Geometry of interaction and uniformity

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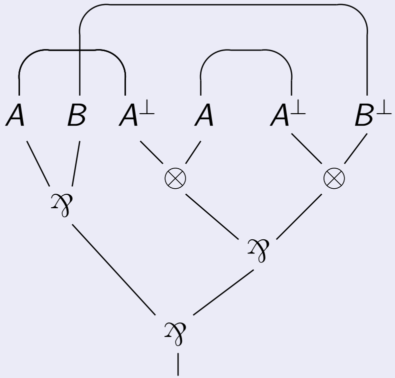
Outline

- Introduction: free compact closure
- Handwaving: how to get a linear exponential comonad
- Serious stuff

Paths in proof-nets

$(A \parr B) \parr (A^\perp \otimes A) \parr (A^\perp \otimes B^\perp)$



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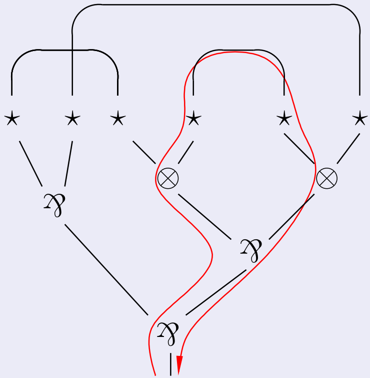

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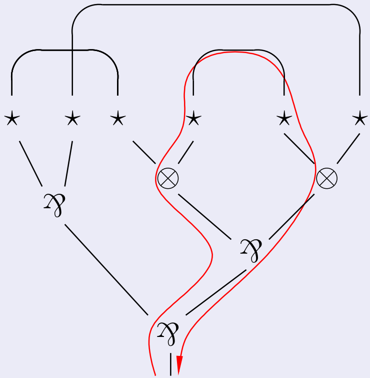
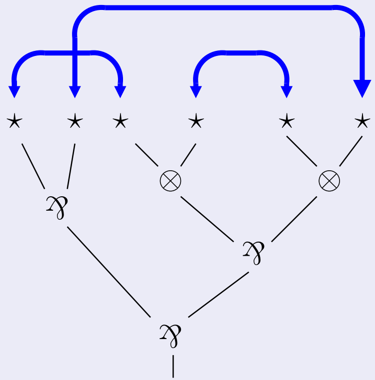


$1.0.1.\star \to 1.1.0.\star$

Paths in proof-nets



$0.0.\star \leftrightarrow 1.0.0.\star$

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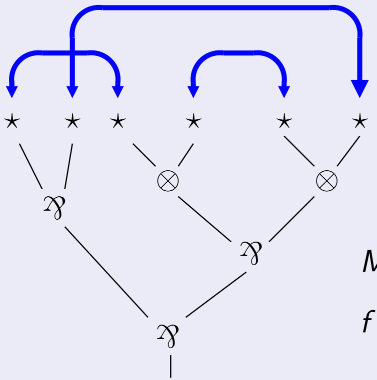

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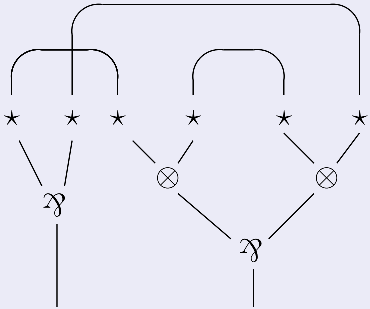
$$M = \left\{ \begin{array}{l} 0.0.\star, 1.0.0.\star, 0.1.\star, \\ 1.1.1.\star, 1.0.1.\star, 1.1.0.\star \end{array} \right\}$$

$f : M \to M$

Paths in proof-nets



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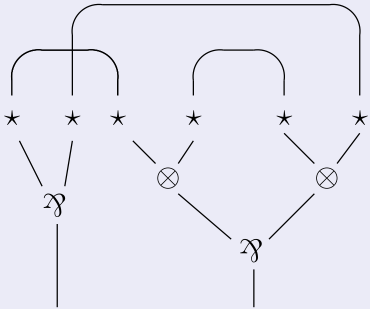
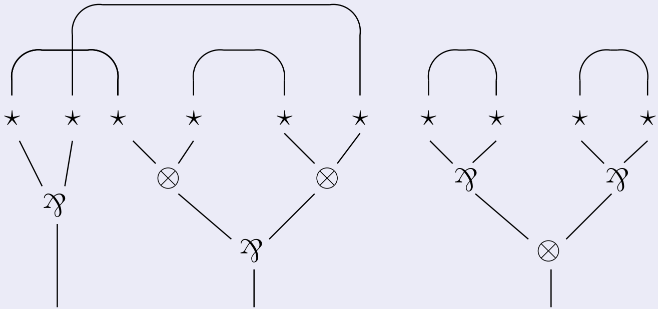


$f : M_1 \uplus M_2 \to M_1 \uplus M_2$

$M_1 = \{0.\star, 1.\star\}$
$M_2 = \{0.0.\star, 0.1.\star, 1.0.\star, 1.1.\star\}$

Paths in proof-nets



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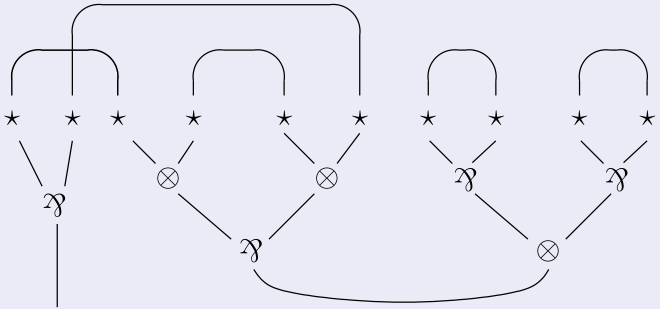

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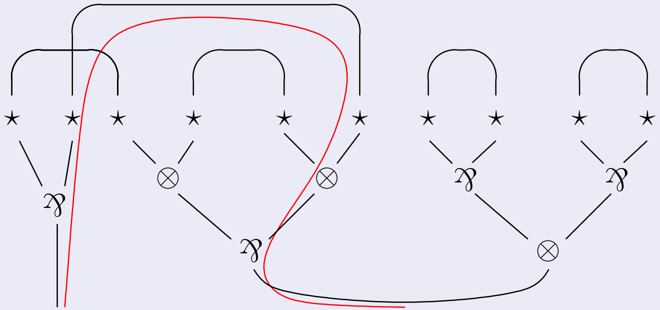

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$f(1.\star) = 1.1.\star$

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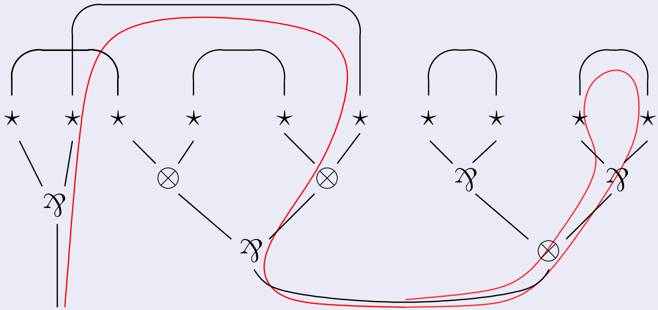

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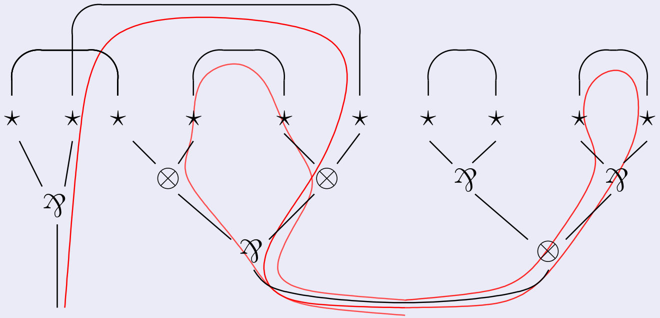

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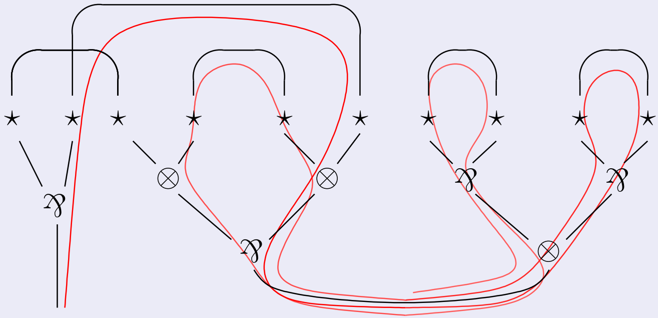

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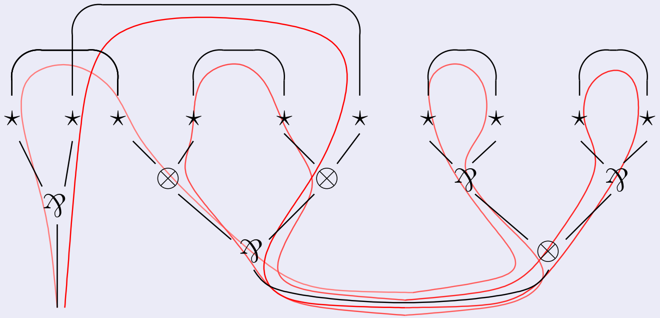

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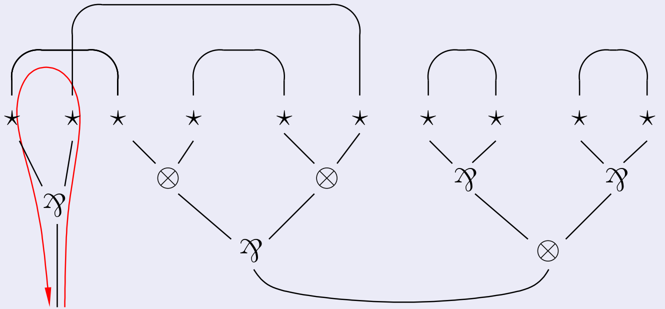

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$0.\star \leftrightarrow 1.\star$

Traced monoidal category

Trace operator

Categorical operator for fixed point, iterator, feedback...

- In a symmetric monoidal category $(\mathbb{C}, \otimes, I)$:

$$f : A \otimes U \to B \otimes U \quad \rightsquigarrow \quad tr^{U}_{A,B}(f) : A \to B$$

satisfying some conditions (naturality, dinaturality...)

- In partial injections $(\mathbf{PInj}, \uplus, \emptyset)$:

$$f : A \uplus U \to B \uplus U \quad \rightsquigarrow \quad tr^{U}_{A,B}(f) = \bigsqcup_{n \geq 0} \pi f \, (\rho f)^{n} \, \iota$$

where:

- $\iota : A \to A \uplus U$
- $\pi : B \uplus U \to B$
- $\rho = \emptyset \uplus id_U : B \uplus U \to A \uplus U$

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Traced monoidal category

Free compact closure

With a TSMC $(\mathbb{C}, \otimes, I, tr)$, **Int** construction of Joyal *et al.*:

- objects: (A^+, A^-) where A^+, A^- are $\mathbb{C}$-objects
- morphisms:

$$\mathbf{Int}(\mathbb{C})((A^+, A^-), (B^+, B^-)) = \mathbb{C}(A^+ \otimes B^-, A^- \otimes B^+)$$

- composition: $f : (A^+, A^-) \to (B^+, B^-)$ and $g : (B^+, B^-) \to (C^+, C^-)$

$$g \circ f = tr^{B^+ \otimes B^-}_{A^+ \otimes C^-, C^+ \otimes A^-}(\sigma(f \otimes g))$$

  σ some canonical symmetry

- compact closed structure:

$$I_{\mathbf{Int}(\mathbb{C})} = (I, I)$$
$$(A^+, A^-) \otimes (B^+, B^-) = (A^+ \otimes B^+, A^- \otimes B^-)$$
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Interpretation of MELL

In $\mathbf{Int}(\mathbf{PInj})$:

- self-dual objects: $\mathbf{A} = (A, A)$

$$f \in \mathbf{PInj}(A, B) \quad \rightsquigarrow \quad \mathcal{N}(f) = \sigma_{A,B}(f \uplus f^{\star}) \in \mathbf{Int}(\mathbf{PInj})(\mathbf{A}, \mathbf{B})$$

- compact closed: multiplicative linear logic
- exponentials: $!\mathbf{A} = (\mathbb{N} \times A, \mathbb{N} \times A)$
  - $e : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ bijection

$$\mathcal{N}(e \times id_A) : !\mathbf{A} \to !!\mathbf{A}$$

  - $d : \mathbb{N} \to \{\star\}, \quad d(0) = \star$

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Adding some structure

Geometry of interaction:

- formula $\rightsquigarrow$ leaves of the syntactic tree
- permutation of these leaves

Category $\mathbb{G}$:

- objects $\mathbf{A}$: $\mathrm{m}_{\mathbf{A}} : |A| \to \mathcal{P}_f(M_A)$
  - $|A|$: positions (the "abstract trees")
  - M_A: token (the "leaves")
- morphisms $\sigma = (p_\sigma, f_\sigma) \in \mathbb{G}(\mathbf{A}, \mathbf{B})$:
  - $p_\sigma \subseteq |A| \times |B|$
  - $f_\sigma : M_A \uplus M_B \to M_A \uplus M_A$ partial injection
  - f_σ preserves the positions in p_σ:

$$\forall (a, b) \in p_\sigma, \quad f_\sigma(\mathrm{m}_{\mathbf{A}}(a) \uplus \mathrm{m}_{\mathbf{B}}(b)) = \mathrm{m}_{\mathbf{A}}(a) \uplus \mathrm{m}_{\mathbf{B}}(b)$$

Adding some structure

Geometry of interaction *with additional structure*:

- formula $\rightsquigarrow$ abstract trees (i.e. points in **Rel**)
- abstract trees $\rightsquigarrow$ set of leaves
- permutation of leaves *preserving the abstract trees*

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Adding some structure

Composition in $\mathbb{G}$: $\sigma \in \mathbb{G}(\mathbf{A}, \mathbf{B})$, $\tau \in \mathbb{G}(\mathbf{B}, \mathbf{C})$

- on positions: relational composition

$$p_{\tau \circ \sigma} = p_\tau \circ p_\sigma \qquad \text{in } \mathbf{Rel}$$

- on token: usual GoI compostion (with the trace)

$$f_{\tau \circ \sigma} = tr(f_\sigma \uplus f_\tau) \qquad \text{in } \mathbf{Int}(\mathbf{PInj})$$

$\mathbb{G}$ is compact closed, and:

- $I = (\{\star\}, \emptyset, \star \mapsto \emptyset)$
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What about exponentials ?

$\mathbf{!A} \in \mathbb{G}$:

- positions: finite subtrees of an infinite branching tree

$$|!A| = \{\, \alpha : \mathbb{N} \rightharpoonup |A| \mid \#d(\alpha) < \infty \,\}$$

- token: $M_{!A} = \mathbb{N} \times M_A$ as usual
- $\mathrm{m}_{!\mathbf{A}}(\alpha) = \bigcup_{n \in d(\alpha)} \{n\} \times \mathrm{m}_{\mathbf{A}}(\alpha(n))$

So:

- positions are ordered by "subtree relation" ($\alpha \sqsubseteq \alpha'$: restriction order)
- still the reindexations to define...

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What positions *really* are

Tree-like structures with partial maps

A category of linear domains pdl:

- objects:
  - algebraic dcpo A
  - $x \in \mathcal{K}(A) \implies \downarrow\{x\}$ finite distributive lattice
  - $A' \subseteq_s A \iff \mathcal{K}(A') \subseteq \mathcal{K}(A)$ and $\mathcal{K}(A')$ closed by $\vee, \wedge$ $(|A| = \mathcal{K}(A))$
- morphisms are partial, linear and c.m.: $f \in \mathrm{pdl}(|A|, |B|)$
  - $f : |A| \rightharpoonup |B|$
  - $d(f) \subseteq_s |A|$
  - $a \uparrow b \implies f(a \vee b) = f(a) \vee f(b),\ f(a \wedge b) = f(a) \wedge f(b)$
- partial cartesian: $|A \times B| = |A| \times |B|$
- monoidal closed: e.g. $\mathbb{N}$ flat

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Tree-like structures with partial maps

A category of linear domains pdl:

- objects:
  - algebraic dcpo A
  - $x \in \mathcal{K}(A) \implies \downarrow\{x\}$ finite distributive lattice
  - $A' \subseteq_s A \iff \mathcal{K}(A') \subseteq \mathcal{K}(A)$ and $\mathcal{K}(A')$ closed by $\vee, \wedge$ $(|A| = \mathcal{K}(A))$
- morphisms are partial, linear and c.m.: $f \in \mathbf{pdl}(|A|, |B|)$
  - $f : |A| \rightharpoonup |B|$
  - $d(f) \subseteq_s |A|$
  - $a \uparrow b \implies f(a \vee b) = f(a) \vee f(b),\ f(a \wedge b) = f(a) \wedge f(b)$
- partial cartesian: $|A \times B| = |A| \times |B|$
- monoidal closed: e.g. $\mathbb{N}$ flat

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What positions *really* are

Relations on positions

Building relations of positions:

- pdl is partial cartesian with a class of subobjects:
  - objects in $Rel(\text{pdl})$: as in pdl
  - $Rel(\text{pdl})(|A|, |B|) = \{R \subseteq_s |A| \times |B|\}$

- $(Rel(\text{pdl}), \times, \bot)$ is (dagger) compact closed

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Which relation between token and positions ?

Glueing of token to positions

- We know that:
  - $\mathbf{PInj}$ is a TSMC
  - $\mathbf{Int}(\mathbf{PInj})$ is compact closed

- Monoidal functor $\mathcal{P}_f : (\mathbf{PInj}, \uplus) \to (\mathsf{pdl}, \times)$

$$f \in \mathbf{PInj}(M_A, M_B) \ \rightsquigarrow \ \mathcal{P}_f(f) : \left\{ \begin{array}{ccc} \mathcal{P}_f(M_A) & \to & \mathcal{P}_f(M_B) \\ m \subseteq_{fin} d(f) & \to & f(m) \end{array} \right.$$

$$\varphi_{M_A,M_B} : \mathcal{P}_f(M_A) \times \mathcal{P}_f(M_B) \to \mathcal{P}_f(M_A \uplus M_B)$$

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Lax glueing

Let $\mathbb{C} = (Id_{\mathrm{pdl}} \downharpoonleft \mathcal{P}_f)$

- objects: $\mathbf{A} = (|A|, M_A, \mathrm{m}_{\mathbf{A}} : |A| \to \mathcal{P}_f(M_A))$ ($\mathrm{m}_{\mathbf{A}}$ total)
- morphisms: pairs p, f such that:

$$\begin{array}{ccc} |A| & \xrightarrow{\mathrm{m}_{\mathbf{A}}} & \mathcal{P}_f(M_A) \\ {\scriptstyle p}\downarrow & \sqsubseteq & \downarrow{\scriptstyle \mathcal{P}_f(f)} \\ |B| & \xrightarrow{\mathrm{m}_{\mathbf{B}}} & \mathcal{P}_f(M_B) \end{array}$$

- monoidal product:

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Let $\mathbb{G} = (Id_{\mathbf{Rel}(\mathrm{pdl})} \downharpoonleft \mathbf{Int}(\mathcal{P}_f))$

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Exponentials

- Interpretation of ! modality:
  - in pdl: $!\_ = \mathbb{N} \rightharpoonup \_$
  - in $\mathbf{PInj}$: $!\_ = \mathbb{N} \times \_$
- We have a "linear distribution": $\kappa : !\mathcal{P}_f(\_) \to \mathcal{P}_f(!\_)$
- $!\mathbf{A} = (\mathbb{N} \rightharpoonup |A|, \mathbb{N} \times M_A, \kappa_{M_A} \circ !\mathrm{m}_{\mathbf{A}})$
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  - $der_A \in \mathbb{G}(!\mathbf{A}, \mathbf{A})$
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Reindexation by conjugation

- Let $f \in \mathbb{C}(\mathbf{A},\mathbf{A})$ an iso; f acts on $\mathbb{C}(\mathbf{A},\mathbf{A})$ by conjugation

$$f : \begin{cases} \mathbb{G}(I,\mathbf{A}) \subseteq \mathbb{C}(\mathbf{A},\mathbf{A}) & \to & \mathbb{G}(I,\mathbf{A}) \\ \sigma & \to & f \cdot \sigma = f \circ \sigma \circ f^{-1} \end{cases}$$

- $F \subseteq \mathbb{C}(\mathbf{A},\mathbf{A})$ a group (of total maps)

$$\overline{F} \simeq \text{ closure of } F \text{ by restriction and directed lub}$$
$$\overline{F} \subseteq_s \mathbb{C}(\mathbf{A},\mathbf{A})$$

- $f \in \overline{F}$ defines a partial action on $\mathbb{G}(I,\mathbf{A})$

$$f \cdot \sigma \text{ is defined } \iff d(\sigma) \subseteq d(f)$$

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Let $F, H \subseteq \mathbb{C}(\mathbf{A}, \mathbf{A})$ be groups

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$$\{ \sigma \in \mathbb{G}(1, \mathbf{A}) \,|\, F \cdot \sigma \subseteq H \cdot \sigma \}$$

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Exponentials

Category with uniformity

Define $\mathbb{U}$ as the category with:

- objects $(\mathbf{A}, F_A, H_A)$:
  - $\mathbf{A} \in \mathbb{G}$
  - $F_A, H_A \subseteq \mathbb{C}(\mathbf{A}, \mathbf{A})$ groups
  - $F_A H_A = H_A F_A$ and $F_A \cap H_A = \{id_{\mathbf{A}}\}$
- morphisms $\sigma : I_{\mathbb{U}} \to (\mathbf{A}, F_A, H_A)$
  - $\sigma \in \mathbb{G}(I, \mathbf{A})$
  - $F_A \cdot \sigma \subseteq H_A \cdot \sigma$
- compact closed structure:
  - $(\mathbf{A}, F_A, H_A) \otimes (\mathbf{B}, F_A, H_A) = (\mathbf{A} \otimes \mathbf{B}, F_A \times F_B, H_A \times H_B)$
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Interpretation of the ! modality: $!(\mathbf{A}, F_A, H_A) = (!\mathbf{A}, F_{!A}, H_{!A})$

- $F_{!A} \simeq F_A \wr \mathfrak{S}(\mathbb{N})$
- $H_{!A} \simeq \prod_{n \in \mathbb{N}} H_A$

Can we quotient by $F_A H_A$?
Is it a congruence for composition ?

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Composition and quotient

Let $\sigma : (\mathbf{A}, F_A, H_A) \to (\mathbf{B}, F_B, H_B)$ and $\tau : I_{\mathbb{U}} \to (\mathbf{A}, F_A, H_A)$
Let $G_A = F_A H_A$ and $G_B = F_B H_B$

- $h \in H_A$, we want $\sigma \circ (h \cdot \tau) \sim \sigma \circ \tau$; i.e.:

$$\forall h \in H_A,\ \exists g_A, g_B \in G_A \times G_B \begin{cases} g_A h \cdot \tau = h \cdot \tau \\ (g_A h, g_B) \cdot \sigma = \sigma \end{cases}$$

  so that $\sigma \circ (h \cdot \tau) = ((g_A h, g_B) \cdot \sigma) \circ (g_A h \cdot \tau) = g_B(\sigma \circ \tau) \sim \sigma \circ \tau$

- we only have by definition:

$$F_A \cdot \tau \subseteq H_A \cdot \tau \implies \forall f \in F_A,\ \exists h' \in H_A,\ h' f \in \mathrm{Stab}_{G_A}(\tau)$$

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$$(H_A \times F_B) \cdot \sigma \subseteq (F_A \times H_B) \cdot \sigma \implies \forall h \in H_A,\ \exists (f', h') \in F_A \times H_B,\ (f' h, h') \in \mathrm{Stab}_{G_A \times G_B}(\tau)$$

Composition and quotient

Uniformity and biorthogonality

Let $\sigma : I \to (\mathbf{A}, F, H)$ and $\tau : I \to (\mathbf{A}^\star, H, F)$:

- orthogonality relation: $\sigma \perp \tau$ iff. $\forall (f_n) \in F^{\mathbb{N}},\ \forall (h_n) \in H^{\mathbb{N}}$

$$\forall i \in \mathbb{N} \begin{cases} h_i \dots h_0 f_0 \in \mathrm{Stab}_{FH}(\sigma) \\ f_{i+1} \dots f_1 h_0 \in \mathrm{Stab}_{FH}(\tau) \end{cases}$$

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  then (h_i), (f_i) is stationnery in $id_{\mathbf{A}}$

- $P_A \subseteq \mathbb{U}(I, (\mathbf{A}, F, H))$ such that $P_A^{\perp\perp} = P_A$
- $\mathcal{T}(\mathbb{U}) \equiv \sigma : (\mathbf{A}, F_A, H_A, P_A) \to (\mathbf{B}, F_B, H_B, P_B)$

$$\forall \tau \in P_A,\ \sigma \circ \tau \in P_B$$

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