Reliability Benefit of Network Coding

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Abstract

The capacity benefit of network coding has been extensively studied in wired and wireless networks. Moreover, it has been shown that network coding improves network reliability by reducing the number of packet retransmissions in lossy networks. However, the extent of the reliability benefit of network coding is not known. In this work, we characterize the reliability benefit of network coding for reliable multicasting. In particular, we show that the expected number of transmissions using link-by-link ARQ compared to network coding to send a packet from the multicast source to K receivers scales as $\Theta(\frac{\log K}{\log \log K})$.

Index Terms

Network coding, multicast, reliability, ARQ, asymptotic analysis.

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I. Introduction

We study the performance of different error control techniques in tree-based reliable multicasting. We make the following simplifying assumptions although it is straightforward to extend our results to more general cases:

1) Each node has exactly K children,
2) The loss probability over all links is independent and equal to p,
3) There is reliable and instantaneous feedback.

In the following subsections, we study the performance of end-to-end and link-by-link error control techniques based on ARQ and FEC[1]. As the measure of performance, we compute the expected number of transmissions of a packet until the packet is received by all nodes of the multicast tree.

The rest of the paper is organized as follows. In sections II and III, we analyze end-to-end and link-by-link error control techniques, respectively, and derive exact expressions for the expected number of transmissions at source and in multicast tree. While these expressions can be numerically evaluated, they do not provide any particular insight about the scaling behavior of different error control techniques. Hence, section IV is devoted to the asymptotic analysis of error control techniques based on order statistics. In section V, we provide numerical examples to show the exact behavior of different error control techniques by evaluating the expressions we derive in sections II and III. Our conclusions as well as future work are discussed in section VI.

II. End-to-End Error Control

A. Probability Distribution of the Number of Transmissions

Let N_r denote the number of transmissions of a packet to the root of a subtree of height r (from its parent) until the packet is received by all nodes of the subtree. For the source of a multicast tree of height h, we interpret N_h as the number of packet transmissions at the source until the packet is received by all the multicast receivers (see Figure 1).

[1] We refer to the case of link-by-link FEC as *network coding*.

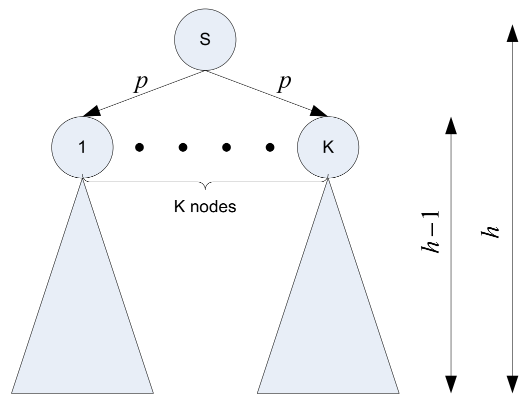


Fig. 1. Tree topology for reliable multicast.

Define $F_r(i)$ as follows:

$$F_r(i) = \mathbb{P}\left\{N_r \leq i\right\}, \qquad 0 \leq r \leq h, i \geq 0 \tag{1}$$

where, $F_r(0) = 0$ according to the definition.

Similar to [1], we develop recursive equations to compute $F_r(i)$ in the case of ARQ and FEC error control. First, consider the case of $r > 0$ and denote the root of the subtree by s. The probability that j packets out of i packets that have been transmitted to node s are received by node s is given by a binomial distribution expressed as

$$\mathbb{P}\left\{j|i\right\} = \binom{i}{j}(1-p)^j p^{i-j}, \qquad 0 \leq j \leq i\,. \tag{2}$$

Note that for the root of the multicast tree the error probability is zero, *i.e.*, $p = 0$, and hence $\mathbb{P}\left\{j|i\right\} = 1$, if $j = i$, and $\mathbb{P}\left\{j|i\right\} = 0$, otherwise. If node s receives j packets, it will broadcast the j received packets to its children. For each child, the probability that all nodes of the subtree rooted at that child receive a packet is given by $F_{r-1}(j)$. Since the children of a node have independent packet losses, the probability that all the nodes of the subtrees rooted at children of node s receive a packet is given by $\left\{F_{r-1}(j)\right\}^K$, which we denote by $F_{r-1}^K(j)$ for notational simplicity. Therefore, by summing over all possible values of j, it is obtained that

$$F_r(i) = \sum_{j=0}^{i} \binom{i}{j}(1-p)^j p^{i-j} F_{r-1}^K(j), \qquad 0 < r < h\,. \tag{3}$$

Hence, we have a recursive equation for computing $F_r(i)$ for $r > 0$. Interestingly, computing $F_r(j)$ for $r > 0$ is independent of the end-to-end error control technique.

Next, we compute $F_0(i)$ for the leaves of the multicast tree as follows:

1) End-to-End ARQ:

The probability that a (leaf) node does not receive any packet out of i transmitted packets is given by p^i. Therefore, with probability $1 - p^i$ the node receives at least a copy of the packet. Therefore,

$$F_0(i) = 1 - p^i\,. \tag{4}$$

2) End-to-End FEC:

We assume that block size for coding is B. Clearly, $F_0(i) = 0$ for $i < B$. Hence, we consider $i \geq B$ in the following. The probability that a node receives at least B coded packets out of i transmitted packets is given by a binomial distribution. Therefore,

$$F_0(i) = \sum_{j=B}^{i} \binom{i}{j} (1-p)^j p^{i-j}, \qquad i \geq B\,. \tag{5}$$

So far, we have determined $F_r(i)$ for all subtrees of height r. As mentioned before, for the source of the multicast we have $\mathbb{P}\left\{i|i\right\} = 1$. Hence, the expression $F_h(i)$ can be simplified as follows

$$F_h(i) = F_{h-1}^K(i)\,. \tag{6}$$

where, $F_{h-1}(i)$ is given by (3).

B. Expected Number of Transmissions

For the source of the multicast, the expected number of transmissions until a packet is received by all receivers is expressed as follows.

1) End-to-End ARQ:

$$\mathbb{E}\left[N_h\right] = \sum_{i=0}^{\infty}(1 - F_h(i)) = 1 + \sum_{i=1}^{\infty}(1 - F_h(i))\,. \tag{7}$$

2) End-to-End FEC:

$$\mathbb{E}\left[N_h\right] = \frac{1}{B}\sum_{i=0}^{\infty}(1 - F_h(i)) = 1 + \frac{1}{B}\sum_{i=B}^{\infty}(1 - F_h(i))\,. \tag{8}$$

Next, we compute the expected number of transmissions in the multicast tree (not just at the source) until a packet is received by all receivers. Let T_h denote the total number of transmissions in the multicast tree until a packet is received by all receivers. First, we compute the expected number of transmissions in the tree per each transmission at the source of the muticast. Let X_r denote the number of transmissions in a subtree of height r per each transmission at the root of the subtree. Then,

$$\begin{aligned} X_r &= 1 + \sum_{j=0}^{K} \binom{K}{j} (1-p)^j p^{K-j} \Big(j X_{r-1} \Big) \\ &= 1 + K(1-p) X_{r-1}, \end{aligned} \tag{9}$$

where,

$$X_0 = 0 \,. \tag{10}$$

It is therefore obtained that

$$X_r = \frac{(Kq)^r - 1}{Kq - 1} \,. \tag{11}$$

where $q = 1 - p$. Therefore, the expected number of transmissions per packet in the multicast tree is given by

$$\begin{aligned} \mathbb{E}\left[T_h\right] &= X_h \mathbb{E}\left[N_h\right] \\ &= \frac{(Kq)^h - 1}{Kq - 1} \mathbb{E}\left[N_h\right] \,. \end{aligned} \tag{12}$$

Note that if $Kq = 1$ then it is simply obtained that

$$\mathbb{E}\left[T_h\right] = h \mathbb{E}\left[N_h\right] \,. \tag{13}$$

III. Link-by-Link Error Control

For link-by-link error control, we consider a simple topology as depicted in Figure 2 in which a source s broadcasts a packet to all its K children. Let N denote the number of transmissions of a packet by the source until the packet is received by all K children. Define $F(i)$ as follows:

$$F(i) = \mathbb{P}\left\{N \le i\right\}, \tag{14}$$

where $F(0) = 0$.

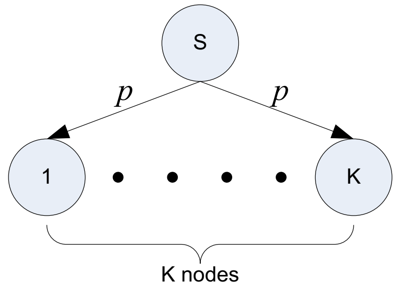


Fig. 2. Model for link-by-link error control.

A. Probability Distribution of the Number of Transmissions

Interestingly, $F_0(i)$ from previous section can be used to compute the expected number of transmissions in the case of link-by-link error control. In particular, we obtain the following expressions.

1) Link-by-Link ARQ:

The probability that a child does not receive any packet out of i transmitted packets is given by p^i. Therefore, with probability $1 - p^i$ the child receives at least a copy of the packet. All K children of a node are independent, therefore, the probability that all children receive at least a packet is given by

$$F(i) = (1 - p^i)^K \,. \tag{15}$$

2) Network Coding:

We assume that block size for coding is B and $i \geq B$. Clearly, $F(i) = 0$ for $i < B$. The probability that a child receives at least B coded packets out of i transmitted packets is given by a binomial distribution. Therefore, the probability that every child receives B packets is given by

$$F(i) = \left\{ \sum_{j=B}^{i} \binom{i}{j} (1-p)^j p^{i-j} \right\}^K \,. \tag{16}$$

B. Expected Number of Transmissions

Similar to the previous section, the expected number of transmissions at the source of the multicast is simply given by

1) Link-by-Link ARQ:

$$\mathbb{E}\left[N\right]=\sum_{i=0}^{\infty}(1-F(i))=1+\sum_{i=1}^{\infty}(1-F(i))\,. \tag{17}$$

2) Network Coding:

$$\mathbb{E}\left[N\right]=\frac{1}{B}\sum_{i=0}^{\infty}(1-F(i))=1+\frac{1}{B}\sum_{i=B}^{\infty}(1-F(i))\,. \tag{18}$$

Next, consider a multicast tree of height h. At height r, there are K^{h-r} nodes. For each of them, the expected number of transmissions is given by $\mathbb{E}\left[N\right]$ because of the link-by-link error control mechanism. Let T_h denote the total number of transmissions in the tree. It is obtained that

$$\mathbb{E}\left[T_h\right]=\mathbb{E}\left[N\right]\sum_{r=1}^{h}K^{h-r}=\frac{K^h-1}{K-1}\mathbb{E}\left[N\right]\,. \tag{19}$$

Note that for $K=1$, we have

$$\mathbb{E}\left[T_h\right]=h\mathbb{E}\left[N\right]\,. \tag{20}$$

IV. Asymptotic Analysis

In previous sections, we derived expressions for the expected number of transmissions for reliable multicasting. However, the exact expressions do not provide insight about the scaling of the number of transmissions with respect to K, B and h. In this section, using order statistics, we derive asymptotic expressions for the expected number of transmissions at the source and in the multicast tree.

A. Link-by-Link Error Control

Consider node s of the tree with K children as depicted in Figure 2. We note that link-by-link ARQ is a special case of network coding in which $B=1$. We use asymptotic results from order statistics to compute bounds on the expected number of transmissions for both ARQ and coding techniques.

1) Link-by-Link ARQ:

Consider child k of node s and let X_k denote the number of transmissions to child k until it receives the packet. Clearly, X_k has a geometric distribution, that is

$$\mathbb{P}\left\{X_k=i\right\}=(1-p)p^{i-1}\,. \tag{21}$$

We are interested in finding

$$\mathbb{E}\left[\max_{1\le k\le K} X_k\right]. \tag{22}$$

Define $Q = \frac{1}{p}$ and $\log x = \frac{\ln x}{\ln Q}$, where $\ln$ denotes the natural logarithm. Using the asymptotic analysis of the maximum statistics of geometric variables [2], we have

$$\overline{X}_{\text{ARQ}} = \mathbb{E}\left[\max_{1\le k\le K} X_k\right] = \Theta(\log K)\,. \tag{23}$$

2) Network Coding:

In this case, X_k has a negative binomial distribution, that is

$$\mathbb{P}\left\{X_k = i\right\} = \binom{i-1}{B-1}(1-p)^B p^{i-B}\,. \tag{24}$$

This meas that $B-1$ packets have been received until packet $i-1$, and packet i is received too. Using the asymptotic analysis of the maximum statistics of negative binomial random variables [3], we have

$$\mathbb{E}\left[\max_{1\le k\le K} X_k\right] = \Theta(\log K + (B-1)\log\log K)\,. \tag{25}$$

By dividing both sides of the equation by B, we obtain the expected number of transmissions per packet:

$$\overline{X}_{\text{Coding}} = \frac{1}{B}\mathbb{E}\left[\max_{1\le k\le K} X_k\right] = \Theta(\frac{1}{B}\log K + \log\log K)\,. \tag{26}$$

For $B = \omega(\log K)$, the above asymptotic expression reduces to:

$$\overline{X}_{\text{Coding}} = \Theta(\log\log K)\,. \tag{27}$$

Using (19) and the above asymptotics, we get the following bounds for the number of transmissions in the tree.

1) Link-by-Link ARQ:

$$\overline{T}_{\text{ARQ}} = \Theta(K^{h-1}\log K)\,. \tag{28}$$

2) Network Coding:

$$\overline{T}_{\text{Coding}} = \Theta(K^{h-1}\log\log K)\,. \tag{29}$$

B. End-to-End Error Control

Let N_r denote the number of transmissions at the root of a subtree of height r until each receiver in the subtree receives the packet. From the asymptotic analysis of negative binomial distribution we have

$$N_r = \Theta(\log K + N_{r-1} \log\log K), \tag{30}$$

where, $N_1 = \Theta(\log K + B \log\log K)$. This simply states that each child of the root of the subtree needs to receive N_{r-1} packets. Therefore, the root has to transmit $\Theta(\log K + N_{r-1} \log\log K)$ packets until every child has received N_{r-1} packets. By expanding the recursive equation (30), it is obtained that

$$\begin{aligned} N_r &\sim \log K\Big(1 + \cdots + (\log\log K)^{r-2}\Big) + N_1(\log\log K)^{r-1} \\ &= \log K \frac{(\log\log K)^r - 1}{\log\log K - 1} + B(\log\log K)^r \\ &= \Theta\Big(\log K(\log\log K)^{r-1} + B(\log\log K)^r\Big). \end{aligned} \tag{31}$$

Therefore, for $B = \omega(\log K)$, the expected number of transmissions at the source is given by

1) End-to-End ARQ:

$$\overline{N}_{\text{ARQ}} = \Theta\Big(\log K(\log\log K)^{h-1}\Big). \tag{32}$$

2) End-to-End FEC:

$$\overline{N}_{\text{FEC}} = \Theta\Big((\log\log K)^h\Big). \tag{33}$$

Using (12) and the above asymptotics, we can derive the following bounds for the number of transmissions in the tree.

1) End-to-End ARQ:

$$\overline{T}_{\text{ARQ}} = \Theta\Big(K^{h-1} \log K(\log\log K)^{h-1}\Big). \tag{34}$$

2) End-to-End FEC:

$$\overline{T}_{\text{FEC}} = \Theta\Big(K^{h-1}(\log\log K)^h\Big). \tag{35}$$

C. Convolutional Coding

In previous sections, we assumed the use of block codes for network coding. We showed that as $K \to \infty$ for $B = \Omega(\log K)$, the expected number of transmissions scales as $\Theta(\log\log K)$. Intuitively, for a block code with block length B, every packet is useful to all the receivers until one of the receivers receives B packets. From that time until the time the last receiver receives B packets, each transmitted packet is only useful to a subset of the receivers and not all of them. In particular, at the beginning, each transmitted packet is useful to K receivers; after one of the receivers receives B packets, each transmitted packet is only useful to $K-1$ receivers; and so on.

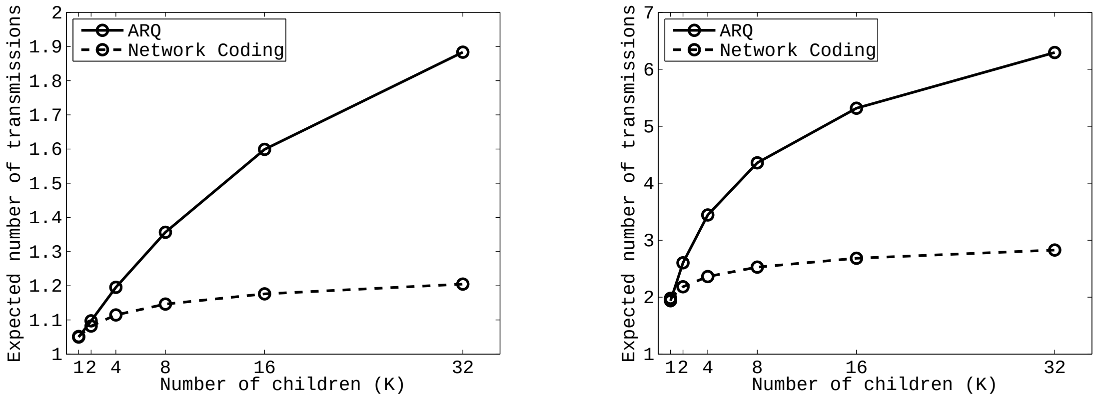
However, if assume the use of convolutional codes that operate on streams of packets instead of blocks of packets, then every transmitted packet is useful to all the receivers. Hence, for each receiver, the expected number of transmissions per packet is simply given by $\frac{1}{1-p}$. In this case, the reliability gain of network coding compared to link-by-link ARQ scales as $\Theta(\log K)$ as $K \to \infty$.

V. Numerical Examples

We have numerically evaluated the exact expressions for the expected number of transmissions at the source and in the multicast tree for different tree heights and error probabilities. A summary of our numerical results are presented in this section. For FEC-based error control techniques, *i.e.*, end-to-end FEC and network coding, we have assumed the use of block codes with block length $B = 16$. We have also generated results for larger values of B which show the same behavior in the number of transmissions, and hence are not presented here.

As a base for comparison, in Figure 3, we have plotted the expected number of transmissions for $h = 1$. This is the case of having a source transmitting packets to K receivers in its transmission range. Since feedback overhead is ignored, ARQ is indeed the optimal error control technique in a non-coded case. Nevertheless, network coding outperforms ARQ in both low-loss and high-loss regimes as shown in Figures 3(a) and 3(b), respectively.

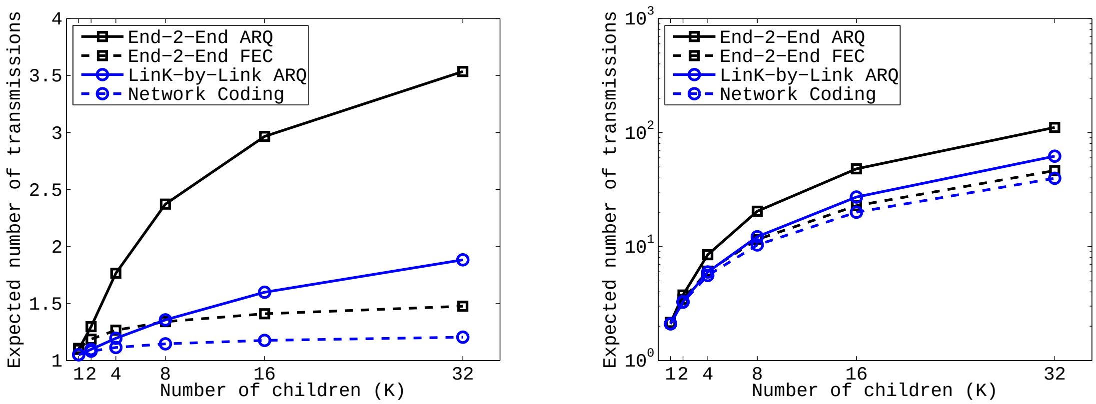
Next, we look at multicast trees with $h \in \{2, 4\}$. In both cases network coding outperforms all the other techniques. Interestingly, link-by-link ARQ and end-to-end FEC show different behavior with different tree heights. Figures 4 and 5 depict the expected number of transmissions for a tree of height $h = 2$ and error probabilities $p \in \{0.05, 0.5\}$. It can be seen from the figures



(a) $p = 0.05$

(b) $p = 0.5$

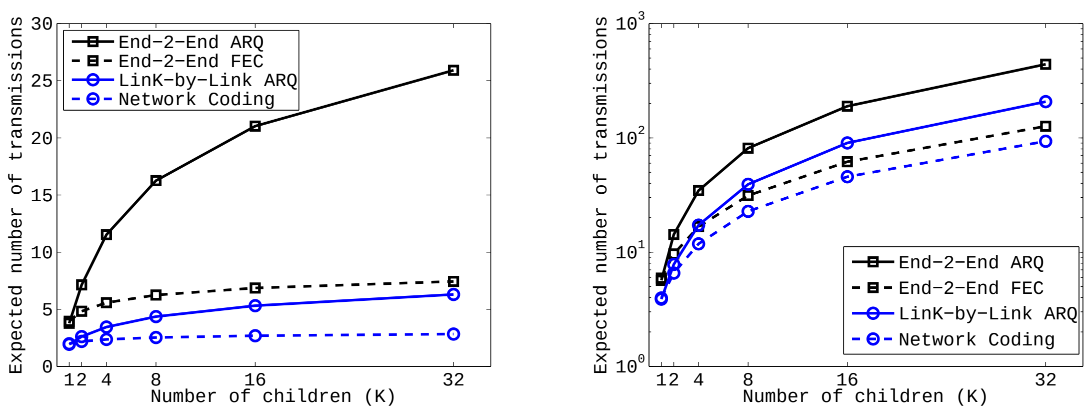
Fig. 3. No. of transmissions at source for $h = 1$.



(a) No. of transmissions at source

(b) No. of transmissions in tree

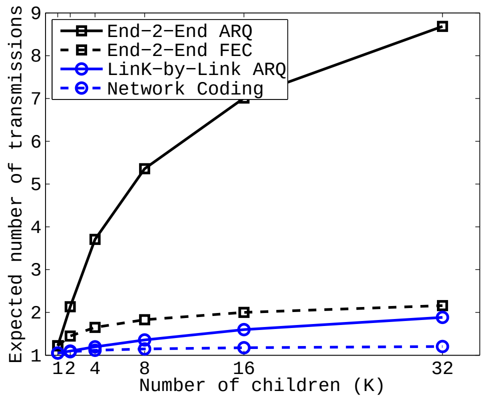
Fig. 4. $P = 0.05$ and $h = 2$.



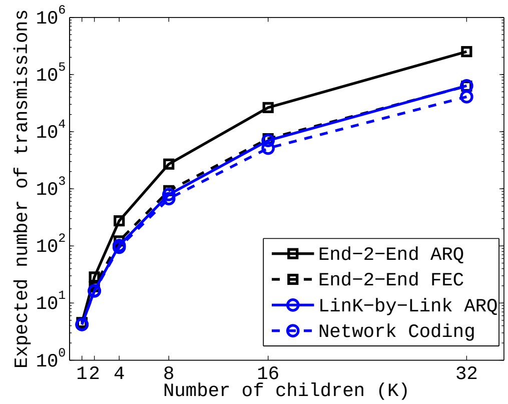
(a) No. of transmissions at source

(b) No. of transmissions in tree

Fig. 5. $P = 0.5$ and $h = 2$.



(a) No. of transmissions at source



(b) No. of transmissions in tree

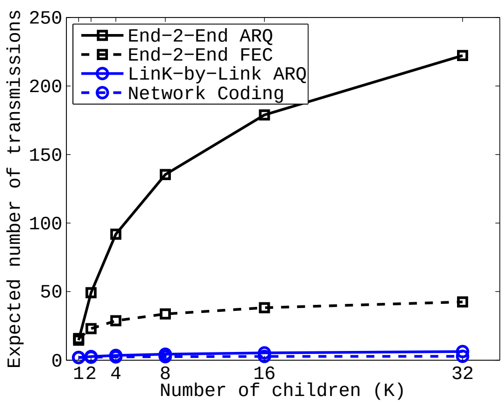
Fig. 6. $P = 0.05$ and $h = 4$.

that, in some cases, end-to-end FEC has better performance than link-by-link ARQ although the difference is not significant. Interestingly, there is a significant difference between end-to-end ARQ (dominant error control technique in Internet) and network coding. In particular, in $p = 0.5$ representing a high-loss regime, the difference in the number of transmissions at source is huge.

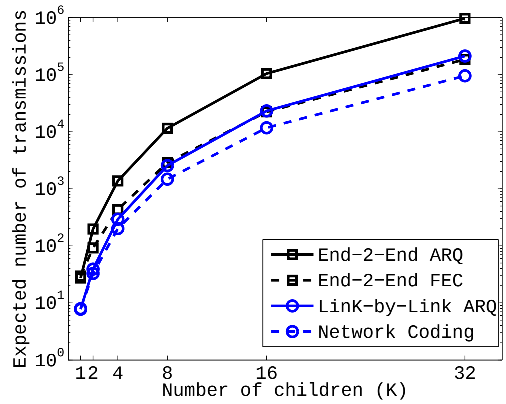
Figures 6 and 7 show the number of transmissions for a tree of height $h = 4$. Interestingly, link-by-link error control techniques always outperform end-to-end techniques. Moreover, the difference between end-to-end ARQ and network coding is significant even for small values of K. Again, in high-loss regimes, there is a huge difference between end-to-end ARQ and Network coding.

VI. Conclusion

In this paper, we studied the reliability benefit of network coding for tree-based reliable multicasting. Four types of error control techniques, namely, end-to-end ARQ, end-to-end FEC, link-by-link ARQ and network coding were considered. We analyzed the expected number of transmissions at source and in multicast tree as the measure of the performance of different error control techniques. We derived exact expressions for the expected number of transmissions. Furthermore, using results from order statistics, asymptotic bounds for the performance of difference error control techniques were derived. In particular, it was shown that the reliability benefit of network coding compared to link-by-link ARQ is $\Theta(\frac{\log K}{\log \log K})$ where K is the fan-out degree of the nodes of the tree. Our results can be readily utilized to compute exact and



(a) No. of transmissions at source



(b) No. of transmissions in tree

Fig. 7. $P = 0.5$ and $h = 4$.

asymptotic delay benefit of network coding as studied in [4]. In the future, we would like to extend our analysis to more complicated multicast topologies such as a grid with significant amount of *path diversity*.

VII. Acknowledgements

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