(-1)}e = (f \otimes ge)\Delta^{(-1)}$ for e a restriction idempotent. Thus, we have

$$\Delta(f \otimes g) = \overline{f}\Delta(f \otimes g) = \Delta(f \otimes \overline{f}g) = \Delta(f \otimes 0) = \Delta 0 = 0,$$

where $\overline{f}g = 0$ follows from the disjointness of f, g by [**Dis.2**] and the remaining equalities are due to $\mathbb{X}$ being an additive tensor category. Dually, as $\nabla = \Delta^{(-1)}$, we have

$$(f \otimes g)\nabla = (f \otimes g)\nabla\hat{f} = (f \otimes g\hat{f})\nabla = (f \otimes 0)\nabla = 0\nabla = 0.$$

$\square$

Lemma 10.3.6. *Given $\mathbb{X}$ is an additive tensor category with biproducts, when f, g are maps in* $\text{CF}\textsc{rob}(\mathbb{X})$ *with $f \perp g$, then $f+g$ is a map in* $\text{CF}\textsc{rob}(\mathbb{X})$.

Proof. We must show $(f+g)\Delta = \Delta((f+g)\otimes(f+g))$ and $\nabla(f+g) = ((f+g)\otimes(f+g))\nabla$. As this is an additive tensor category and both f, g are in $\text{CF}\textsc{rob}(\mathbb{X})$ and using Lemma 10.3.5, we have

$$\Delta((f+g)\otimes(f+g)) = \Delta(f\otimes f + f\otimes g + g\otimes f + g\otimes g) =$$
$$\Delta(f\otimes f) + \Delta(f\otimes g) + \Delta(g\otimes f) + \Delta(g\otimes g) =$$
$$\Delta(f\otimes f) + \Delta(g\otimes g) = f\Delta + g\Delta = (f+g)\Delta.$$

Hence, the biproduct of f, g preserves Δ. Similarly, $f+g$ preserves ∇:

$$((f+g)\otimes(f+g))\nabla = (f\otimes f + f\otimes g + g\otimes f + g\otimes g)\nabla =$$
$$(f\otimes f)\nabla + (f\otimes g)\nabla + (g\otimes f)\nabla + (g\otimes g)\nabla =$$
$$(f\otimes f)\nabla + (g\otimes g)\nabla = \nabla f + \nabla g = \nabla(f+g).$$

□

Proposition 10.3.7. *Given $\mathbb{X}$ is an additive tensor category with biproduct $\oplus$, and $f \perp g$, then $f \sqcup g := f+g$ is a disjoint join.*

Proof. We need to show the four axioms of disjoint join from Definition 6.3.1.

[**DJ.1**] : $f \leq f \sqcup g$ and $g \leq f \sqcup g$. As $f+g = g+f$, we need only show the first part of the axiom. As $f \perp g$, we know $\overline{f}g = 0$ by the definition of disjointness and therefore we have:

$$\overline{f}(f+g) = \overline{f}f + \overline{f}g = f + 0 = f$$

and thus $f \leq f \sqcup g$ and [**DJ.1**] is true.

[**DJ.2**]: $f \leq h,\ g \leq h$ and $f \perp g$ implies $f \sqcup g \leq h$. We calculate

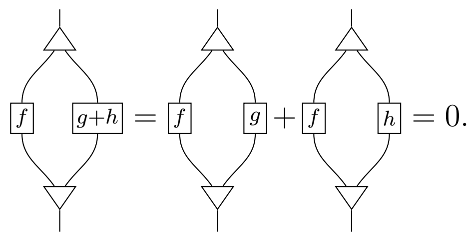
$$\overline{f+g}h = (\overline{f}+\overline{g})h = \overline{f}h + \overline{g}h = f+g$$

giving us the required inequality and [**DJ.1**] is true.

[**DJ.3**]: Disjoint join is stable, i.e., $h(f \sqcup g) = hf \sqcup hg$. This is immediate as $h(f+g) = hf + hg$.

[**DJ.4**]: $\perp \{f, g, h\} \iff f \perp (g \sqcup h)$. Consider the $\Longleftarrow$ direction first. We immediately have $g \perp h$ as we are able to form the disjoint join. By [**DJ.1**], we have both $g \leq g \sqcup h$ and $h \leq g \sqcup h$ and therefore by [**Dis.3**], $f \perp g$ and $f \perp h$. Thus the $\Longleftarrow$ direction is true.

For the $\implies$ direction, we compute



This gives us both directions and all of the axioms have been shown to be true, hence, the addition of maps is a disjoint join. $\square$

10.4 Disjoint sums in CFrob($\mathbb{X}$)

Now that we have shown we have a disjoint join, our last remaining task is to show that the biproduct in $\mathbb{X}$ provides a disjoint sum. For the remainder of this section, we define the following objects and maps:

$$
\begin{aligned}
A \oplus B &:= A + B \text{ (the biproduct of } A \text{ and } B\text{)},\\
\epsilon_{A+B} &:= [\epsilon_A, \epsilon_B] : A + B \to I,\\
\eta_{A+B} &:= \langle \eta_A, \eta_B \rangle : I \to A + B,\\
\Delta_{A+B} &:= (\Delta_A i_1 + \Delta_B i_2) : A + B \to A \otimes A + A \otimes B + B \otimes A + B \otimes B,\\
\nabla_{A+B} &:= (\pi_1 \nabla_A + \pi_2 \nabla_B) : (A \otimes A + A \otimes B) + (B \otimes A + B \otimes B) \to A + B.
\end{aligned}
$$

Note that we have $A \otimes A + A \otimes B + B \otimes A + B \otimes B \cong (A+B) \otimes (A+B)$ via the isomorphism $b = \langle i_1 \otimes i_1 | i_1 \otimes i_2 | i_2 \otimes i_1 | i_2 \otimes i_2 \rangle$.

Lemma 10.4.1. *Given $\mathbb{X}$ is an additive tensor category with biproduct $+$, then the injection maps i_1, i_2 and projection maps π_1, π_2 of the biproduct are maps in* $\text{CFrob}(\mathbb{X})$.

Proof. We must show each of the injections and projections preserve Δ and ∇.

Consider

$$\begin{tikzcd}
A \arrow[rr, "i_1"] \arrow[d, "\Delta"'] & & A+B \arrow[d, "\Delta_{A+B}(=\Delta_A i_1 + \Delta_B i_2)"] \\
A \otimes A \arrow[rr, "i_1 \otimes i_1"'] \arrow[dr, "i_1"'] & & (A+B)\otimes(A+B) \\
& A\otimes A + A \otimes B \arrow[r, "i_1"'] & A\otimes A + A\otimes B + B\otimes A + B\otimes B. \arrow[u, "b"']
\end{tikzcd}$$

The outer arrows commute as $i_1(f+g) = fi_1$. The bottom half commutes due to the isomorphism hence the top rectangle commutes and i_1 preserves Δ. Similarly, i_2 preserves Δ. By dualizing the diagram, we also have π_1, π_2 preserve ∇.

Next, we show the projections preserve ∇,

$$\begin{tikzcd}
A+B \arrow[rr, "\pi_1"'] \arrow[d, "\Delta"'] & & A \arrow[d, "\Delta_A"] \\
(A+B)\otimes(A+B) \arrow[rr, "\pi_1 \otimes \pi_1"] \arrow[d, "b^{-1}"'] & & A \otimes A \\
A\otimes A + A\otimes B + B\otimes A + B\otimes B \arrow[r, "\pi_1"] & A\otimes A + A\otimes B \arrow[ur, "\pi_1"] &
\end{tikzcd}$$

which, by the same argument as above, shows that Δ is preserved by π_1 and similarly by π_2. Once again, dualizing this diagram gives us that ∇ is preserved by i_1 and i_2. $\square$

Lemma 10.4.2. *Suppose A and B are Frobenius algebras in* $\text{CFrob}(\mathbb{X})$ *where $\mathbb{X}$, an additive tensor category, has the biproduct $+$. Then $A \oplus B$ is a Frobenius algebra and is therefore in* $\text{CFrob}(\mathbb{X})$.

Proof. To show $A \oplus B$ is a Frobenius algebra and therefore in $\text{CFrob}(\mathbb{X})$, we must show it is separable, the unit laws hold and the Frobenius condition hold for $A \oplus B$.

For the requirement that it is separable:

$$\begin{aligned}
\Delta_{A+B}\nabla_{A+B} &= (\Delta_A i_1 + \Delta_B i_2)(\pi_1\nabla_A + \pi_2\nabla_B)\\
&= (\Delta_A i_1\pi_1\nabla_A + \Delta_B i_2\pi_2\nabla_B)\\
&= (\Delta_A\nabla_A + \Delta_B\nabla_B)\\
&= (1_A + 1_B) = 1_{A+B}.
\end{aligned}$$

To show the comultiplication unit law,

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "i_1"'] & A\otimes A \arrow[d, "i_1\otimes i_1"] \arrow[dr, equal] & \\
A+B \arrow[r, "\Delta"] \arrow[ddr, equal] & (A+B)\otimes(A+B) \arrow[d, "1\otimes\epsilon"] & A\otimes A \arrow[l, "i_1\otimes i_1"] \arrow[d, "1\otimes\epsilon"] \\
& (A+B)\otimes I \arrow[d, "u^l_\otimes"] & A\otimes I \arrow[l, "i_1\otimes 1"] \arrow[d, "u^l_\otimes"] \\
& A+B & A. \arrow[l, "i_1"]
\end{tikzcd}$$

The outer path shows the unit law for the Frobenius algebra A and is what happens to the A component in the inner path. There is a similar diagram where the outer path A's are replaced with a B and the i_1 with i_2. As these outer paths commute, they show that the inner path commutes as each component of it commutes.

For the Frobenius law, as we are in a commutative world, we need only show $\nabla\Delta = (1\otimes\Delta)(\nabla\otimes 1)$:

$$\begin{tikzcd}
& A\otimes A \arrow[rr, "1\otimes\Delta"] \arrow[d, "i_1\otimes i_1"] \arrow[dl, equal] & & A\otimes A\otimes A \arrow[d, "i_1\otimes i_1\otimes i_1"] \arrow[dr, equal] & \\
A\otimes A \arrow[r, "i_1\otimes i_1"] \arrow[dr, "\nabla"'] & (A+B)\otimes(A+B) \arrow[rr, "1\otimes\Delta"] \arrow[dr, "\nabla"] & & (A+B)\otimes(A+B)\otimes(A+B) \arrow[dd, "\nabla\otimes 1"] & A\otimes A\otimes A \arrow[l, "i_1\otimes i_1\otimes i_1"] \arrow[dd, "\nabla\otimes 1"] \\
& A \arrow[r, "i_1"] \arrow[ddrr, "\Delta"'] & A+B \arrow[dr, "\Delta"] & & \\
& & & (A+B)\otimes(A+B) & A\otimes A \arrow[l, "i_1\otimes i_1"'] \arrow[dl, equal] \\
& & & A\otimes A. \arrow[u, "i_1\otimes i_1"'] &
\end{tikzcd}$$

By the same reasoning as the previous two arguments, the diagram commutes and $A+B$ is a Frobenius algebra. $\square$

Now that we have that $A \oplus B$ is in $\mathrm{CF{\scriptsize ROB}}(\mathbb{X})$ when, A, B are in $\mathrm{CF{\scriptsize ROB}}(\mathbb{X})$, we can show that the biproduct projections and injections are maps in $\mathrm{CF{\scriptsize ROB}}(\mathbb{X})$.

Proposition 10.4.3. *Suppose A and B are Frobenius algebras in* $\mathrm{CF{\scriptsize ROB}}(\mathbb{X})$ *where $\mathbb{X}$, an additive tensor category, has the biproduct $+$. Then $A \oplus B$ as defined in Lemma 10.3.6 is a disjoint sum in* $\mathrm{CF{\scriptsize ROB}}(\mathbb{X})$.

Proof. Next, we must give maps i_1, i_2, x_1, x_2 in $\mathrm{CF{\scriptsize ROB}}(\mathbb{X})$ that satisfy the disjoint sum diagram,

$$\begin{tikzcd} A \arrow[r, "i_1"] & A+B \arrow[l, bend left, "x_1"] \arrow[r, bend right, "x_2"'] & B \arrow[l, "i_2"'] \end{tikzcd} \tag{10.6}$$

where

(i) i_1 and i_2 are monic,

(ii) ${i_1}^{(-1)} = {i_1}^{(-1)}$ and ${i_2}^{(-1)} = {i_2}^{(-1)}$, and

(iii) ${i_1}^{(-1)} i_1 \perp {i_2}^{(-1)} i_2$ and ${i_1}^{(-1)} i_1 \sqcup {i_2}^{(-1)} i_2 = 1_X$.

By Lemma 10.4.1, setting i_1, i_2 to be the injections of the biproduct and ${i_1}^{(-1)}, {i_2}^{(-1)}$ to be the projections will immediately give us Diagram (10.6), as all those maps are in $\mathrm{CF{\scriptsize ROB}}(\mathbb{X})$.

For the three conditions, as i_1 and i_2 are total maps, they are monic in the inverse category. We know from above that $i_j \pi_j = 1$ and therefore ${i_j}^{(-1)} = \pi_j$. Additionally we know that $\pi_1 i_1 + \pi_2 i_2 = 1$, but as $+$ is the disjoint join, this shows the third condition is true and $A \oplus B$ is a disjoint sum. $\square$

Chapter 11

Turing categories and PCAs

In this chapter, we review the definition and properties of a Turing category and partial combinatory algebras [18, 20]. Because of the theorems of the earlier chapters, we will be able to transfer these ideas in a straightforward way from discrete Cartesian restriction categories to discrete inverse categories. Inverse Turing categories are defined below and correspond to Turing categories using Theorem 11.2.3: This provides the link between reversible computation and standard models of computation as promised in the introduction.

As noted in the introduction, Bennett [6] showed how a reversible Turing machine can emulate a standard Turing machine. As Turing machines can perform the applications of a partial combinatory algebra, we have a link between inverse Turing categories through Turing categories to reversible Turing machines.

11.1 Turing categories

Turing categories provide a categorical formulation for computability and includes partial combinatory algebras, the partial lambda calculus, and various other models as given in [18].

Definition 11.1.1 (Turing category)**.** Given $\mathbb{X}$ is a Cartesian restriction category:

(i) For a map $\tau_{X,Y} : A \times X \to Y$, a map $f : B \times X \to Y$ *admits a* $\tau_{X,Y}$*-index* when there is a total $\ulcorner f \urcorner : B \to A$ such that

$$\begin{tikzcd} A \times X \arrow[r, "\tau_{X,Y}"] & Y \\ B \times X \arrow[u, dotted, "\ulcorner f \urcorner \times 1_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

commutes.

(ii) A map $\tau_{X,Y} : A \times X \to Y$ is called a *universal application* if all $f : B \times X \to Y$ admit a $\tau_{X,Y}$-index.

(iii) If A is an object in $\mathbb{X}$ such that for every pair of objects X, Y in $\mathbb{X}$ there is a universal application $\tau : A \times X \to Y$, then A is called a *Turing object*.

(iv) A Cartesian restriction category that contains a Turing object is called a *Turing category*.

Note there is no requirement in the definition for the map $\ulcorner f \urcorner$ to be unique. When $\ulcorner f \urcorner$ is unique for a specific $\tau_{X,Y}$, then that $\tau_{X,Y}$ is called *extensional*. In the case where the object B is the terminal object, then the map $\ulcorner f \urcorner$ is a point of A (with $f = (\ulcorner f \urcorner \times 1)\tau_{X,Y}$) and $\ulcorner f \urcorner$ is referred to as a *code* of f.

Example 11.1.2. This example is due to Cockett and Hofstra [18].

We start with a "suitable" enumeration of partial recursive functions $f : \mathbb{N} \to \mathbb{N}$. Based on the fact that functions such as these can be described by Turing machines, and that Turing machines may be enumerated as $\{\phi_0, \phi_1, \ldots\}$, each of these functions can be coded into a single number. This may be extended to partial recursive functions of n variables which may similarly be enumerated, $\{\phi_0^{(n)}, \phi_1^{(n)}, \ldots\}$. When f is given by ϕ_e we say e is a *code* for f.

Two facts we will need about the family $\phi_m^{(n)}$:

- **Universal Functions:** There are partial recursive functions such that for each $n > 0$,

$$\Phi^{(n)}(e, x_1, \ldots, x_n) = \phi_e(x_1, \ldots, x_n)$$

  which are called the *universal functions*.

- **Parameter Theorem:** There are primitive recursive functions S_m^n for each $n, m > 0$ such that:

$$\Phi^{(n+m)}(e, x_1, \ldots, x_m, u_1, \ldots, u_n) = \Phi^{(m)}(S_m^n(e, x_1, \ldots, x_m), u_1, \ldots, u_n).$$

Suppose we choose such an enumeration and use it for the *Kleene-application* on the natural numbers, i.e., $\mathbb{N} \times \mathbb{N} \xrightarrow{\bullet} \mathbb{N}$ will be defined as

$$n \bullet x = \phi_n(x).$$

Now, consider the category with objects being the finite powers of $\mathbb{N}$ and a map $\mathbb{N}^k \to \mathbb{N}^m$ is an m-tuple of partial recursive function of k variables. When k is zero, the map simply picks a specific m-tuple of $\mathbb{N}$. We denote this category by $\textsc{Comp}(\mathbb{N})$. Then, $\textsc{Comp}(\mathbb{N})$ is a Turing category with $\mathbb{N}$ being a Turing object, using Kleene-application as the application map. We know $\mathbb{N}$ is isomorphic to $\mathbb{N} \times \mathbb{N}$ and hence we have $\mathbb{N} \times \mathbb{N} \triangleleft \mathbb{N}$. Application is guaranteed to be partial recursive by the universal functions item above and the weak universal property of $\bullet$ is a result of the Parameter Theorem.

Definition 11.1.3. Given $\mathbb{T}$ is a Turing category and A is an object of $\mathbb{T}$,

(i) If $\Upsilon = \{\tau_{X,Y} : A \times X \to Y | X, Y \in \mathbb{T}_o\}$, then Υ is called an *applicative family* for A.

(ii) An applicative family Υ is called *universal for A* when each $\tau_{X,Y}$ is a universal application. This is also referred to as a *Turing structure* on A.

(iii) A pair (A, Υ) where Υ is universal for A is called a *Turing structure* on $\mathbb{T}$.

Lemma 11.1.4. *If $\mathbb{T}$ is a Turing category with Turing object T, then every object B in $\mathbb{T}$ is a retract of T.*

Proof. As T is a Turing object, we have a diagram for $\tau_{1,B}$ and $\pi_0 : B \times 1 \to B$:

$$\begin{tikzcd}
T \times 1 \arrow[r, "\tau_{1,B}"] & B \\
B \times 1. \arrow[u, dotted, "\ulcorner \pi_0 \urcorner \times 1_1"] \arrow[ur, "\pi_0"'] &
\end{tikzcd}$$

Note we also have $u_r : B \to B \times 1$ is an isomorphism and therefore we have $1_B = u_r \pi_0 = (u_r(\ulcorner \pi_0 \urcorner \times 1))\tau_{1,B}$. Hence, we have $B \triangleleft^{\tau_{1,B}}_{u_r(\ulcorner \pi_0 \urcorner \times 1)} T$. $\square$

This allows for various recognition criteria for Turing categories.

Theorem 11.1.5. *A Cartesian restriction category $\mathbb{D}$ is a Turing category if and only if $\mathbb{D}$ has an object T for which every other object of $\mathbb{D}$ is a retract and T has a universal self-application map $\bullet$, written as $T \times T \xrightarrow{\bullet} T$.*

Proof. The "only if" portion follows immediately from setting T to be the Turing object of $\mathbb{D}$ and $\bullet = \tau_{T,T}$.

For the "if" direction, we need to construct the family of universal applications $\tau_{X,Y} : T \times X \to Y$ for each pair of objects X, Y in $\mathbb{D}$.

Let us choose pairs of maps that witness the retractions of X, Y of T, that is:

$$X \triangleleft^{r_X}_{m_X} T \quad \text{and} \quad Y \triangleleft^{r_Y}_{m_Y} T.$$

Define $\tau_{X,Y} = (1_T \times m_X) \bullet r_Y$. Suppose we are given $f : B \times X \to Y$. Consider

$$\begin{tikzcd}
T \times X \arrow[r, "1_T \times m_X"] & T \times T \arrow[r, "\bullet"] & T \arrow[r, "r_Y"] & Y \\
B \times X \arrow[u, "h \times 1_X"] \arrow[r, "1_B \times m_X"'] & B \times T \arrow[u, dotted, "h \times 1_T"] \arrow[r, "1_B \times r_X"'] & B \times X \arrow[u, "f m_Y"] \arrow[ur, "f"'] &
\end{tikzcd}$$

where h is the index for the composite map $(1_B \times r_X) f m_Y$. The middle square commutes as $\bullet$ is a universal application for T, T. The right triangle commutes as $m_Y r_Y = 1$. The left square commutes as each composite is $h \times m_X$. Noting that the bottom path from $B \times X$ to Y is $(1_B \times m_X)(1_B \times r_X) f = f$ and the top path from $T \times X$ to Y is our definition of $\tau_{X,Y}$, this means f admits the $\tau_{X,Y}$-index h. $\square$

Note that different splittings (choices of (m, r) pairs) would lead to different $\tau_{X,Y}$ maps. In fact there is no requirement that this is the only way to create a universal applicative family for T.

There is another criteria that also gives a Turing category:

Lemma 11.1.6. *A Cartesian restriction category $\mathbb{T}$ is a Turing category if:*

(i) $\mathbb{T}$ *has an object* T *for which every other object of* $\mathbb{T}$ *is a retract;*

(ii) $T \times T$ *has a map* $T \times T \xrightarrow{\circ} T$ *and for all* $f : T \to T$ *there exists an element,* $f_\bullet : 1 \to T$ *(which is total) such that*

$$\begin{tikzcd}
T \times T \arrow[r, "\circ"] & T \\
T \arrow[u, "\langle !f_\bullet,1\rangle"] \arrow[ur, "f"'] &
\end{tikzcd}$$

is a commutative diagram.

Proof. We need only show that T has a universal self-application map and then use Theorem 11.1.5.

T having a universal self-application map, $\bullet$, means for every map $f : B \times T \to T$ there is a map, $\ulcorner f \urcorner : B \to T$ such that

$$\begin{tikzcd}
T \times T \arrow[r, "\bullet"] & T \\
B \times T \arrow[u, "\ulcorner f \urcorner \times 1"] \arrow[ur, "f"'] &
\end{tikzcd}$$

commutes.

Let $T \times T \triangleleft^r_m T$. Then, consider

$$\begin{tikzcd}
T \times T \arrow[r, "r\times 1"] & T \times T \times T \arrow[r, "1\times m"] & T \times T \arrow[rr, "\circ"] & & T \\
T \times T \times T \arrow[u, "m\times 1"] \arrow[ur, equal] & & & & \\
T \times T \arrow[u, dotted, "\langle (rf)_\bullet,\pi_0,\pi_1\rangle"] \arrow[uur, dotted, "\langle (rf)_\bullet,\pi_0,\pi_1\rangle"'] \arrow[rr, "m"'] \arrow[rrr, equal, bend right] & & T \arrow[uu, dotted, "\langle (rf)_\bullet,1\rangle"'] \arrow[r, "r"'] & T \times T. \arrow[uur, "f"'] &
\end{tikzcd}$$

The rightmost quadrilateral commutes by assumption of this lemma. The middle quadrilateral commutes due to the properties of the product map and π_0 and π_1. The top left triangle commutes as $mr = 1$ and the remaining triangle has the same map on both dotted lines.

Thus, we may conclude that $\bullet := (r \times 1)(1 \times m)\circ$ and $\ulcorner f \urcorner := \langle !(rf)_\bullet, 1\rangle m$ satisfy the requirements of Theorem 11.1.5 and therefore T is a Turing object in a Turing category. $\square$

11.2 Inverse Turing categories

Now, we define inverse Turing categories. The idea is that an inverse Turing category should be a discrete inverse category $\mathbb{X}$ such that $\widetilde{\mathbb{X}}$ is a Turing category. A concrete description of this is developed below.

Definition 11.2.1. A discrete inverse category $\mathbb{X}$ is an *inverse Turing category* when there is a universal object T (i.e., every $B \in \mathbb{X}_o$ is a retract of T) in $\mathbb{X}$ with a map $\diamond : T \otimes T \to T \otimes T$ such that for every map $f : T \to T \otimes T$ there is a total map $\ulcorner f \urcorner : I \to T$ and a map $h_f : T \otimes T \to T \otimes T$ with $h_f \in T^{\triangle}_{\triangledown}$ such that $f \overset{h_f}{\simeq} {u^l_\otimes}^{(-1)}(\ulcorner f \urcorner \otimes 1)\diamond$, i.e., the diagram

$$\begin{tikzcd}
 & & T \otimes T \arrow[dd, dotted, "h_f"] \\
 & T \otimes T \arrow[ur, "\diamond"] & \\
T \arrow[ur, "{u^l_\otimes}^{(-1)}(\ulcorner f \urcorner \otimes 1)"] \arrow[rr, "f"'] & & T \otimes T
\end{tikzcd} \tag{11.1}$$

commutes.

First, we observe:

Lemma 11.2.2. *When $\mathbb{T}$ is a discrete Turing category then* $\mathbf{INV}(\mathbb{T})$ *is an inverse Turing category.*

Proof. By Lemma 4.3.7, we know that $\mathbf{INV}(\mathbb{T})$ is a discrete inverse category. Thus, all that remains is to show:

(i) There is a map $\diamond : T \otimes T \to T \otimes T$ in $\mathbf{INV}(\mathbb{T})$;

(ii) for a map f in $\mathbf{INV}(\mathbb{T})$, there is another map $\ulcorner f \urcorner$ which makes Diagram (11.1) commute.

As we are in a Turing category, we know that we have the diagram

$$\begin{tikzcd}
T \times T \arrow[r, "\circ"] & T \\
T \arrow[u, "\langle !f_\bullet, 1\rangle"] \arrow[ur, "f"'] &
\end{tikzcd}$$

in $\mathbb{T}$. The map $\langle !f_\bullet, 1\rangle$ is invertible by 3.10.7 as we are in a discrete Cartesian restriction category.

Expressing this in $\mathbf{INV}(\mathbb{T})$, for some $h'_f \in T_{\nabla}^{\Delta}$ we have:

$$\begin{tikzcd}
& & T \otimes T \otimes C \arrow[dd, dotted, "h'_f"] \\
& T \otimes T \arrow[ur, "\circ"] & \\
T \arrow[ur, "{u_\otimes^l}^{(-1)}(f_\bullet \otimes 1)"] \arrow[rr, "f"'] & & T \otimes D.
\end{tikzcd} \tag{11.2}$$

Recall from Lemma 4.4.3 that T_{∇}^{Δ} is closed under composition and that $1 \otimes f \in T_{\nabla}^{\Delta}$ for any f. In particular, for $f : T \to T \otimes D$ in $\mathbf{INV}(\mathbb{T})$, as $D \triangleleft T$, we have f is in the same equivalence class as $f(1 \otimes m_D)$. We see this as $\overline{f} = \overline{f(1 \otimes m_D)}$, $1 \otimes m_D \in T_{\nabla}^{\Delta}$ and the diagram

$$\begin{tikzcd}
& T \otimes D \arrow[d, dotted, "1\otimes m_D"] \\
T \arrow[ur, "f"] \arrow[r, "f(1\otimes m_D)"'] & T \otimes T
\end{tikzcd}$$

obviously commutes.

We use this and the fact that we have both $T \otimes C \triangleleft T$ and $D \triangleleft T$, to add to Diagram (11.2):

$$\begin{tikzcd}
 & & & & T \otimes T \arrow[d, dotted, "1\otimes r_{T\otimes C}"] \\
 & & & T \otimes T \otimes C \arrow[ur, "1\otimes m_{T\otimes C}"] \arrow[r, equal] \arrow[ddd, dotted, "h'_f"] & T \otimes T \otimes C \arrow[ddd, dotted, "h'_f"] \\
 & & T \otimes T \arrow[ur, "\circ"] & & \\
 & T \arrow[ur, "u_\otimes^{l\,(-1)}((f(1\otimes m_D))_\bullet \otimes 1)"] \arrow[drr, "f"'] & & & \\
 & & & T \otimes D \arrow[dr, "1\otimes m_D"'] & T \otimes D \arrow[d, "1\otimes m_D"] \\
 & & & & T \otimes T.
\end{tikzcd}$$

But this is the required diagram for an inverse Turing category with $h_f = (1 \otimes r_{T\otimes C})h'_f(1 \otimes m_D)$, and with $\ulcorner f \urcorner = (f(1 \otimes m_D))_\bullet$ and with $\diamond = \circ(1 \otimes m_{T\otimes C})$. Therefore $\mathbf{INV}(\mathbb{T})$ is an inverse Turing category. $\square$

We know that applying the Cartesian Completion to $\mathbb{X}$, an inverse Turing category, results in $\widetilde{\mathbb{X}}$, a discrete Cartesian restriction category. Moreover, if $A \triangleleft^{r_A}_{m_A} T$ in $\mathbb{X}$, then $A \triangleleft^{r_A u_\otimes^{r\,(-1)}}_{m_A u_\otimes^{r\,(-1)}} T$ in $\widetilde{\mathbb{X}}$ and thus T will remain universal in $\widetilde{\mathbb{X}}$. Hence, we have the basic requirements for a Turing category as specified in Theorem 11.1.5 and Lemma 11.1.6. All that remains to be shown is that we have a self-application map and a code for each map $f : 1 \to T$ as in Lemma 11.1.6.

Theorem 11.2.3. *When $\mathbb{X}$ is an inverse Turing category, $\widetilde{\mathbb{X}}$ is a Turing category.*

Proof. From the discussion, we need to specify the self-application map $\circ : T \times T \to T$ and $f_\bullet : 1 \to T$ in $\widetilde{\mathbb{X}}$.

The diagram of Definition 11.2.1, when raised to $\widetilde{\mathbb{X}}$ translates to:

$$\begin{tikzcd}
T \times T \arrow[r, "(\diamond, T)"] & T \\
T. \arrow[u, "\langle \ulcorner f \urcorner, 1\rangle"] \arrow[ur, "(f, T)"'] &
\end{tikzcd}$$

But this corresponds exactly to the requirement of Lemma 11.1.6 with $\circ = (\diamond, T)$ and $(f,T)_\bullet = \ulcorner f \urcorner$. Finally, noting that T is universal in $\mathbb{X}$, if we have $(f,B) : T \to T$ in $\widetilde{\mathbb{X}}$, where $B \lhd^{r_B}_{m_B} T$ in $\mathbb{X}$, we recall that $(f,B) \simeq (f(1 \otimes m_B), T)$ in $\mathbb{X}$. Therefore, it may be written as above and we therefore have shown that $\widetilde{\mathbb{X}}$ is a Turing category. $\square$

11.3 Partial combinatory algebras

In a Cartesian restriction category, for any operation $f : A \times A \to A$ define $f^{(n)}$ for $n \geq 1$ recursively by:

(i) $f^{(1)} = f$,

(ii) $f^{(n+1)} = (f \times 1)f^{(n)}$.

Definition 11.3.1. A Cartesian restriction category has a *partial combinatory algebra* when it has an object A together with:

(i) A partial map $\bullet : A \times A \to A$,

(ii) two total elements $1 \xrightarrow{k} A$ and $1 \xrightarrow{s} A$ which satisfy

$$\begin{tikzcd}
A \times A \times A \arrow[r, "(\bullet \times 1)\bullet"] & A \\
A \times A \arrow[u, "k \times 1 \times 1"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times A \times A \times A \arrow[rr, "\bullet^{(3)}"] & & A \\
& & A \times A \arrow[u, "\bullet"'] \\
A \times A \times A \arrow[uu, "s \times 1 \times 1 \times 1"] \arrow[rr, "\theta'_A"'] & & (A \times A) \times (A \times A), \arrow[u, "\bullet \times \bullet"']
\end{tikzcd}$$

(iii) $A \times A \xrightarrow{s \times 1 \times 1} A \times A \times A \xrightarrow{\bullet^2} A$ is total.

In the above $\theta' = (1 \times 1 \times \Delta)(1 \times c \times 1)a$ where a sets the parenthesis as in the diagram.

Of course, this is more familiarly given equationally by:

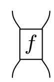
$$(k \bullet x) \bullet y = x \qquad ((s \bullet x) \bullet y) \bullet z = (x \bullet z) \bullet (y \bullet z).$$

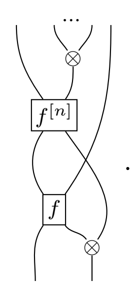
These are the equations of a combinatory algebra where partiality is not considered. As we have partiality, we also add the requirement that $s \bullet x \bullet y$ is a total map for any x, y.

Note that if we have a Turing object T in a Cartesian restriction category, it is a partial combinatory algebra. All we need to do is to actually define the element k and s by using the commuting diagrams of Definition 11.3.1.

Now, we want to consider what are the conditions required for an inverse category $\mathbb{X}$ such that $\widetilde{\mathbb{X}}$ has a partial combinatory algebra.

In a discrete inverse category, we define the notation $f^{[n]}$. For any operation $f : A \otimes A \to A \otimes A$ define $f^{[n]}$ recursively by:

(i) $f^{[1]} : A \otimes A \to A \otimes A = f =$  .

(ii) $f^{[n+1]} : A \otimes (\otimes_n A) \otimes A \to A \otimes (\otimes_{n+1} A) =$  .

Definition 11.3.2. A discrete inverse category $\mathbb{X}$ has an *inverse partial combinatory algebra* when there is an object A in $\mathbb{X}$ with a map $A \otimes A \xrightarrow{\bullet} A \otimes A$ and two total elements:

$$1 \xrightarrow{k} A \qquad 1 \xrightarrow{s} A$$

and maps $h_k : A \otimes A \otimes A \to A \otimes A, h_s : A \otimes A \otimes A \otimes A \to A \otimes A \otimes A \otimes A$ in A^{Δ}_{∇} which satisfy the following three axioms:

[iCPA.1]

$$\begin{tikzcd}
 & & A \otimes A \otimes A \arrow[dd, dotted, "h_k"] \\
 & A \otimes A \otimes A \arrow[ur, "\bullet^{[2]}"] & \\
A \otimes A \arrow[ur, "{u_\otimes^l}^{(-1)}(k \otimes 1 \otimes 1)"] \arrow[rr, "1"'] & & A \otimes A.
\end{tikzcd}$$

[**iCPA.2**]

$$\begin{tikzcd}
 & & & A\otimes A\otimes A\otimes A \arrow[dddd, dotted, "h_s"] \\
 & & A\otimes A\otimes A\otimes A \arrow[ur, "\bullet^{[3]}"] & \\
A\otimes A\otimes A \arrow[urr, "{u_\otimes^{l}}^{(-1)}(s\otimes 1)"] \arrow[dr, "\theta'_A"'] & & & \\
 & (A\otimes A)\otimes(A\otimes A) \arrow[dr, "(\bullet\otimes\bullet)(1\otimes c\otimes 1)"'] & & \\
 & & A\otimes A\otimes A\otimes A \arrow[r, "(\bullet\otimes 1)"'] & A\otimes A\otimes A\otimes A.
\end{tikzcd}$$

[**iCPA.3**] $I\otimes A\otimes A \xrightarrow{s\otimes 1\otimes 1} A\otimes A\otimes A \xrightarrow{\bullet^{[2]}} A\otimes A\otimes A$ is total.

Proposition 11.3.3. *A discrete inverse category $\mathbb{X}$ has an inverse partial combinatory algebra if and only if $\widetilde{\mathbb{X}}$ has a partial combinatory algebra.*

Proof. When we have a discrete inverse category $\mathbb{X}$ with an inverse partial combinatory algebra, we see immediately the map $\bullet : A\otimes A \to A\otimes A$ in $\mathbb{X}$ becomes the map $(\bullet, A) : A\times A \to A$, satisfying (i) of Definition 11.3.1. The commutative diagrams [**iCPA.1**] and [**iCPA.2**], when lifted to $\widetilde{\mathbb{X}}$, become the diagrams for a partial combinatory algebra as given in (ii), where $(k{u_\otimes^{l}}^{(-1)}, I)$ and $(s{u_\otimes^{l}}^{(-1)}, I)$ are the k, s of the partial combinatory algebra. Finally, the totality requirement, [**iCPA.3**], gives (iii) of the partial combinatory algebra definition.

Hence, we have shown that an inverse partial combinatory algebra in $\mathbb{X}$ gives a partial combinatory algebra in $\widetilde{\mathbb{X}}$.

For the reverse, when we have a partial combinatory algebra over A in $\widetilde{\mathbb{X}}$, a discrete Cartesian restriction category, by Lemma 3.10.7 we know that the map $\langle\bullet, 1\rangle$ is invertible and hence is in $\mathbb{X}$. The two maps $\langle k, 1\rangle$ and $\langle s, 1\rangle$ are also invertible and therefore are in $\mathbb{X}$.

Given this, the diagrams of the partial combinatory algebra in $\widetilde{\mathbb{X}}$ translate directly to the [**iCPA.1**] and [**iCPA.2**] where $\bullet$ in $\mathbb{X}$ is the invertible map $\langle\bullet, 1\rangle$.

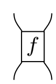
The totality of $s\bullet^{(2)}$ in $\widetilde{\mathbb{X}}$ then immediately gives us [**iCPA.3**], the totality of $s\bullet^{[2]}$ in $\mathbb{X}$.

$\square$

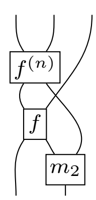
However, we can simplify the definition of an inverse partial combinatory algebra when A is powerful in $\mathbb{X}$. (Here, powerful means that $1 \triangleleft A$, $A \otimes A \triangleleft A$, $A \otimes A \otimes A \triangleleft A$, $\ldots$). Note that if A is a partial combinatory algebra in $\widetilde{\mathbb{X}}$, that guarantees it is powerful in $\widetilde{\mathbb{X}}$. Assuming the retractions are $A^n \triangleleft_{m_n}^{r_n} A$, we have $m_j r_j = 1$ and $r_j m_j = \overline{r_j m_j}$ for each j. Thus, each of the maps are partial inverses in $\widetilde{\mathbb{X}}$ and therefore in $\mathbb{X}$. Thus, A is a powerful object in $\mathbb{X}$.

Note that our definition of "powerful" does not relate to resource usage or a particular complexity class. An example of a powerful object is the natural numbers with the Cantor enumeration.

In a discrete inverse category, we redefine the notation $f^{(n)}$. For any map $f : A \otimes A \to A \otimes A$ where A is a powerful object define $f^{(n)}$ recursively by:

(i) $f^{(1)} : A \otimes A \to A \otimes A = f =$ 

;

(ii) $f^{(n+1)} : \otimes_{n+2} A \to A \otimes A =$ 

.

Lemma 11.3.4. *Suppose a discrete inverse category $\mathbb{X}$ has a inverse partial combinatory algebra over A and A is a powerful object in $\mathbb{X}$, with $\otimes^n A \triangleleft_{m_n}^{r_n} A$. Then [**iCPA.1**], [**iCPA.2**] and [**iCPA.3**] may be simplified to:*

[***iCPA′.1***]

$$
\begin{tikzcd}
 & & A \otimes A \arrow[dd, dotted, "h'_k"] \\
 & A \otimes A \otimes A \arrow[ur, "\bullet^{(2)}"] & \\
A \otimes A \arrow[ur, "{u^{l}_{\otimes}}^{(-1)}(k\otimes 1\otimes 1)"] \arrow[rr, "1"'] & & A \otimes A.
\end{tikzcd}
$$

[***iCPA′.2***]

$$
\begin{tikzcd}
 & & & A \otimes A \arrow[dddd, dotted, "h'_s"] \\
 & A \otimes A \otimes A \otimes A \arrow[urr, "\bullet^{(3)}"] & & \\
A \otimes A \otimes A \arrow[ur, "{u^{l}_{\otimes}}^{(-1)}(s\otimes 1)"] \arrow[dr, "\theta'_A"'] & & & \\
 & (A \otimes A) \otimes (A \otimes A) \arrow[dr, "(\bullet\otimes\bullet)(1\otimes c\otimes 1)(1\otimes 1\otimes m_2)"'] & & \\
 & & A \otimes A \otimes A \arrow[r, "(\bullet\otimes 1)(1\otimes m_2)"'] & A \otimes A.
\end{tikzcd}
$$

[***iCPA′.3***] $I \otimes A \otimes A \xrightarrow{s\otimes 1\otimes 1} A \otimes A \otimes A \xrightarrow{\bullet^{(2)}} A \otimes A$ *is total.*

11.4 Computable functions

Given a partial combinatory algebra A in a Cartesian restriction category, one can form $\textsc{Comp}(A)$, the category of computable partial functions generated by A. These are the maps with an index:

$$\begin{tikzcd}
A \times (\times_n A) \arrow[r, "\bullet^{(n)}"] & A \\
(\times_n A). \arrow[u, "\ulcorner f \urcorner \times 1"] \arrow[ur, "f"'] &
\end{tikzcd}$$

We would like $\textsc{Comp}(A)$ to be a discrete Turing category so that $\mathbf{INV}(\textsc{Comp}(A))$ is an inverse Turing category by Lemma 11.2.2. Unfortunately, there is no guarantee that $\textsc{Comp}(A)$ is a discrete Turing category. However, we can define conditions so that it is true:

Definition 11.4.1. Given a discrete object A in a Cartesian restriction category, A has a *discrete partial combinatory algebra* when:

(i) A has a partial combinatory algebra;

(ii) there exists $e : 1 \to A$, a total element, such that

$$\begin{tikzcd}
A \times A \times A \arrow[r, "\bullet^{(2)}"] & A, \\
A \times A \arrow[u, "e \times 1"] \arrow[ur, "\Delta^{(-1)}"'] &
\end{tikzcd}$$

meaning there is a code for $\Delta^{(-1)}$.

We immediately have:

Lemma 11.4.2. *When A has a discrete partial combinatory algebra, then* $\textsc{Comp}(A)$ *is a discrete Cartesian restriction category.*

By Lemma 11.2.2, this means that $\mathbf{INV}(\textsc{Comp}(A))$ is an inverse Turing category. Note it is still the case that there can be a map of $\textsc{Comp}(A)$ which is invertible in $\widetilde{\mathbb{X}}$ (i.e., is in $\mathbb{X}$), but is *not* invertible in $\textsc{Comp}(A)$.

Chapter 12

Conclusions and future work

This thesis has studied inverse categories [21], providing conditions for a tensor to act like a product — the inverse product — and similarly for a second tensor to behave like a coproduct — the disjoint sum.

The following are the main results of this thesis.

1. **Restriction Products and Coproducts in Inverse Categories**

   We showed in Proposition 4.2.1 that an inverse category with restriction products is a restriction preorder. Similarly, in Proposition 6.1.5, an inverse category with coproducts is a preorder.

2. **Discrete Inverse Categories**

   We introduced the inverse product, Definition 4.3.1 and showed that this provided meets for the inverse category in Proposition 4.3.6. We showed that the inverse subcategory of a discrete Cartesian restriction category is a discrete inverse category in Lemma 4.3.7.

   Given a discrete inverse category $\mathbb{X}$, we constructed a category $\widetilde{\mathbb{X}}$, the Cartesian Completion of $\mathbb{X}$ with the same objects as $\mathbb{X}$ and maps being equivalence classes of maps of $\mathbb{X}$. We showed this was a restriction category in Lemmas 5.1.7 and 5.1.8 and in fact was a discrete Cartesian restriction category as shown in Theorem 5.1.12.

   Finally, in Theorem 5.2.6 we provided an equivalence between the category of discrete inverse categories and the category of discrete Cartesian restriction categories. This result, together with its consequences, is the main theoretical contribution of this thesis.

3. **Disjointness Relations and Disjoint Joins**

   In Definition 6.2.1 we defined what a disjointness relation is in an inverse category, followed by Definition 6.2.4 which defined disjointness on the restriction idempotents of the inverse category. Theorem 6.2.7 shows that these two definitions are equivalent, allowing us to define disjointness in whichever way is most convenient.

   Disjoint joins are introduced with Definition 6.3.1, providing an analogue to joins in a restriction category.

4. **Disjoint Sums**

   Disjoint sums in an inverse category are given in Definition 7.1.1 and we show that an inverse category having all disjoint sums has a symmetric monoidal tensor in Proposition 7.1.4.

   Disjointness tensors, Definition 7.2.1, are shown in Lemma 7.2.9 to provide enough structure to allow the creation of a disjointness relation. However, to create a disjoint join requires a disjoint sum tensor. Disjoint sum tensors are given in Definition 7.3.1 as additional conditions on a disjointness tensor. Given a disjoint sum tensor, Proposition 7.3.5 defines a disjoint join.

   Section 7.4 proves, in Proposition 7.4.1 and Proposition 7.4.2, that a disjoint sum enables the creation of a disjoint sum tensor and conversely, a disjoint sum tensor produces disjoint sums.

   In Chapter 8, Definition 8.1.2 gives us $\textsc{iMat}(\mathbb{X})$, a matrix category over the inverse category $\mathbb{X}$ with disjoint joins. Theorem 8.1.4 proves this is an inverse category with disjoint sums. Lemma 8.2.2 shows that there is a functor M from an inverse category with disjoint joins to its matrix category. The matrix construction gives an adjoint between DSum, the category of inverse categories with disjoint sums and DJoin, the category of inverse categories with disjoint joins. Furthermore, Proposition 8.2.5 shows that an inverse category $\mathbb{X}$ with disjoint sums is equivalent to iMat($\mathbb{X}$).

5. **Distributive Inverse Categories**

   We define distributive inverse categories in Definition 9.2.1 and show that distributivity of the inverse product over the disjoint join is equivalent to distributing over a disjoint sum tensor in Lemma 9.2.6.

   Then, in Theorem 9.3.3 we show that Cartesian Completion turns a disjoint sum tensor into a coproduct. From this we conclude that $\widetilde{\mathbb{X}}$ is a distributive restriction category in Corollary 9.3.4.

6. **Linkage to quantum computation**

   Chapter 10 starts with a review of Frobenius algebras and how they appear in models of quantum computation. Theorem 10.1.17 shows the category of commutative Frobenius algebras, CFrob($\mathbb{X}$), in a symmetric monoidal category $\mathbb{X}$ is actually a discrete inverse category. Furthermore, when the symmetric monoidal category $\mathbb{X}$ is an additive tensor category with zero maps and biproducts, then CFrob($\mathbb{X}$) is shown in Lemma 10.2.3 to possess a disjointness relation, given by $f \perp g \iff \Delta(f \otimes g)\Delta^{(-1)} = 0$. Proposition 10.3.7 shows that the addition of maps is a disjoint join in CFrob($\mathbb{X}$) and Proposition 10.4.3 shows the biproduct of objects is a disjoint sum.

7. **Inverse Turing Categories and Inverse PCAs**

   Inverse Turing categories and inverse partial combinatory algebras are defined in Definition 11.2.1 and Definition 11.3.2 respectively. First, we show in Lemma 11.2.2 that if we have a discrete Turing category $\mathbb{T}$, then **INV**($\mathbb{T}$), the inverse subcategory of $\mathbb{T}$, is an inverse Turing category. Then we show that the Cartesian Completion of an inverse Turing category gives a Turing category. In Proposition 11.3.3 we show that a discrete inverse category $\mathbb{X}$ has an inverse PCA if and only if $\widetilde{\mathbb{X}}$ has a PCA. Furthermore, we discuss the category of computable function based on a PCA in a Cartesian restriction category and show the conditions required for that to be a discrete Cartesian restriction category in Lemma 11.4.2.

This thesis demonstrates that discrete inverse categories provide a convenient intermediate link between standard models of computing (i.e., via inverse Turing categories to Turing categories) and quantum computing (via special commutative Frobenius algebras).

12.1 Future directions

The obvious next step is to use discrete inverse categories to provide a categorical semantics for existing reversible languages such as **Inv** [55], and Theseus [37,38]. The examples of the category of partial injective functions used throughout this thesis provides the basis for the semantics of **Inv**.

It would be interesting to understand how to formulate datatypes, higher order structures, etc. at the inverse category level. The obvious correspondences are from product types to the inverse product as well as between sum types and the disjoint sum. From there, one would study the trace and explore infinite and recursive types. Note that some of this work would follow naturally from exploring the semantics of Theseus from [37,38].

An interesting direction, in my opinion, would be to further investigate the work on reversible CCS as in [28] and [59]. This could be combined with the work of Chakraborty on MPL [13] to describe the communication of multiple processes and generalize to a series of reversible processes.

The connection to quantum computing, as exampled in Chapter 10 would benefit from further investigation. As well, the bulk of current literature on semantics of quantum computing is based on finite dimensional Hilbert spaces. Our construction is not limited to the finite dimensional case, and as such, may be of use when considering infinite dimensions. To pursue this area, one would likely start with a consideration of Frobenius algebras in a linearly distributive category, as described by Egger in [29].

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