Chapter 4

Fuzzy Systems

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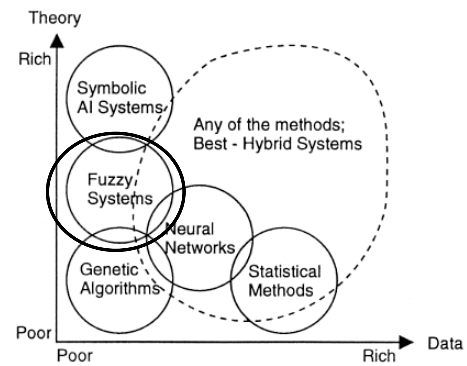
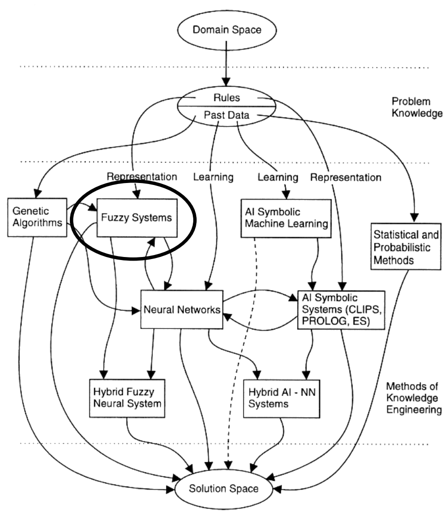
Fuzzy Systems in Knowledge Engineering



[Kasabov, 1996]

Fuzzy Systems in Knowledge Engineering



[Kasabov, 1996]

4. Fuzzy Systems

What Does Fuzzy Logic Mean?

- Fuzzy logic was introduced by Lotfi Zadeh (UC Berkeley) in 1965.
- Fuzzy logic attempts to formalize "approximate knowledge" and approximate reasoning.
- Fuzzy logic is based on fuzzy set theory, an extension of classical set theory.
- Fuzzy logic did not attract any attention until the 1980s (fuzzy controller applications)

Fuzzy Logic: Just Human ...

- Humans primarily use fuzzy terms:
  large, small, fast, slow, warm, cold, ...
- We say:
  "If the weather is nice and I have a little time, I will probably go for a hike in the Rockies."
- We don't say:
  "If the temperature is above 24 degrees and the cloud cover is less than 10% and I have 3 hours time, I will go for a hike with a probability of 0.47."

Fuzzy Logic: Motivation

- Lotfi Zadeh: “Make use of the leeway of fuzziness.”
- Fuzziness as a principle of economics:
  - Precision is expensive.
  - Only apply as much precision to a problem as necessary.
- Example (1): Backing into a parking space
  How long would it take if we had to park the car with a precision of ±2 mm?
- Example (2): Temperature control
  How much effort would be involved in controlling the temperature of the water flowing into your bathtub by ±1°C ?

4. Fuzzy Systems

Basics of Fuzzy Sets

- Example: the set of “young people”

$$\text{young} = \{x \in P \mid \text{age}(x) \leq 20\}$$

- We can also define a characteristic function for this set:

$$\mu_{\text{young}}(x) = \begin{cases} 1 : \text{age}(x) \leq 20 \\ 0 : 20 < \text{age}(x) \end{cases}$$

Basics of Fuzzy Sets (2)

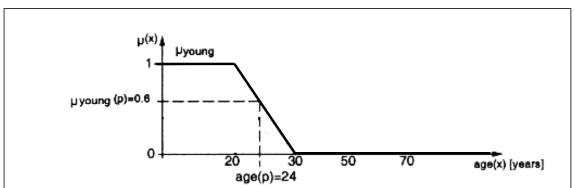
- Fuzzy set theory offers a variable notion of membership:
  - A person of age 21 could still belong to the set of young people, but only to a degree of less than one, maybe 0.9.

$$\mu_{\text{young}}(x) = \begin{cases} 1 & : \quad \text{age}(x) \leq 20 \\ 1 - \frac{\text{age}(x)-20}{10} & : \quad 20 < \text{age}(x) \leq 30 \\ 0 & : \quad 30 < \text{age}(x) \end{cases}$$

  - Now the set *young* contains people with ages between 20 and 30 with a linearly decreasing *degree of membership*.

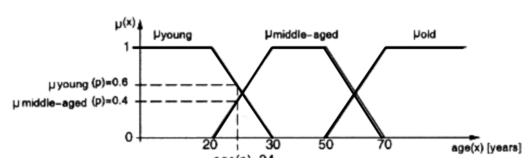
Fuzzy Membership Function

$$\mu_{\text{young}}(x) = \begin{cases} 1 & : \quad \text{age}(x) \leq 20 \\ 1 - \frac{\text{age}(x)-20}{10} & : \quad 20 < \text{age}(x) \leq 30 \\ 0 & : \quad 30 < \text{age}(x) \end{cases}$$



Linguistic Variables

- Covering the domain of a variable with several fuzzy sets, together with a corresponding semantics, defines a *linguistic variable*.



- Linguistic variable *age*

Linguistic Variables (2)

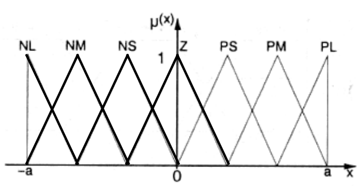
- Using fuzzy sets allows us to incorporate the fact that no sharp boundaries between these groups exist.
- The corresponding fuzzy sets overlap in certain areas, forming non-crisp or fuzzy boundaries.
- This way of defining fuzzy sets over the domain of a variable is referred to as *granulation*,
  - in contrast to the division into crisp sets (quantization).

Fuzzy Granules

- Granulation results in a grouping of objects into imprecise clusters of *fuzzy granules*.
- The objects forming a granule are drawn together by similarity.
- This can be seen as a form of fuzzy data compression.
- Often granulation is obtained manually through expert interviews.

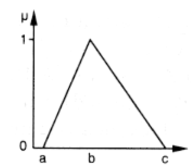
Finding Fuzzy Granules

- If expert knowledge on a domain is not available, an automatic granulation approach can be used.

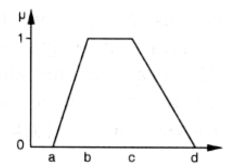


- Standard granulation using an odd number of membership functions:
  - NL: negative large, NM: negative medium, NS: negative small,Z: zero, ...

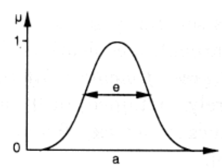
Shapes for Membership Functions



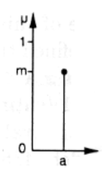
Triangle: [a,b,c]



Trapezoid: [a,b,c,d]



Gaussian: [a,θ]



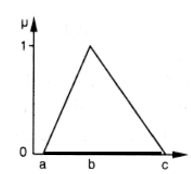
Singleton: [a,m]

Parameters of Fuzzy Membership Fcts.

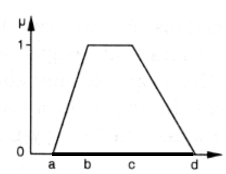
- **Support:** $s_A := \{ x : \mu_A(x) > 0 \}$
  - The area where the membership function is greater than zero.
- **Core:** $c_A := \{ x : \mu_A(x) = 1 \}$
  - The area for which elements have maximum degree of membership to the fuzzy set A.
- **α-Cut:** $A_\alpha := \{ x : \mu_A(x) = \alpha \}$
  - The cut through the membership function of A at height α.
- **Height:** $h_A := \max_x \{ \mu_A(x) \}$
  - The maximum value of the membership function of A.

$s_A := \{ x : \mu_A(x) > 0 \}$

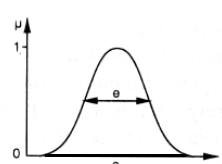
Membership Functions: Support



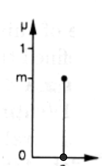
Triangle: [a,b,c]



Trapezoid: [a,b,c,d]



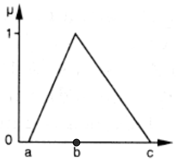
Gaussian: [a,θ]



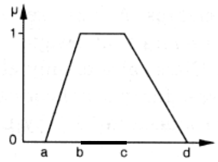
Singleton: [a,m]

$c_A := \{ x : \mu_A(x) = 1 \}$

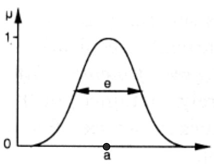
Membership Functions: Core



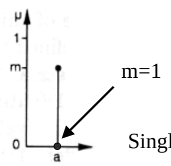
Triangle: [a,b,c]



Trapezoid: [a,b,c,d]



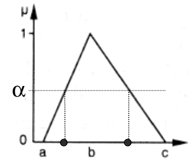
Gaussian: [a,θ]



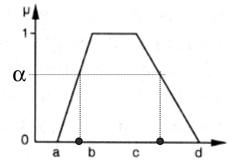
Singleton: [a,m]

$A_\alpha := \{ x : \mu_A(x) = \alpha \}$

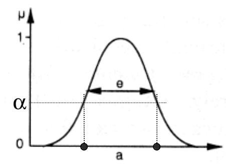
Membership Functions: α-Cut



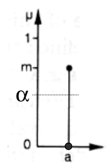
Triangle: [a,b,c]



Trapezoid: [a,b,c,d]



Gaussian: [a,θ]

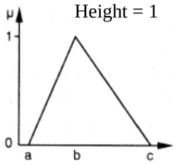


Singleton: [a,m]

$h_A := \max_x \{ \mu_A(x) \}$

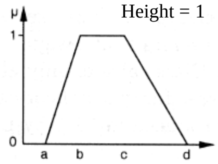
Membership Functions: Height

Height = 1



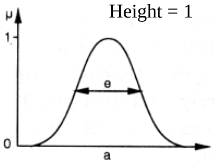
Triangle: [a,b,c]

Height = 1

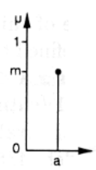


Trapezoid: [a,b,c,d]

Height = 1



Gaussian: [a,θ]



Height = m

Singleton: [a,m]

4. Fuzzy Systems

Fuzzy Numbers

- Real-world measurements are always imprecise in nature.
- Usually such measurements are modeled through a crisp number *x*, denoting the most typical value, together with an interval describing the amount of imprecision.
- In a linguistic sense, this could be described as “about *x*”.
- Using fuzzy sets we can incorporate this information directly.

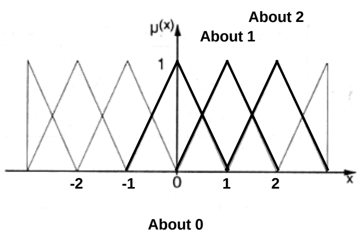
Fuzzy Numbers as Fuzzy Sets

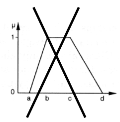
- Fuzzy numbers are a special type of fuzzy sets, restricting the possible types of membership functions:
  - μ_A must be *normalized* (i.e., the core is non-empty, $c_A \neq \emptyset$).
  - μ_A must be *singular*, i.e., there is precisely one point which lies inside the core, modeling the typical value (= *modal value*) of the fuzzy number.
  - μ_A must be monotonically increasing left of the core and monotonically decreasing on the right.

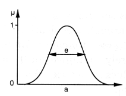
  This makes sure that there is only one peak, and therefore only one typical value exists.

Fuzzy Numbers: Example

- Typically triangular membership functions are chosen for fuzzy numbers.

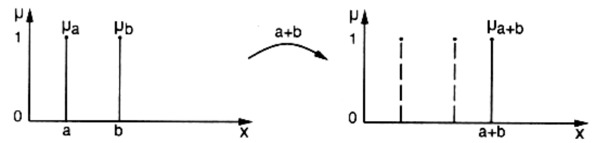






Adding Numbers

- Let us first consider the classical crisp version of addition.



$$\mu_{a+b}(x) = \begin{cases} 1 & \text{if } \exists\, y, z \in \mathbb{R} : y + z = x \wedge \mu_a(y) = 1 \wedge \mu_b(z) = 1 \\ 0 & \text{else} \end{cases}$$

Operations on Fuzzy Numbers

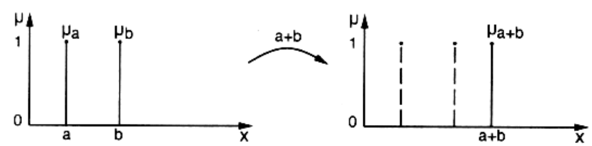
- Using the so-called *extension principle*, we extend classical operators (addition, multiplication) to their fuzzy counterparts, such that we can also handle intermediate degrees of membership.
- For an arbitrary binary operator $\otimes$:

$$\mu_{A \otimes B}(x) = \max_{y,z \in \mathbb{R}} \{\min\{\mu_A(y), \mu_B(z)\} \mid y \otimes z = x\}$$

  - For a value x a degree of membership is derived which is the maximum of $\min\{\mu_A(y), \mu_B(z)\}$ over all possible pairs of y,z for which $y \otimes z = x$ holds.

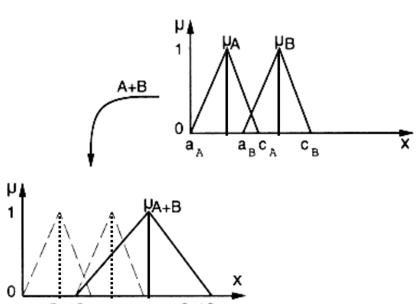
Fuzzy Addition on Crisp Numbers

- We check whether the fuzzy addition is consistent with "normal" addition on crisp numbers.



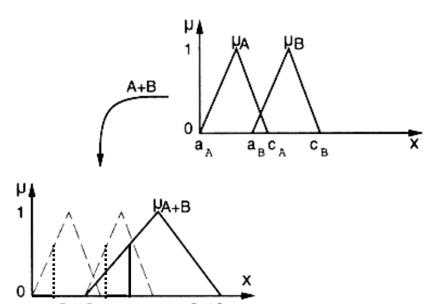
$$\mu_{A \otimes B}(x) = \max_{y,z \in \mathbb{R}} \{\min\{\mu_A(y), \mu_B(z)\} \mid y \otimes z = x\}$$

Fuzzy Addition



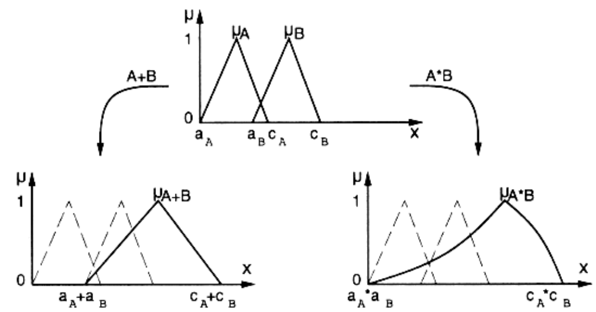
$$\mu_{A \otimes B}(x) = \max_{y,z \in \mathbb{R}} \{\min\{\mu_A(y), \mu_B(z)\} \mid y \otimes z = x\}$$

Fuzzy Addition



$$\mu_{A \otimes B}(x) = \max_{y,z \in \mathbb{R}} \{\min\{\mu_A(y), \mu_B(z)\} \mid y \otimes z = x\}$$

Fuzzy Addition and Multiplication



$$\mu_{A \otimes B}(x) = \max_{y,z \in \mathbb{R}} \{\min\{\mu_A(y), \mu_B(z)\} \mid y \otimes z = x\}$$

Fuzzy Number Operations: How to ...

- For practical purposes we can calculate the result of applying a monotonical operation on fuzzy numbers as follows:
  - Subdivide the membership functions $\mu_A(x)$ and $\mu_B(x)$ into monotonically increasing and decreasing parts.
  - Then perform the operation jointly on the increasing (decreasing) parts of numbers A and B.
  - Plateaus can be dealt with in a single computation step.

Fuzzy Number Operations: How to ... (2)

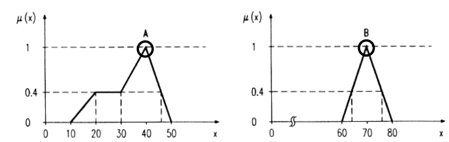
- Let A and B be fuzzy numbers and $\otimes$ a strongly monotonical operation.
- Let $[a_1, a_2]$ and $[b_1, b_2]$ be the intervals in which $\mu_A(x)$ and $\mu_B(x)$ are monotonically increasing (decreasing).
- Now if there exist subintervals $[\alpha_1, \alpha_2] \subseteq [a_1, a_2]$ and $[\beta_1, \beta_2] \subseteq [b_1, b_2]$ such that

$$\mu_A(x_A) = \mu_B(x_B) = \lambda \qquad \forall x_A \in [\alpha_1, \alpha_2], \forall x_B \in [\beta_1, \beta_2],$$

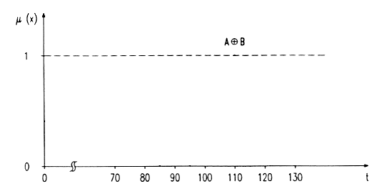
then:

$$\mu_{A \otimes B}(t) = \lambda \qquad \forall t \in [\alpha_1 \otimes \beta_1, \alpha_2 \otimes \beta_2],$$

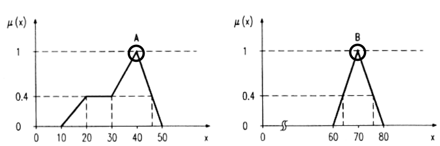
Again: Adding Fuzzy Numbers

- 

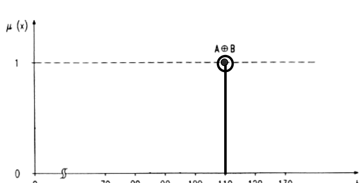
$\mu_{A+B}(t) = 1.0 \qquad t = ...$



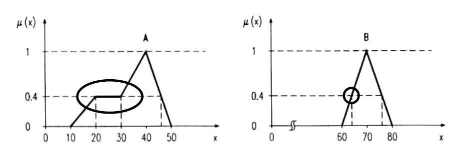
Again: Adding Fuzzy Numbers

- 

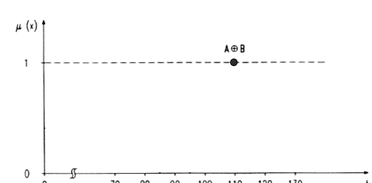
$\mu_{A+B}(t) = 1.0 \qquad t = 40 + 70 = 110$



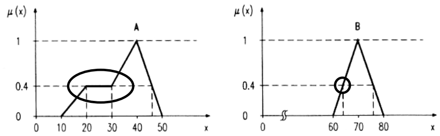
Again: Adding Fuzzy Numbers

- 

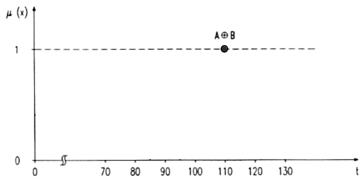
$\mu_{A+B}(t) = 0.4 \qquad \forall t \in [..., ...]$



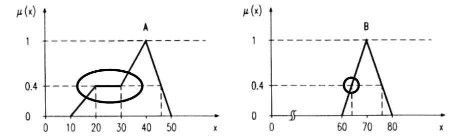
Again: Adding Fuzzy Numbers

- 

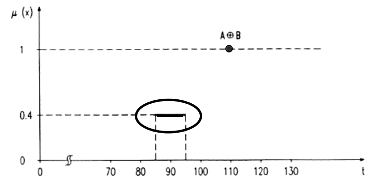
$\mu_{A+B}(t) = 0.4 \qquad \forall\, t \in [20+64,\ 30+64]$



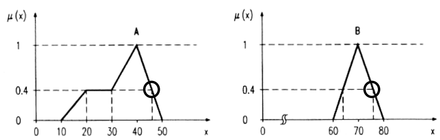
Again: Adding Fuzzy Numbers

- 

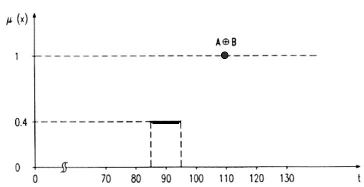
$\mu_{A+B}(t) = 0.4 \qquad \forall\, t \in [20+64,\ 30+64] = [84,\ 94]$



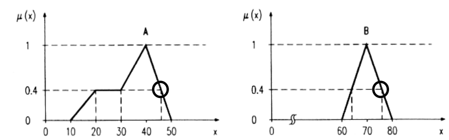
Again: Adding Fuzzy Numbers

- 

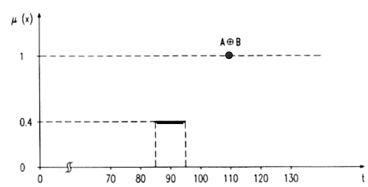
$\mu_{A+B}(t) = 0.4 \qquad t = \ldots$



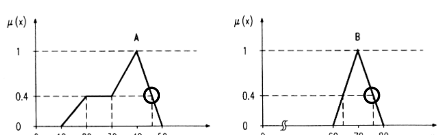
Again: Adding Fuzzy Numbers

- 

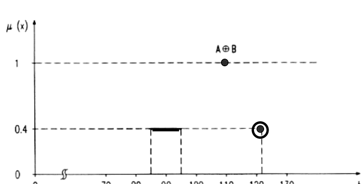
$\mu_{A+B}(t) = 0.4 \qquad t = 46 + 76 = 122$



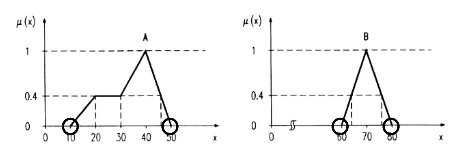
Again: Adding Fuzzy Numbers

- 

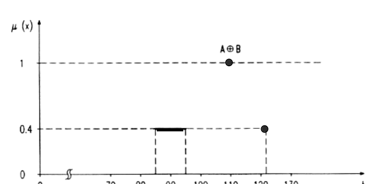
$\mu_{A+B}(t) = 0.4 \qquad t = 46 + 76 = 122$



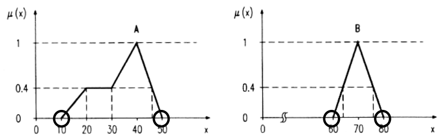
Again: Adding Fuzzy Numbers

- 

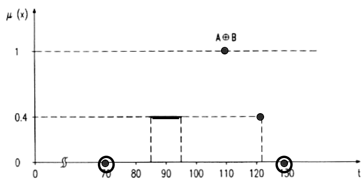
Support: $s_{A+B} = [\ldots,\ \ldots]$



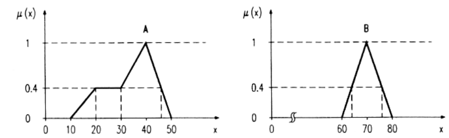
Again: Adding Fuzzy Numbers

❏ 

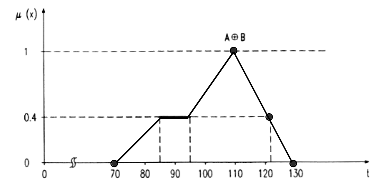
Support: s_{A+B} = [10+60, 50+80] = [70, 130]



Again: Adding Fuzzy Numbers

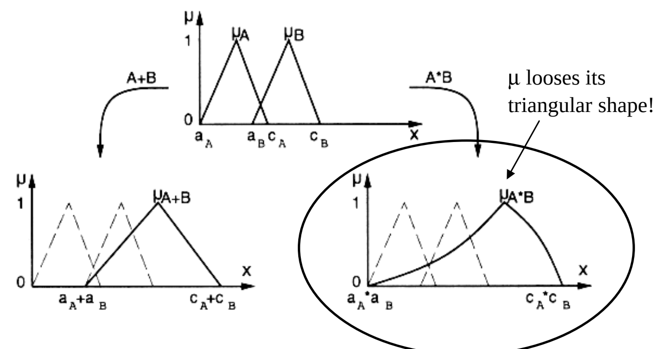
❏ 

… and the rest by linear interpolation:



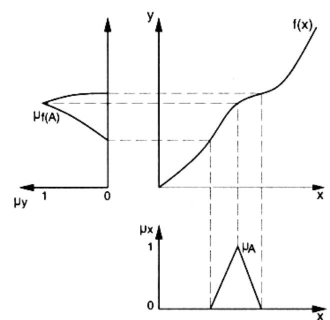
Done!

Fuzzy Addition and Multiplication



$$\mu_{A \otimes B}(x) = \max_{y,z \in \mathbb{R}} \{\min\{\mu_A(y), \mu_B(z)\} \mid y \otimes z = x\}$$

Applying a Function to a Fuzzy Number



$$\mu_{f(A)}(y) = \max\{\mu_A(x) \mid \forall x : f(x) = y\}$$

❏ Examples from the *Mathematica* Fuzzy Logic Package

4. Fuzzy Systems

Intersection, Union, and Complement

- Let A and B be fuzzy sets.
  - Intersection (conjunction):

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

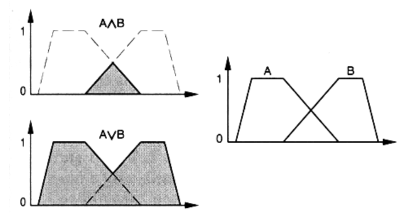
  - Union (disjunction):

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

  - Complement:

$$\mu_{\neg A}(x) = 1 - \mu_A(x)$$

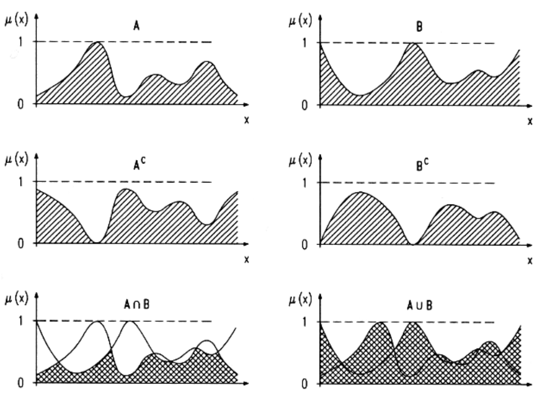
Fuzzy Union and Intersection



$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

Fuzzy Union and Intersection (2)



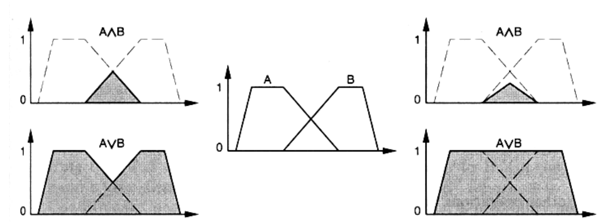
Fuzzy Set Laws (Min-Max Operators)

- Commutativity:
  - $A \cap B = B \cap A$
  - $A \cup B = B \cup A$
- Associativity:
  - $(A \cap B) \cap C = A \cap (B \cap C)$
  - $(A \cup B) \cup C = A \cup (B \cup C)$
- Distributivity:
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Fuzzy Set Laws (Min-Max Operators) (2)

- Adjunctivity:
  - $A \cap (B \cup A) = A$
  - $A \cup (B \cap A) = A$
- De Morgan Laws:
  - $\neg(A \cap B) = \neg A \cup \neg B$
  - $\neg(A \cup B) = \neg A \cap \neg B$
- However, the laws of complementarity do <u>not</u> hold:
  - $A \cap \neg A \neq \varnothing$
  - $A \cup \neg A \neq 1$

Union and Intersection: Alternatives



$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

$$\mu_{A\cap B}(x) = \mu_A(x) \cdot \mu_B(x)$$

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

$$\mu_{A\cup B}(x) = \min\{\mu_A(x) + \mu_B(x), 1\}$$

4. Fuzzy Systems

Fuzzy Inference

- As the complementarity laws do not hold in fuzzy logic, we can not simply use laws from classical logic to derive other operators, such as *implication*.
- Consequently, implication has to be defined rather than derived:

$$\boxed{\mu_{A \to B}(x) = \max \{1 - \mu_A(x), \min \{\mu_A(x), \mu_B(x)\}\}}$$

– (motivated by A → B = ¬A ∨ (A ∧ B))

Generalized Modus Ponens

- In classical logic, conclusions can be drawn from known facts and implications based on these facts.
- In fuzzy logic, this process of inference can be extended also to partial true facts:

x IS *A'*
IF *x* IS *A* THEN *y* IS *B*
y IS *B'*

Generalized Modus Ponens

- In classical logic, conclusions can be drawn from known facts and implications based on these facts.
- In fuzzy logic, this process of inference can be extended also to partial true facts:

x IS *A'*	$\mu_{A'}(x)$
IF *x* IS *A* THEN *y* IS *B*	A → B
y IS *B'*	$\mu_{B'}(y)$

Generalized Modus Ponens (2)

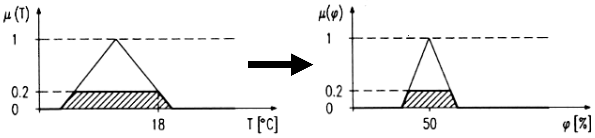
- The interpretation of GMP depends on the definition of implication.
- One possible GMP interpretation: Joint Constraint:
  - Using min-max norms, the implication can be seen as forming a constraint on a joint variable (x, y) ∈ A × B.
  - Cartesian product: $\boxed{\mu_{A \times B}(x, y) = \min \{\mu_A(x), \mu_B(y)\}}$
  - *B'* can then be obtained through *B'* = *A'* ∧ (*A* × *B*), or

$$\boxed{\mu_{B'}(y) = \sup_x \{\min \{\mu_A(x), \mu_{A \times B}(x, y)\}\}}$$

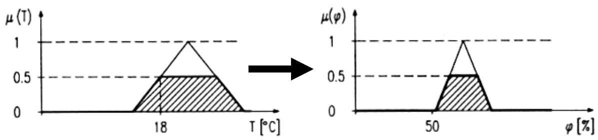
4. Fuzzy Systems

Fuzzy Rules

- IF *temperature* = *low* THEN *cooling valve* = *half open*.

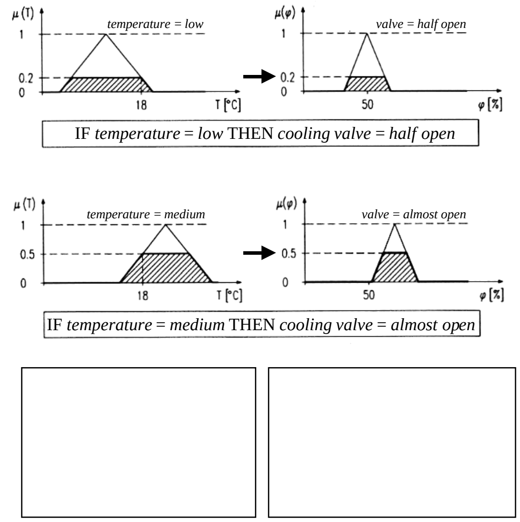


- IF *temperature* = *medium* THEN *cooling valve* = *almost open*.



Max-Min Inference

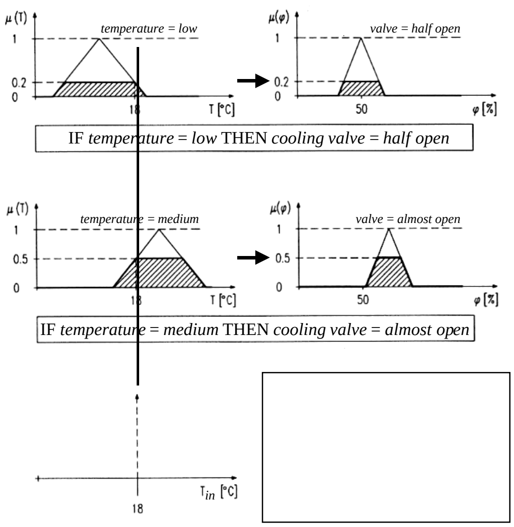
Approximate Reasoning



Max-Min Inference

Approximate Reasoning

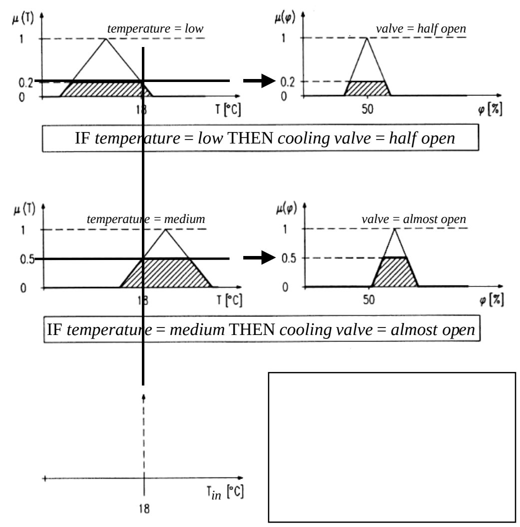
1. Input of crisp value



Max-Min Inference

Approximate Reasoning

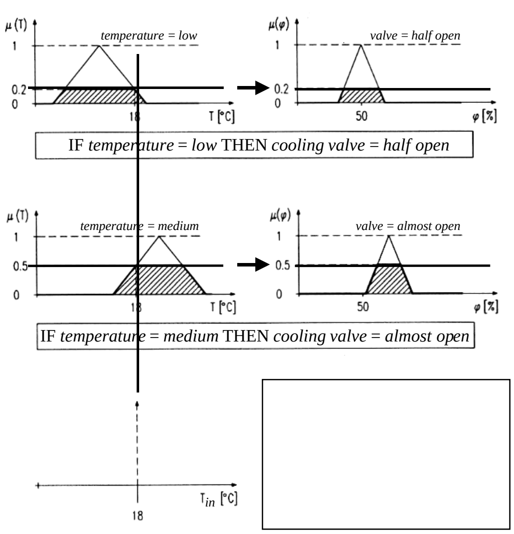
1. Input of crisp value
2. Fuzzification



Max-Min Inference

Approximate Reasoning

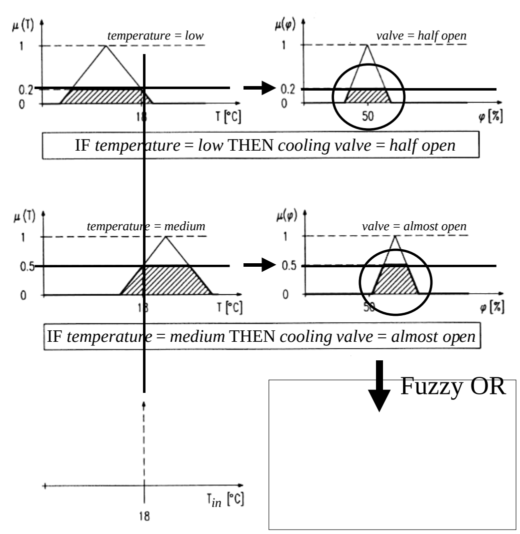
1. Input of crisp value
2. Fuzzification
3. Inference



Max-Min Inference

Approximate Reasoning

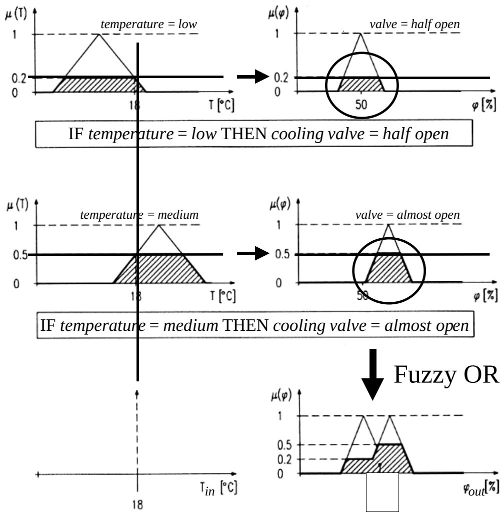
1. Input of crisp value
2. Fuzzification
3. Inference
4. Output set



Max-Min Inference

Approximate Reasoning

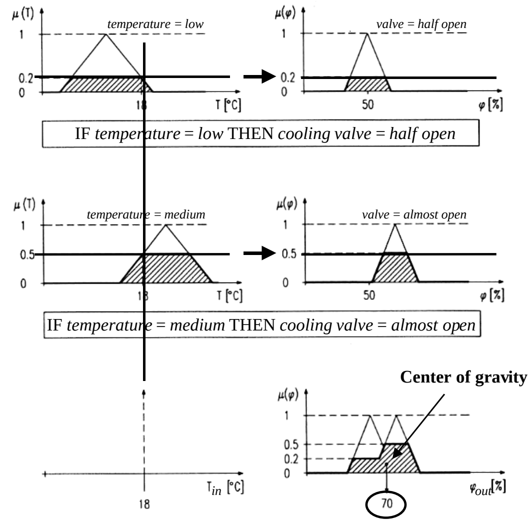
1. Input of crisp value
2. Fuzzification
3. Inference
4. Output set



Max-Min Inference

Approximate Reasoning

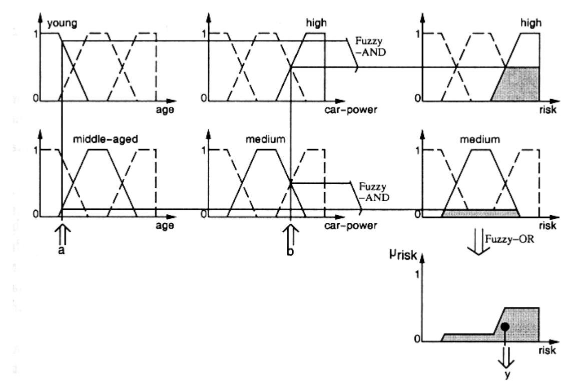
1. Input of crisp value
2. Fuzzification
3. Inference
4. Output set
5. Defuzzification



Another Inference Example

- IF *age* IS *young* AND *car power* IS *high* THEN *risk* IS *high*.
- IF *age* IS *middle aged* AND *car power* IS *medium* THEN *risk* IS *medium*.

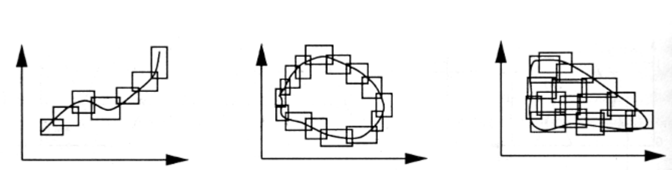
Another Inference Example (2)



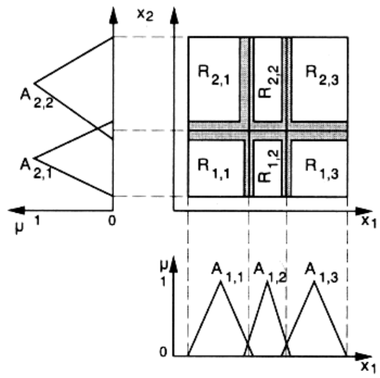
4. Fuzzy Systems

Extracting Fuzzy Models: Graphs

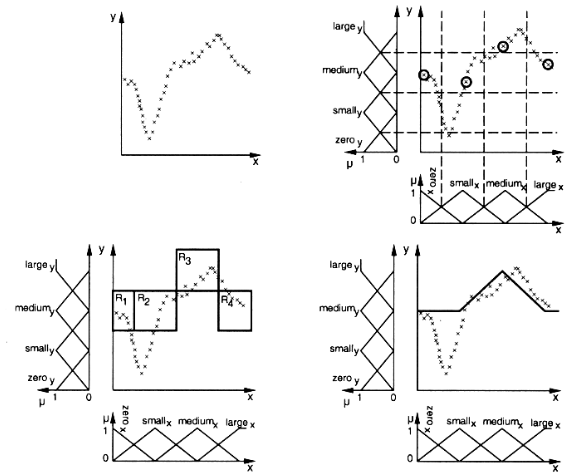
- Approximate representation of functions, contours, and relations:



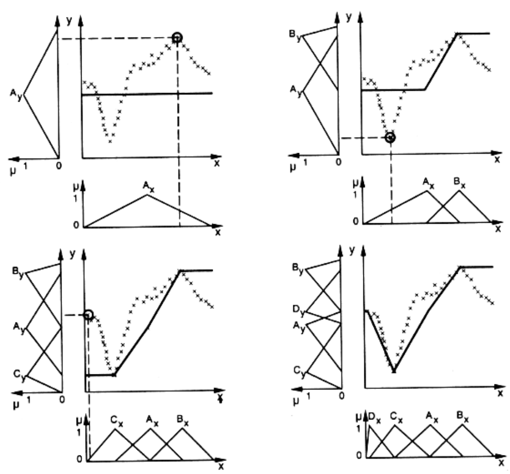
Global Granulation of Input Space



Extracting Fixed-Grid Fuzzy Rules



Building Adaptive Grid-Based Fuzzy Rules



4. Fuzzy Systems

References

- Berthold, M. and Hand, D. J. (1999). *Intelligent Data Analysis*. Berlin, Springer.
- Kasabov, N. (1996). *Foundations of Neural Networks, Fuzzy Systems, and Knowledge Engineering*. Cambridge, MA, MIT Press.
- Bothe, H. (1995). *Fuzzy Logic — Einführung in Theorie und Anwendungen*. Berlin, Springer.