Using Lipschitz Constants

For Rendering Implicit Surfaces

By

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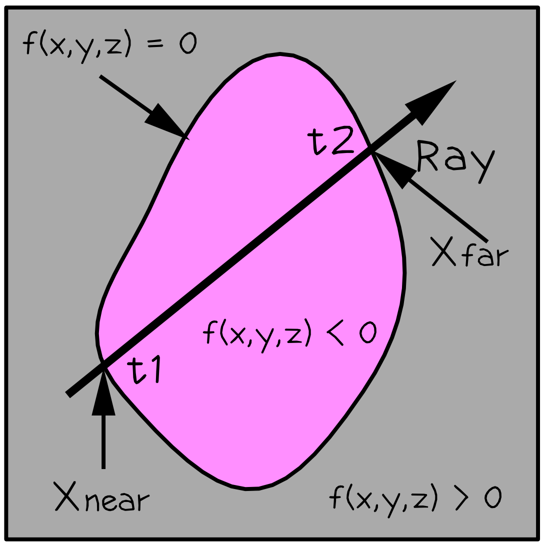
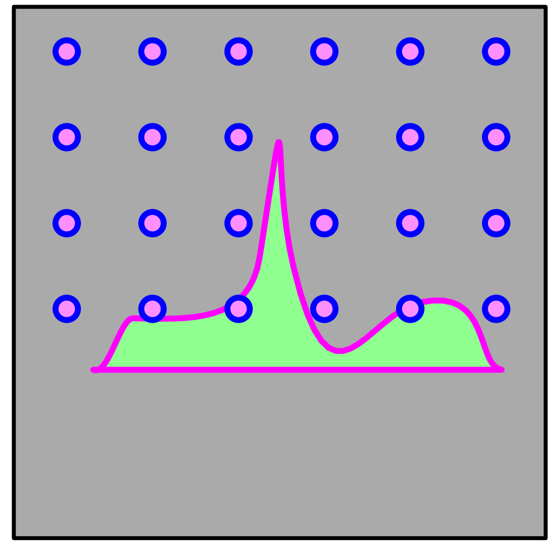
Lipschitz Constants and Ray Tracing Implicit Surfaces

Reference: Siggraph 89. Kalra and Barr, pp 297-306
Guaranteed Ray Intersections with Implicit Surfaces





"It is impossible to create an algorithm solely based on evaluation of the implicit function which is guaranteed to correctly intersect a ray with an implicit surface."

<u>Sampling Algorithms</u>

It is possible to miss spikes on the surface falling between sampled points.
Note problem in animation.

LG-Surfaces

1. Guarantees that smallest features of the surface are sampled.

2. Obtains the nearest intersection of a ray from the origin of the ray with the implicit surface S represented by f(x).

Implicit Surface:

$f(x,y,z) = 0$ or in vector notation

$f(\mathbf{x}) = 0$

Ray Definition:

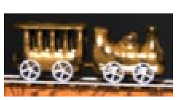
$$\mathbf{x} = \alpha t + \beta, \quad t >= 0$$

substitute into $f(\mathbf{x}) = 0$

gives: $F(t) = f(\alpha t + \beta)$

L – limits the net rate of change of f(x,y,z)

G – limits the net rate of change of the gradient



Directional Derivative

$F(t) = f(\alpha t + \beta)$

Define $g(t) = \frac{dF}{dt}$

$= \alpha \, . \nabla f(\mathbf{x})|_{\alpha t + \beta}$

α is the direction of the ray origin β

$g(t)$ is the directional derivative of $f(\mathbf{x})$ along the ray direction α thus 'g' for gradient.

Lipschitz Constant

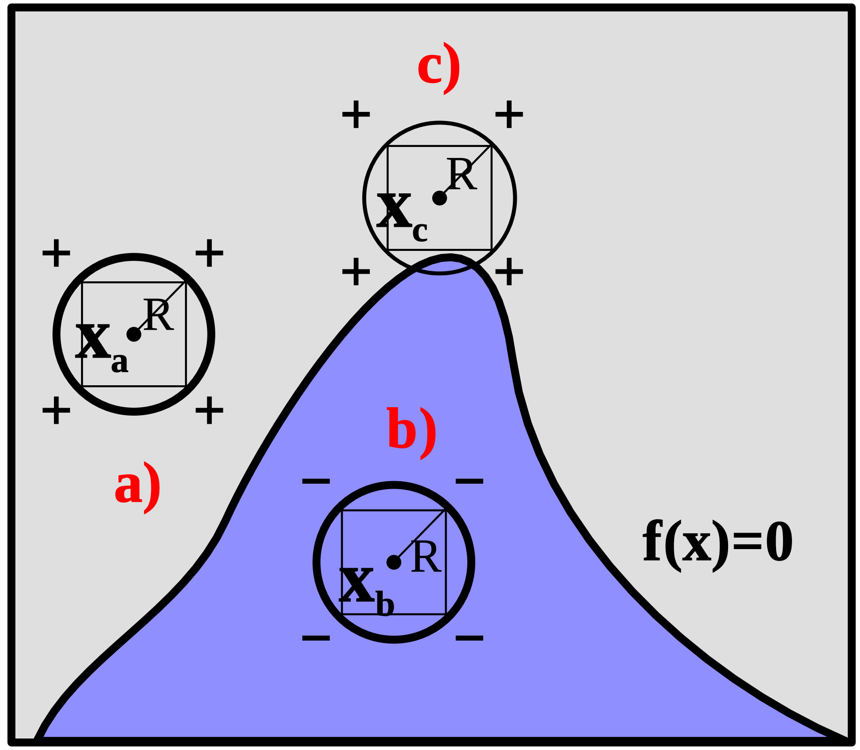
A (+)ve real number L is called a Lipschitz constant on a function f($\mathbf{x}$) in a region R, if given any two points $\mathbf{x_1}$ and $\mathbf{x_2}$ in R, the following condition holds:

$$\|f(\mathbf{x_1}) - f(\mathbf{x_2})\| < L \; \|\mathbf{x_1} - \mathbf{x_2}\|$$

where || vector norm||
f($\mathbf{x}$) could be an n dimensional function
n=1 in this case.

Given a point X_0, $r = f(\mathbf{x_0}) / L$ is the radius of the sphere, S around X_0 such that $f(\mathbf{x})$ does not change sign in S.

L is a measure of the maximum rate of change of a function in a region over which the function is defined.



Spheres a) and b) are guaranteed not to intersect the surface since

Lipschitz radii $r_a = f(x_a) / L_a < R$ and $r_b = f(x_b) / L_b < R$

However $r_c > R$ and S_c may intersect.

LG Surfaces (continued)

An LG surface is defined to be an implicit function f(x,y,z) which has bounds on the net rate of change of the function and its directional derivatives (these are called L and G they are Lipschitz constants).

Definition: Let L be the Lipschitz constant for the function $f(\mathbf{x})$ in a 3D region R and G be the Lipschitz constant for the corresponding function $g(t)$ in a closed interval $T = [t_1, t_2]$

$$||f(\mathbf{x_1})-f(\mathbf{x_2})|| < L\ ||\mathbf{x_1}-\mathbf{x_2}||$$

$$||g(\mathbf{t_a})-g(\mathbf{t_b})|| < G\ ||\mathbf{t_a}-\mathbf{t_b}||$$

For any $\mathbf{x_1}$, $\mathbf{x_2} \in R$ and any $\mathbf{t_a}, \mathbf{t_b} \in T$.

L and G must exist and be computable for an implicit surface to be an LG surface.

How To Compute L and G

A Lipschitz constant is a measure of the maximum rate of change of a function in a region over which the function is defined.

$$\|f(\mathbf{x_1}) - f(\mathbf{x_2})\| < L \, \|\mathbf{x_1} - \mathbf{x_2}\|$$

divide through by : $\|\mathbf{x_1} - \mathbf{x_2}\|$ gives $\dfrac{\|f(\mathbf{x_1}) - f(\mathbf{x_2})\|}{\|\mathbf{x_1} - \mathbf{x_2}\|} \rightarrow \dfrac{df}{dx} < L$

limit $\mathbf{x_1} \rightarrow \mathbf{x_2}$

$$L >= \max_{R} |\nabla f(\mathbf{x})|$$

L is greater than or equal to max rate of change of $f(\mathbf{x})$ in 3D region R.

$$G >= \max_{T} \left| \frac{dg}{dt} \right|$$

G is greater than or equal to max rate of change of $g(t)$ in the 1D interval $T=[t_1, t_2]$.

Ray LG Surface Intersection

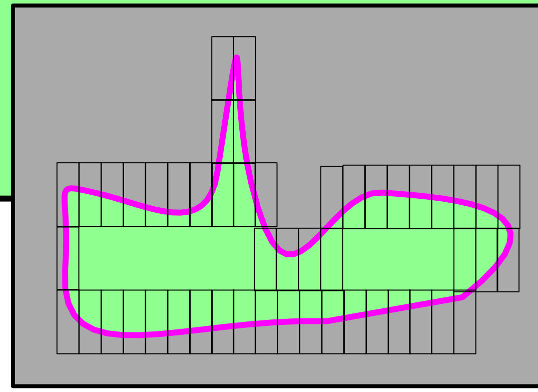
A) Space Pruning (For efficiency only)

Take away parts of space not containing the surface.

Leave volume V that could contain the surface.

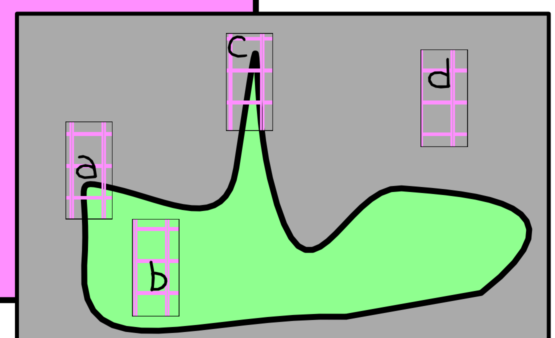
B) Intersection Test

Search V for intersections.



A) We obtain a volume V of non-overlapping boxes which straddle the surface.

At least one vertex inside and one outside.

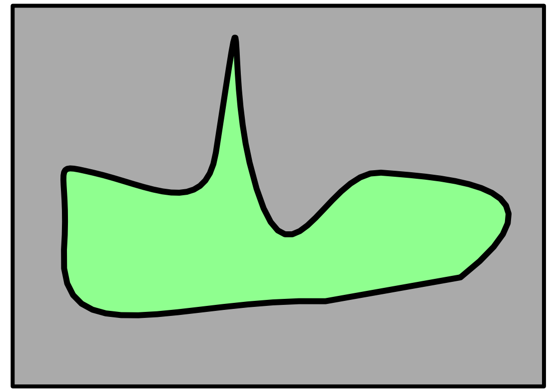


Only a is straddling

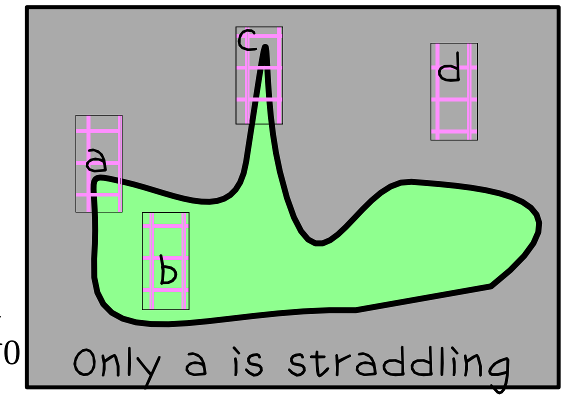
Space Pruning

continued

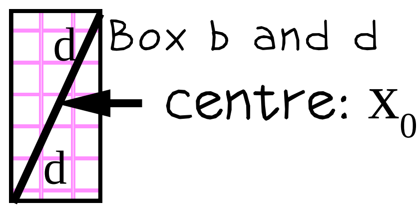
Start with a bounding box then sub-divide to level n. Discard all boxes that do not contain a part of the surface.



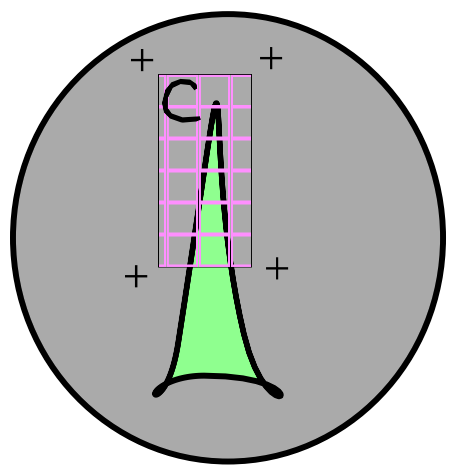
Is a box acceptable?
Box a can be accepted immediately by checking vertices. All of the others may contain part of the surface.



L tells us if the box could contain part of the surface.



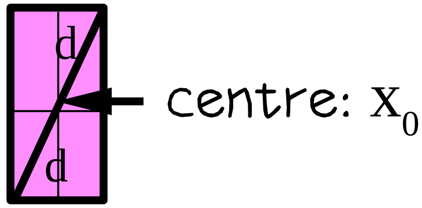
Max rate of change is L
Max distance to any point in the box from x_0 is d, the maximum change in the value of $f(\mathbf{x})$ in the box from $f(\mathbf{x_0})$ is Ld.



box b: $|f(\mathbf{x_0})| > Ld$ then $f(\mathbf{x})$ stays the same sign otherwise subdivide the box as in C

Sub-Division Termination

if $|f(\mathbf{x_0})| > L\mathrm{d}$ then $f(\mathbf{x})$ is guaranteed to stay the same sign that it has at $\mathbf{x_0}$ so box can be discarded else subdivide into 8.



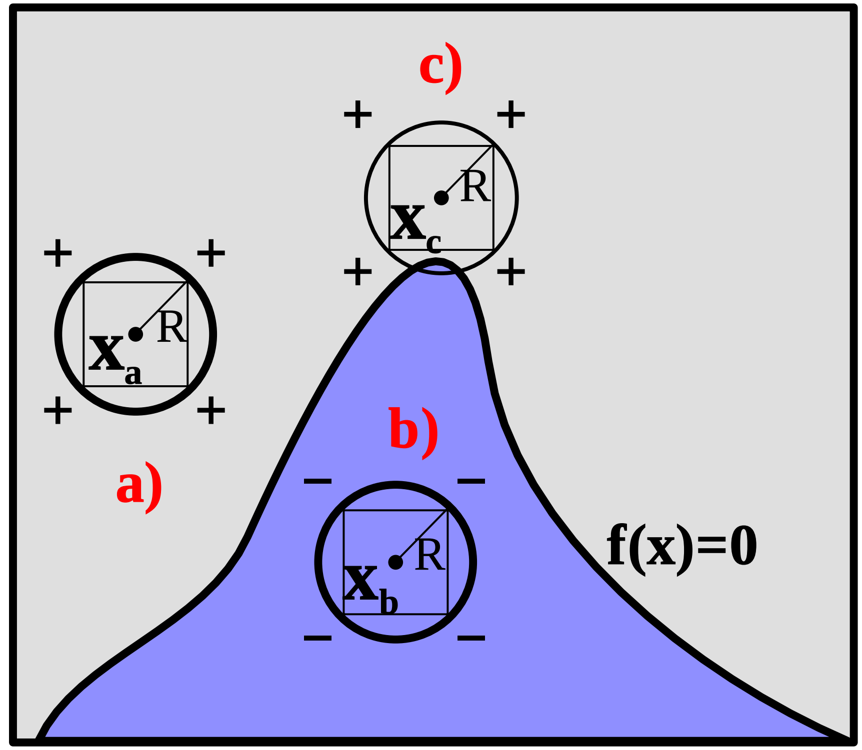
If any vertex of the box lies exactly on the surface then the algorithm won't terminate.

i.e. $|f(\mathbf{x_0})| = L\mathrm{d}$

Algorithm must stop at certain box size i.e. numerical precision limit met - does not happen in practice.

Finding Ray Intersections

We have a set of boxes V containing straddling vertices (and ray intersections). We need to find intersection nearest to origin of the ray with the surface in the box.



a) Find box in V nearest to origin of ray

b) compute intersection in the box, if none get next nearest box.

Either ensure no intersection or find the nearest intersection. F(t) represents the behaviour of f(x) along the ray.

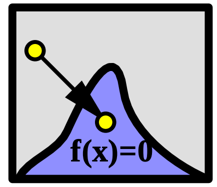
Ray $F(t)=f(\alpha t + \beta)$

$$g(t) = \frac{dF}{dt} \qquad F(t) = 0 \text{ at the intersection,}$$

$$= \alpha \,.\nabla f(\mathbf{x})|_{\alpha t+\beta}$$

Define $g(t) = \frac{dF}{dt}$

we wish to compute the intersections along the ray at $\alpha t+\beta$ we need to determine if g(t) becomes zero in an interval.



1. if F(t1) and F(t2) are opposite signs t1<t2
at least one intersection between t1 and t2.

2. if F(t1) and F(t2) have the same sign at t1 and t2 t1< t2 and g(t)=df/dt is not zero between t1 and t2 there is no intersection.

Ray $F(t)=f(\alpha t + \beta)$

G is the Lipschitz constant for g(t) in [t1, t2]

tm = (t1+t2)/2   d = (t2-t1)/2

Define $g(t) = \frac{dF}{dt}$

if |g(tm)|>Gd

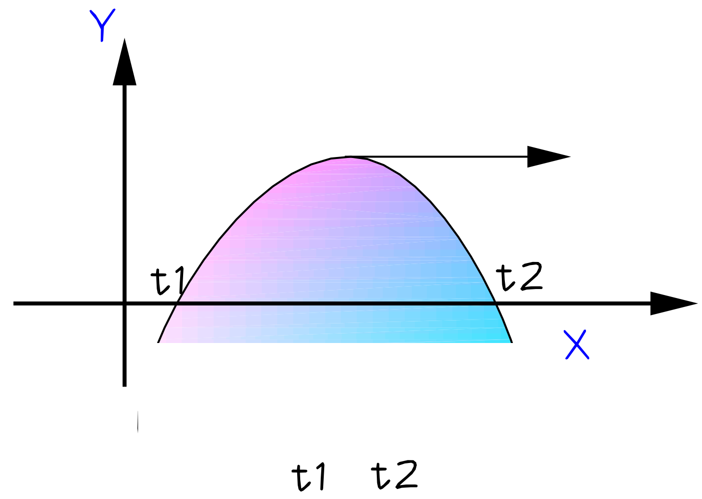
in [t1, t2] then g(t) never becomes zero in the interval - one intersection.

G >= max rate of change of g(t) and d is the max dist. along the ray from tm.

Gd is the max possible change in g(t) from g(tm)

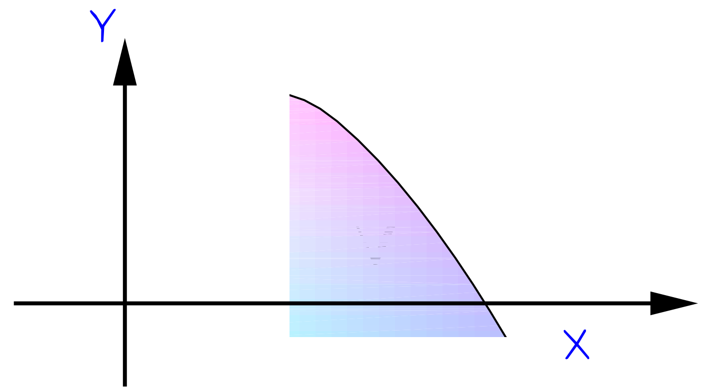
Mean Value Theorum

If a continuous function attains two consecutive zero values at t1 and t2, there exists at least one point between t1 and t2 where df/dt = 0.



If g(t) does not become zero there is exactly one intersection between the ray and the surface

1. if F(t1) and F(t2) are opposite signs t1<t2 at least one intersection between t1 and t2. Compute Regula Falsi.



2. if F(t1) and F(t2) have the same sign at t1 and t2 t1< t2 and ||g(t)|| > Gd between t1 and t2 there is no intersection.

Consider the next box along the ray, no more boxes then ray rejected.

if ||g(t)|| < Gd subdivide the ray interval.

Ray Intersection Algorithm

We are looking for the interval [t1, t2] nearest to the origin of the ray with exactly one intersection.

Given a box B ray intersects at p1, p2 (t1, t2). tm is the midpoint:

$$tm = (t1+t2)/2 \quad d = (t2-t1)/2$$