Experimental Results on Class Groups of Real Quadratic Fields

(Extended Abstract)

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In an effort to expand the body of numerical data for real quadratic fields, we have computed the class groups and regulators of all real quadratic fields with discriminant $\Delta < 10^9$. We implemented a variation of the group structure algorithm for general finite Abelian groups described in [2] in the C++ programming language using built-in types together with a few routines from the LiDIA system [12]. This algorithm will be described in more detail in a forth-coming paper. The class groups and regulators of all 303963581 real quadratic fields were computed on 20 workstations (SPARC-classics, SPARC-4's, and SPARC-ultra's) by executing the computation for discriminants in intervals of length 10^5 on single machines and distributing the overall computation using PVM [8]. The entire computation took just under 246 days of CPU time (approximately 3 months real time), an average of 0.07 seconds per field.

In this contribution, we present the results of this experiment, including data supporting the truth of Littlewood's bounds on the function $L(1, \chi_\Delta)$ [13] and Bach's bound on the maximum norm of the prime ideals required to generate the class group [1]. Data supporting several of the Cohen-Lenstra heuristics [6,7] is presented, including results on the percentage of non-cyclic odd parts of class groups, percentages of odd parts of class numbers equal to small odd integers, and percentages of class numbers divisible by small primes p. We also give new examples of irregular class groups, including examples for primes $p \leq 23$ and one example of a rank 3 5-Sylow subgroup (3 non-cyclic factors), the first example of a real quadratic class group which has a p-Sylow subgroup with rank greater than 2 and $p > 3$.

1 The $L(1, \chi_\Delta)$ Function

Much interest has been shown in extreme values of the $L(1, \chi_\Delta)$ function [3,14,10,4]. A result of Littlewood [13] and Shanks [14] shows that under the Extended Riemann Hyptothesis (ERH)

$$\{1 + o(1)\} (c_1 \log\log \Delta)^{-1} < L(1, \chi_\Delta) < \{1 + o(1)\} c_2 \log\log \Delta, \qquad (1)$$

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where the values of the constants c_1 and c_2 depend upon the parity of Δ:

$$\begin{aligned} c_1 = 12e^\gamma/\pi^2 \quad &\text{and} \quad c_2 = 2e^\gamma \quad \text{when } 2 \nmid \Delta \\ c_1 = 8e^\gamma/\pi^2 \quad &\text{and} \quad c_2 = e^\gamma \quad \text{when } 2 \mid \Delta \ . \end{aligned}$$

For a fixed Δ, Shanks [14] defines the *upper* and *lower Littlewood indices* as

$$ULI = L(1, \chi_\Delta) / (c_2 \log\log \Delta) \tag{2}$$
$$LLI = L(1, \chi_\Delta)\, c_1 \log\log \Delta \ . \tag{3}$$

If (1) is true, then as Δ increases, we would expect that extreme values of the ULI and LLI would tend to approach 1. A ULI value greater than 1 or LLI value less than 1 would probably indicate a violation of the ERH [14].

Following [11] and [14], we define the function

$$L_\Delta(1) = \prod_{p \text{ prime}} \left(\frac{p}{p - \left(\frac{4\Delta}{p}\right)} \right) \ . \tag{4}$$

Note that this function is essentially $L(1, \chi_\Delta)$ with the 2-factor divided out, i.e.,

$$L_\Delta(1) = \begin{cases} L(1, \chi_\Delta) & \text{if } \Delta \equiv 0 \pmod 4 \\ (1/2) L(1, \chi_\Delta) & \text{if } \Delta \equiv 1 \pmod 8 \\ (3/2) L(1, \chi_\Delta) & \text{if } \Delta \equiv 5 \pmod 8 \end{cases} \ .$$

Since the 2-factor is determined by the congruence class of Δ modulo 8, dividing it out allows us to compare the quadratic residuosity of all discriminants regardless of their congruence modulo 8. In [14], Shanks derives bounds for $L_\Delta(1)$ analogous to (1) (also under ERH)

$$\{1 + o(1)\} \left(\frac{8}{\pi^2} \log\log 4\Delta \right)^{-1} < L_\Delta(1) < \{1 + o(1)\}\, e^\gamma \log\log 4\Delta, \tag{5}$$

and the corresponding indices

$$ULI_\Delta = L_\Delta(1) / (e^\gamma \log\log 4\Delta) \tag{6}$$
$$LLI_\Delta = L_\Delta(1) \tfrac{8}{\pi^2} \log\log 4\Delta \ . \tag{7}$$

If (5) is true, then as Δ increases, we would also expect the extreme values of the ULI_Δ and LLI_Δ to approach 1.

We have recorded the successive $L(1, \chi_\Delta)$ maxima and minima for even Δ, $\Delta \equiv 1 \pmod 8$, and $\Delta \equiv 5 \pmod 8$ where $\Delta < 10^9$, together with ULI values and $L_\Delta(1)$ and ULI_Δ values where appropriate. The maximum $L(1, \chi_\Delta)$ value found was $7.07046680\ldots$ ($ULI = 0.65623747\ldots$) for $\Delta = 872479969$ and the maximum $L_\Delta(1)$ value was $3.74995980\ldots$ ($ULI_\Delta = 0.68501570\ldots$) for $\Delta = 612380869$. The minimum $L(1, \chi_\Delta)$ value found was $0.18948336\ldots$

($LLI = 1.12478715\ldots$) for $\Delta = 5417453$ and the minimum $L_\Delta(1)$ value was $0.27822361\ldots$ ($LLI_\Delta = 1.20515814$) for $\Delta = 133171673$. We found no surprises here — the $L(1, \chi_\Delta)$, ULI, and LLI values seem to behave similarly to those of imaginary quadratic fields [3] and correspond to previous observations [10]. At first glance, it may appear that the LLI value for $\Delta = 1592$ indicates a violation of the ERH. However, as discussed by Shanks in [14], the apparent violation can almost certainly be accounted for by the $o(1)$ term in (1), since this discriminant is so small.

In [4], Buell also looked at the mean values of $L(1, \chi_\Delta)$ for imaginary quadratic fields of both even and odd discriminant. His computations suggest that the mean value of $L(1, \chi_\Delta)$ is approximately 1.186390 for even discriminants and 1.581853 for odd discriminants. Our computations show that the same mean values probably hold for real quadratic fields. We have computed a mean value of 1.18639 for the even discriminants less than 10^9, and our computed value of 1.58154 for odd discriminants less than 10^9 is close to Buell's value. We suppose that the difference can be accounted for by the fact that Buell has considered over twice as many fields as we have ($|\Delta| < 2.2 \times 10^9$). Indeed, it seems that at $\Delta \approx 10^9$ the mean value of $L(1, \chi_\Delta)$ is still slowly approaching Buell's value in our case.

2 Odd Parts of Class Numbers

Let Cl^*_Δ be the odd part of the class group of $\mathbb{Q}(\sqrt{\Delta})$. Cohen and Lenstra [6,7] provide some heuristics on the distribution of various Cl^*_Δ. For example, if we define

$$w(n) = \prod_{p^\alpha \,\|\, n} \frac{1}{p^\alpha(1-p^{-1})(1-p^{-2})\ldots(1-p^{-\alpha})}, \tag{8}$$

$$\eta_\infty(p) = \prod_{i=1}^{\infty}(1-p^{-i}) \quad (\eta_\infty(2) = 0.288788095\ldots), \tag{9}$$

$$C_\infty = \prod_{j=1}^{\infty} \zeta(j+1) = 2.294856589\ldots, \tag{10}$$

$$C = \frac{1}{2\eta_\infty(2)C_\infty} = 0.754458173\ldots, \tag{11}$$

the probability that $h^*_\Delta = |Cl^*_\Delta|$ is equal to k is

$$\mathrm{Prob}(h^*_\Delta = k) = \frac{Cw(k)}{k} \ . \tag{12}$$

This gives us $\mathrm{Prob}(h^*_\Delta = 1) = 0.754458173\ldots$, $\mathrm{Prob}(h^*_\Delta = 3) = 0.125743028\ldots$, and $\mathrm{Prob}(h^*_\Delta = 5) = 0.037722908\ldots$ for the first few small values of k. Using this heuristic assumption, Lukes, Williams, and the author were also able to derive [10]

$$\mathrm{Prob}(h^*_\Delta > x) = \frac{1}{2x} + O\left(\frac{\log x}{x^2}\right), \tag{13}$$

a generalization of a conjecture of Hooley for prime discriminants [9] and

$$k+1 = \frac{1}{2}\left(\frac{1}{1-\mathrm{Prob}(h^*_\Delta \le k)}\right) + O\left(\frac{\log k}{k^2}\right), \tag{14}$$

which can be used to test the validity of (13).

We have used our computation of all class groups of fields $\mathbb{Q}(\sqrt{\Delta})$ where $\Delta < 10^9$ to extend the numerical evidence supporting (12) and (13) presented in [10]. Define $q_i(x)$ to be the observed ratio of odd discriminants less than x with $h^*_\Delta = i$ divided by the conjectured asymptotic probability given by (12). Similarly, we define $s_i(x)$ to be the observed ratio of odd discriminants less than x with $h^*_\Delta \le i$ and

$$t_i(x) = \frac{1}{2}\left(\frac{1}{1-s_i(x)}\right) .$$

Tables 1 and 2 contain values of $q_i(x)$ and $t_i(x)$ for various i and x for $\Delta \equiv 1 \pmod 4$, $\Delta < 10^9$. If (12) is correct, we would expect the values in Tab. 1 ($q_i(x)$ values) to approach 1 for each value of i as x increases. Similarly, if (13) is correct, by (14) we would expect the values in Tab. 2 ($t_i(x)$ values) to approach $i+1$ for each value of i as x increases. As observed in [10], this does appear to happen in both cases. Our extended computation also supports this, although the convergence is still rather slow. The corresponding tables for even discriminants are so similar that in the interest of brevity we do not include them here.

3 Divisibility of h_Δ by Odd Primes

Another heuristic presented in [6,7] is the probability that h_Δ is divisible by an odd prime p is given by

$$\mathrm{Prob}(p \mid h_\Delta) = 1 - \frac{\eta_\infty(p)}{1-p^{-1}}, \tag{15}$$

where $\eta_\infty(p)$ is defined in (9). For example, $\mathrm{Prob}(3 \mid h_\Delta) = 0.159810883\ldots$, $\mathrm{Prob}(5 \mid h_\Delta) = 0.049584005\ldots$, and $\mathrm{Prob}(7 \mid h_\Delta) = 0.023738691\ldots$ for the first few small odd primes.

Define $p_p(x)$ to be the observed ratio of odd discriminants less than x with $p \mid h_\Delta$ divided by the conjectured asymptotic probability given by (15). As x increases, we expect $p_p(x)$ to approach 1 for a specific odd prime p. In Tab. 3 we provide values of $p_p(x)$ for various p and x for $\Delta \equiv 1 \pmod 4$, $\Delta < 10^9$. Unlike the case in imaginary fields [4], the values of $p_p(x)$ seem to approach 1 fairly smoothly from below. The corresponding table for $\Delta \equiv 0 \pmod 4$ is very similar and hence not included here.

4 Non-cyclic p-Sylow Subgroups

As above, let Cl^*_Δ be the odd part of Cl_Δ. Then, under the heuristic assumptions in [6,7] one can easily derive the probability that Cl^*_Δ is cyclic, namely

$$\mathrm{Prob}(Cl^*_\Delta \text{ cyclic}) = C \prod_{p \text{ odd prime}} \frac{p^3 - p^2 + 1}{(p-1)(p^2-1)} = 0.997630528\ldots \qquad (16)$$

where C is given by (11). Define $c(x)$ to be the observed ratio of odd (or even) discriminants less than x with Cl^*_Δ cyclic divided by the conjectured asymptotic probability given by (16). This function should approach 1 as x increases if (16) is true.

Table 4 provides values of $c(x)$ for various values of x and both even and odd Δ. The total number of fields with discriminant less than x and the number of non-cyclic Cl^*_Δ are also listed for even and odd Δ. As expected, the values of $c(x)$ appear to approach 1 in both cases.

For an odd prime p, define the p-rank of Cl_Δ to be the number of non-cyclic factors of the p-Sylow subgroup of Cl_Δ. Yet another heuristic of Cohen and Lenstra [6,7] states that the probability that the p-rank of Cl_Δ is equal to r is given by

$$\mathrm{Prob}(p\text{-rank of } Cl_\Delta = r) = \frac{\eta_\infty(p)}{p^{r(r+1)}(1 - p^{-(r+1)}) \prod_{1 \le k \le r}(1 - p^{-k})^2} \ . \qquad (17)$$

For example, Prob(3-rank of $Cl_\Delta = 2) = 0.002272146\ldots$, Prob(3-rank of $Cl_\Delta = 3) = 0.000003277\ldots$, and Prob(5-rank of $Cl_\Delta = 2) = 0.000083166\ldots$. Define $pr_{p,r}(x)$ to be the observed ratio of odd discriminants less than x with p-rank $= r$ divided by the conjectured asymptotic probability given by (17). As x increases, we expect $pr_{p,r}(x)$ to approach 1 for a specific odd prime p and p-rank r if (17) is true.

In Tab. 5 we provide values of $pr_{p,r}(x)$ for various values of p, r, and x for $\Delta \equiv 1 \pmod 4$, $\Delta < 10^9$. These values do seem to approach 1, but due to the scarcity of examples the convergence is extremely slow, especially for $pr_{3,3}(x)$. The corresponding table for $\Delta \equiv 0 \pmod 4$ is very similar and hence not included here.

5 First Occurrences of Non-cyclic p-Sylow Subgroups

Following Buell [4], we list the total number and first occurrences of discriminants for which the p-Sylow subgroup is non-cyclic for various primes p. For the prime 2, we consider only the principal genus (the subgroup of squares) instead of the whole class group, since much of the information on the 2-Sylow subgroup of Cl_Δ is easily obtainable from the factorization of Δ.

In Tab. 6 and 7 we present those discriminants for which the p-Sylow subgroup has rank 2, and in particular has the structure $C(p^{e_1}) \times C(p^{e_2})$. Table 6 contains the data corresponding to the principal genus, and Tab. 7 contains data for odd primes p. In both tables, we list the smallest discriminant and the total number of discriminants $\Delta < 10^9$ whose class groups contain the specified p-Sylow subgroup, odd and even discriminants being tabulated separately. We have found class groups with non-cyclic p-Sylow subgroups for primes $p \leq 23$. There are obviously not as many examples as in the case of imaginary fields [4], as one would expect from the heuristic p-rank probabilities derived by Cohen and Lenstra [6,7].

In Tab. 8 and 9 we present the corresponding data for class groups with p-Sylow subgroups of rank 3, i.e., having structure $C(p^{e_1}) \times C(p^{e_2}) \times C(p^{e_3})$. Once again, we consider the 2-Sylow subgroup of the principal genus, not Cl_Δ. Again, we have significantly fewer examples as in the case of imaginary fields [4], and no examples with rank greater than 3. However, the discriminant 999790597 has class group isomorphic to $C(5) \times C(5) \times C(40)$ and is believed to be the only discriminant known with p-rank greater than 2 for an odd prime $p > 3$.

The smallest discriminants and total number of discriminants $\Delta < 10^9$ whose class groups contain 2 non-cyclic p-Sylow subgroups are presented in Tab. 10. When one examines the probability that the p-Sylow subgroup is non-cyclic presented in the last section, it is easy to see why so few examples of fields with doubly non-cyclic class groups were found.

6 The Number of Generators Required

In [1], Bach gives a theorem which states that under the ERH, the prime ideals of norm $6 \log^2 \Delta$ are sufficient to generate the class group. In practice, it has been observed that this bound does not seem to be tight, i.e., fewer generators are sufficient [5]. During the course of our computation, we have kept track of the maximum norm of the prime ideals required to generate the class group of each discriminant $\Delta < 10^9$. Of all 303963581 fields considered, the field $\mathbb{Q}(\sqrt{519895977})$ required the prime ideal with largest norm to construct a full generating system, namely 197.

For a specific Δ, define $\max_p(\Delta)$ to be the largest norm of the prime ideals required to generate the class group of $\mathbb{Q}(\sqrt{\Delta})$. If Bach's theorem is true, we would expect that $\max_p(\Delta)/\log^2 \Delta$ should always be less than 6. For $\Delta < 10^9$, this is in fact the case, and indeed if we exclude the very smallest discriminants (like $\Delta = 5$), the maximum value obtained for this ratio is $0.55885\ldots$ for $\Delta =$ 519895977. As one would expect due to the high probability of cyclic odd parts of class groups, the average value of this ratio is significantly less than 6 — for $\Delta < 10^9$ we have obtained a value of $0.01984\ldots$.

It has been conjectured [5] that a tighter bound of the form $c \log^{1+\epsilon} \Delta$ for any $\epsilon > 0$ may hold in this case. Hence, in order to get an idea of the order of magnitude of the constant c, we also considered the ratio $\max_p(\Delta)/\log \Delta$. For $\Delta < 10^9$, the largest value we obtained was $9.81607\ldots$ for the discriminant 519895977 and the average value was $0.38982\ldots$.

A Appendix

Table 1. Values of $q_i(x)$ for $\Delta \equiv 1 \pmod 4$.

x	$q_1(x)$	$q_3(x)$	$q_5(x)$	$q_7(x)$	$q_9(x)$	$q_{11}(x)$	$q_{27}(x)$
1000000	1.06119	0.85263	0.95644	0.94918	0.70424	0.90228	0.47347
10000000	1.03676	0.89604	0.99125	0.99564	0.83023	0.97519	0.69086
20000000	1.03178	0.90683	0.99465	1.00142	0.84625	0.98812	0.74718
30000000	1.02923	0.91246	0.99592	1.00250	0.85705	0.99247	0.76587
40000000	1.02752	0.91613	0.99663	1.00194	0.86264	0.99791	0.78753
50000000	1.02634	0.91893	0.99664	1.00315	0.86638	0.99846	0.79660
60000000	1.02541	0.92078	0.99588	1.00446	0.87092	0.99982	0.80705
70000000	1.02461	0.92235	0.99632	1.00504	0.87567	1.00148	0.81494
80000000	1.02389	0.92374	0.99637	1.00623	0.87874	1.00372	0.82014
90000000	1.02333	0.92480	0.99702	1.00608	0.88182	1.00418	0.82863
100000000	1.02284	0.92605	0.99695	1.00581	0.88409	1.00528	0.83205
200000000	1.01994	0.93304	0.99698	1.00554	0.89658	1.00676	0.86198
300000000	1.01839	0.93699	0.99776	1.00567	0.90286	1.00637	0.87221
400000000	1.01739	0.93972	0.99796	1.00537	0.90680	1.00635	0.87994
500000000	1.01662	0.94173	0.99830	1.00476	0.91021	1.00679	0.88370
600000000	1.01604	0.94313	0.99839	1.00498	0.91269	1.00725	0.88921
700000000	1.01558	0.94444	0.99867	1.00457	0.91438	1.00665	0.89328
800000000	1.01515	0.94556	0.99887	1.00473	0.91642	1.00663	0.89664
900000000	1.01480	0.94654	0.99903	1.00491	0.91783	1.00686	0.89807
1000000000	1.01449	0.94739	0.99925	1.00484	0.91907	1.00690	0.90041

Table 2. Values of $t_i(x)$ for $\Delta \equiv 1 \pmod 4$.

x	$t_1(x)$	$t_3(x)$	$t_5(x)$	$t_7(x)$	$t_9(x)$	$t_{11}(x)$	$t_{27}(x)$
1000000	2.50786	5.42530	8.91565	12.81041	17.88166	22.96408	109.65097
10000000	2.29561	4.75574	7.38079	10.02841	13.58368	16.60010	55.01249
20000000	2.25667	4.64952	7.14116	9.61024	12.91103	15.64977	48.43620
30000000	2.23723	4.59746	7.02378	9.40226	12.59204	15.19731	45.43814
40000000	2.22443	4.56286	6.94593	9.26159	12.36781	14.88841	43.53718
50000000	2.21560	4.54032	6.89384	9.17287	12.22767	14.68742	42.23651
60000000	2.20874	4.52115	6.84708	9.09414	12.10904	14.52054	41.36938
70000000	2.20287	4.50462	6.81076	9.03188	12.02043	14.39801	40.61104
80000000	2.19765	4.48987	6.77728	8.97656	11.93639	14.28389	39.94645
90000000	2.19354	4.47810	6.75275	8.93314	11.87337	14.19501	39.48465
100000000	2.18998	4.46955	6.73308	8.89798	11.82131	14.12367	39.02412
200000000	2.16921	4.41793	6.61671	8.69513	11.51779	13.69636	36.62777
300000000	2.15828	4.39186	6.56095	8.59942	11.37594	13.49526	35.52398
400000000	2.15123	4.37589	6.52601	8.53874	11.28573	13.36847	34.83250
500000000	2.14593	4.36363	6.49985	8.49242	11.21844	13.27520	34.32766
600000000	2.14189	4.35361	6.47794	8.45561	11.16400	13.20015	33.93718
700000000	2.13868	4.34659	6.46330	8.42964	11.12534	13.14471	33.63894
800000000	2.13575	4.33986	6.44904	8.40581	11.09178	13.09783	33.39694
900000000	2.13331	4.33435	6.43741	8.38650	11.06361	13.05911	33.19010
1000000000	2.13123	4.32982	6.42809	8.37053	11.04056	13.02709	33.01701

Table 3. Values of $p_p(x)$ for $\Delta \equiv 1 \pmod 4$.

x	$p_3(x)$	$p_5(x)$	$p_7(x)$	$p_{11}(x)$	$p_{13}(x)$	$p_{17}(x)$	$p_{19}(x)$
1000000	0.79263	0.85146	0.81554	0.75676	0.78022	0.64981	0.64482
10000000	0.86203	0.92211	0.90990	0.87157	0.88008	0.83734	0.82371
20000000	0.87781	0.93583	0.92884	0.89645	0.90250	0.86862	0.85824
30000000	0.88602	0.94186	0.93593	0.90931	0.91832	0.88125	0.87796
40000000	0.89166	0.94644	0.93995	0.91964	0.92549	0.89038	0.88702
50000000	0.89565	0.94941	0.94450	0.92469	0.92899	0.89572	0.89286
60000000	0.89875	0.95110	0.94890	0.92865	0.93287	0.90293	0.90142
70000000	0.90138	0.95319	0.95187	0.93207	0.93397	0.90733	0.90522
80000000	0.90359	0.95474	0.95493	0.93650	0.93619	0.91241	0.91046
90000000	0.90548	0.95692	0.95641	0.93841	0.93792	0.91248	0.91254
100000000	0.90723	0.95769	0.95775	0.94103	0.93852	0.91572	0.91663
200000000	0.91756	0.96437	0.96503	0.95327	0.95101	0.93335	0.93290
300000000	0.92311	0.96849	0.96937	0.95884	0.95706	0.94071	0.94133
400000000	0.92681	0.97071	0.97204	0.96205	0.95890	0.94696	0.94566
500000000	0.92958	0.97254	0.97360	0.96488	0.96211	0.95253	0.94938
600000000	0.93165	0.97381	0.97533	0.96747	0.96460	0.95654	0.95244
700000000	0.93340	0.97503	0.97625	0.96881	0.96618	0.95852	0.95441
800000000	0.93494	0.97597	0.97733	0.97022	0.96828	0.96099	0.95553
900000000	0.93627	0.97676	0.97837	0.97156	0.96979	0.96284	0.95672
1000000000	0.93736	0.97748	0.97896	0.97250	0.97123	0.96415	0.95859

Table 4. Number of non-cyclic odd parts of class groups.

x	$\Delta \equiv 0 \pmod 4$ total	non-cyclic	$c(x)$	$\Delta \equiv 1 \pmod 4$ total	non-cyclic	$c(x)$
1000000	101322	50	1.00188	202635	114	1.00181
10000000	1013213	919	1.00147	2026440	2088	1.00134
20000000	2026421	2129	1.00132	4052851	4627	1.00123
30000000	3039631	3385	1.00126	6079260	7365	1.00116
40000000	4052850	4733	1.00120	8105666	10137	1.00112
50000000	5066064	6108	1.00117	10132117	13008	1.00109
60000000	6079270	7595	1.00112	12158544	15999	1.00106
70000000	7092461	9048	1.00110	14184949	19007	1.00103
80000000	8105723	10519	1.00107	16211387	22000	1.00101
90000000	9118933	12028	1.00105	18237802	25091	1.00100
100000000	10132112	13508	1.00104	20264212	28150	1.00098
200000000	20264226	28941	1.00094	40528477	60347	1.00088
300000000	30396405	44996	1.00089	60792687	93517	1.00083
400000000	40528481	61286	1.00086	81056963	127467	1.00080
500000000	50660585	78144	1.00083	101321188	161867	1.00077
600000000	60792730	94989	1.00081	121585380	197074	1.00075
700000000	70924833	112001	1.00079	141849691	232554	1.00073
800000000	81056948	129369	1.00078	162113906	267801	1.00072
900000000	91189082	146508	1.00076	182378148	303469	1.00071
1000000000	101321191	164246	1.00075	202642390	339554	1.00070

Table 5. Values of $pr_{p,r}(x)$ for $\Delta \equiv 1 \pmod 4$.

x	$pr_{3,2}(x)$	$pr_{3,3}(x)$	$pr_{5,2}(x)$	$pr_{7,2}(x)$	$pr_{11,2}(x)$	$pr_{13,2}(x)$
1000000	0.24109	0.00000	0.17802	0.00000	0.00000	0.00000
10000000	0.43263	0.00000	0.49842	0.58526	0.00000	0.00000
20000000	0.47803	0.00000	0.61116	0.60965	0.00000	0.00000
30000000	0.50844	0.00000	0.62303	0.61778	0.26275	0.00000
40000000	0.52500	0.03765	0.64825	0.59745	0.59119	0.54645
50000000	0.53875	0.06023	0.67525	0.55600	0.63060	0.43716
60000000	0.55223	0.05019	0.69423	0.53649	0.52550	0.36430
70000000	0.56236	0.06454	0.70186	0.58526	0.45043	0.31225
80000000	0.56963	0.05647	0.70610	0.61574	0.49265	0.54645
90000000	0.57777	0.08366	0.70742	0.64487	0.43791	0.48573
100000000	0.58360	0.09035	0.71026	0.63891	0.55177	0.65574
200000000	0.62506	0.13552	0.78116	0.70231	0.63060	0.54645
300000000	0.64581	0.15058	0.80539	0.75434	0.68315	0.58288
400000000	0.66070	0.14682	0.81291	0.76815	0.68971	0.54645
500000000	0.67127	0.18973	0.82311	0.79498	0.69366	0.56830
600000000	0.68148	0.19325	0.83002	0.78604	0.70942	0.51002
700000000	0.68942	0.21082	0.83817	0.79985	0.74320	0.43716
800000000	0.69479	0.20893	0.84376	0.80290	0.73898	0.40983
900000000	0.70012	0.20747	0.84409	0.81069	0.73570	0.46144
1000000000	0.70513	0.20780	0.84898	0.81936	0.74883	0.56830

Table 6. Non-cyclic rank 2 2-Sylow subgroups.

e_1	e_2	first odd Δ	# odd Δ	first even Δ	# even Δ
1	1	26245	625278	12104	437912
2	1	134249	233132	69064	164617
2	2	1717505	8914	1781004	6132
3	1	563545	57267	796552	39791
3	2	2044369	3267	5324556	2138
3	3	22325605	111	34560024	82
4	1	1397321	13789	1542748	9535
4	2	8443681	742	19369756	496
4	3	48365305	34	103252696	18
4	4	*	*	683376268	1
5	1	7182401	3053	10562504	2091
5	2	82670065	138	107723544	83
5	3	327805705	3	522315292	2
6	1	18727689	603	31610632	353
6	2	256055305	13	592435596	9
6	3	938900353	1	887803144	1
7	1	64209289	73	187432072	35
7	2	351270505	4	*	*
8	1	216442945	6	325080904	2
9	1	438986305	1	*	*

Table 7. Non-cyclic rank 2 p-Sylow subgroups.

p	e_1	e_2	first odd Δ	# odd Δ	first even Δ	# even Δ
3	1	1	32009	279754	94636	135945
3	2	1	255973	39982	626264	19100
3	2	2	8739521	313	25725176	147
3	3	1	2178049	4184	1559644	1771
3	3	2	49831633	33	82435336	15
3	3	3	395659153	1	*	*
3	4	1	4822921	381	51236956	115
3	4	2	*	*	793667548	1
3	5	1	125609177	13	412252408	2
3	6	1	604420177	2	*	*
5	1	1	244641	13691	1277996	6929
5	2	1	3874801	605	52929592	220
5	3	1	225225057	12	569204156	2
7	1	1	1633285	1652	3626536	799
7	2	1	30883361	28	96847468	8
11	1	1	26967253	95	81903208	54
13	1	1	39186673	25	41912572	14
13	2	1	900384041	1	*	*
17	1	1	810413473	1	361880744	3
19	1	1	65028097	4	*	*
23	1	1	763945277	1	*	*

Table 8. Non-cyclic rank 3 2-Sylow subgroups.

e_1	e_2	e_3	first odd Δ	# odd Δ	first even Δ	# even Δ
1	1	1	5764805	1409	12490568	879
2	1	1	17737705	620	38922248	396
2	2	1	110255245	32	270453068	22
3	1	1	100282145	133	87572168	80
3	2	1	230818741	8	155979976	7
4	1	1	154877545	27	37970248	12
4	2	1	689289745	1	387642264	2
5	1	1	499871221	7	216461884	4
6	1	1	*	*	708776776	1

Table 9. Non-cyclic rank 3 p-Sylow subgroups.

p	e_1	e_2	e_3	first odd Δ	# odd Δ	first even Δ	# even Δ
3	1	1	1	39345017	122	66567068	44
3	2	1	1	88215377	15	157753592	10
3	3	1	1	545184113	1	*	*
5	1	1	1	999790597	1	*	*

Table 10. Doubly non-cyclic p-Sylow subgroups.

p_1	p_1	first odd Δ	# odd Δ	first even Δ	# even Δ
2	3	10876805	1299	9622408	908
2	5	66376409	43	200600008	20
2	7	230181505	3	630353080	1
3	5	57586597	15	492371864	4
3	7	204242449	3	*	*

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