DIFFERENTIAL CALCULUS ON PFAs

STARTER PACK

Federica Pasqualone

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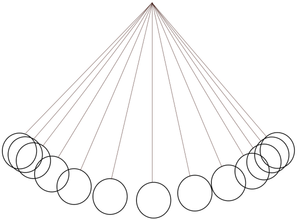
Once upon a time ...

This story starts with a simple innocent object: The pendulum



$$\ddot{\theta} = -\frac{g}{l} sin\theta$$

For $sin\theta \sim \theta$, $\ddot{\theta} = -\frac{g}{l}\theta$

WHY?

1. The last equation is completely equivalent to the one of an harmonic oscillator;
2. If we freeze time, particles form a crystal, whose vibrational modes are exactly described by the same physics of a pendulum;
3. It provides an example, from classical mechanics, of how we study a physical system. In particular, it is clear that if we want to measure some *observable*, such as momentum or velocity, it must be a function over the trajectories cut by the system and not over the whole space-time.

In general, observables are functions over trajectories, i.e. functions over solutions of the Euler-Lagrange equations, a system of pdes. Equivalenty, by Hamilton's principle, they can be seen as functions over the stationary points of the action functional S

$$S[q] := \int_{t_1}^{t_2} dt \ \mathcal{L}(q(t), \dot{q}(t), t)$$

where $\mathcal{L}$ denotes the Lagrangian of the system.

Modelling Observables: Wishlist

We are studying solutions of pdes, that we know exist at least locally, therefore it is resonable to ask for the following:

Let $U \subseteq M$ be an open set of a manifold M, that is our space-time,

1 An assignment of a 'vector space' of quantum observables to every connected open set

$$U \mapsto \mathrm{Obs}^q(U)$$

2. For U_1, U_2 disjoint open connected sets contained in another open V of M

$$\mathrm{Obs}^q(U_1) \otimes \mathrm{Obs}^q(U_2) \to \mathrm{Obs}^q(V)$$

i.e. we require the possibility of combining observables defined over disjoint regions of the space-time.

3. Deformation quantization constraint:

$$\lim_{\hbar \to 0} \frac{1}{\hbar} \mathrm{Obs}^q(U) = \mathrm{Obs}^{cl}(U)$$

(!) SR disclaimer: Observables defined over space-like separated regions are uncorrelated.

Prefactorization Algebras and Disjoint Opens

Definition

[Costello & Gwilliam [CG1] (2016), §3.1.2, Definition 1.2.1] Let $\mathbf{Disj}_M$ denote the following - *symmetric* - multicategory associated to M.

1. The objects consist of all *connected* open subsets of M;
2. For every (possibly empty) finite collection of open sets $\{U_\alpha\}_{\alpha \in A}$ and open set V, there is a set of maps $\mathbf{Disj}_M\left(\{U\}_{\alpha \in A} | V\right)$.
   If the U_α are pairwise disjoint and all contained in V, then the set of maps is a single point. Otherwise, the set of maps is empty;
3. The composition of maps is defined in the obvious way.

Definition

[ibid., §1.2, 40, line 6] A prefactorization algebra is just an algebra over this - *symmetric* - coloured operad $\mathbf{Disj}_M$.

Open connected sets as thin multicategory

Definition

Let $\left(\mathrm{Open}_X^c, \subseteq\right)$ be the ordered set of connected open parts of a topological space X with set-theoretical inclusion as preorder. The associated *symmetric* poset multicategory $\mathbf{Open}_X^c$ consists of the following:

1. $\left(\mathbf{Open}_X^c\right)_0$ as objects;
2. For any finite string $\left(U_1, \ldots, U_n\right) \in \prod^n \left(\mathbf{Open}_X^c\right)_0$ an hom-set $\mathbf{Open}_X^c\left(U_1, \ldots, U_n; V\right)$, where:

$$\mathbf{Open}_X^c\left(U_1, \ldots, U_n; V\right) = \begin{cases} \{\emptyset\} & \iff \dot\cup_{i=1}^n U_i \nsubseteq V \\ \{f\} & \iff \dot\cup_{i=1}^n U_i \subseteq V \ \wedge \ U_i \cap U_j = \emptyset \ \ \forall\, i \neq j \end{cases} \tag{1}$$

3. An operation of composition: $\forall n, k_1, \ldots, k_n \in \mathbb{N}, V, U_i, U_i^j \in \left(\mathbf{Open}_X^c\right)_0$

$$\mathbf{Open}_X^c\left(U_1, \ldots, U_n; V\right) \times \mathbf{Open}_X^c\left(U_1^1, \ldots, U_1^{k_1}; U_1\right) \times \cdots \times \mathbf{Open}_X^c\left(U_1^1, \ldots U_n{}^{k_n}; U_n\right)$$

$$\downarrow$$

$$\mathbf{Open}_X^c\left(U_1^1, \ldots, U_1^{k_1}, \ldots, U_n^1, \ldots, U_n^{k_n}; V\right)$$

$$(f, \ldots, f_n) \mapsto f \circ \left(f_1, \ldots f_n\right) \tag{2}$$

whenever the arrows exist and are sequentially composable.

(iv) An identity arrow: $\forall\, U \in \left(\mathbf{Open}_X^c\right)_0, \exists\, 1_U \in \mathbf{Open}_X^c\,(U; U)$

satisfying associativity and identity law.

The category of prefactorization algebras 1/2

Definition

Let $\mathbf{C}$ a symmetric multicategory, a **prefactorization algebra** with values in $\mathbf{C}$ is a multifunctor

$$\mathbf{Open}_X^c \xrightarrow{\mathcal{F}} \mathbf{C} \tag{3}$$

Definition

Let $\mathcal{F} : \mathbf{Open}_X^c \to \mathbf{C}$, $\mathcal{G} : \mathbf{Open}_X^c \to \mathbf{C}$ be two PFAs taking values in the symmetric multicategory $\mathbf{C}$, an **arrow of prefactorization algebras** is a natural trasformation between them

$$\mathcal{F} \overset{\phi}{\Rightarrow} \mathcal{G} \tag{4}$$

is a family of maps

$$\left\{ \mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U) \right\}_{U \in \left(\mathbf{Open}_X^c\right)_0} \tag{5}$$

such that

$$\phi_V \circ (\mathcal{F}(f)) = \mathcal{G}(f) \circ (\phi_{U_1}, \ldots, \phi_{U_n}) \tag{6}$$

for all $f : \mathcal{F}(U_1), \ldots, \mathcal{F}(U_n) \to \mathcal{F}(V)$.

The category of Prefactorization Algebras 2/2

Definition

Let X be a topological space, **C** be a symmetric multicategory, the **category of prefactorization algebras over X with values in C** consists of an objects class made out of PFAs and, as morphisms, natural transformation between them. We denote such category by the symbol $\mathbf{PFA}_X(\mathbf{C})$.

Factorization Algebras

If we have a global solution of our pdes this descends to local ones, but what about gluing local solutions to a global one? This is exactly what factorization algebras model.

Definition

A factorization algebra is a prefactorization algebra $\mathcal{F}$ satisfying two additional axioms:

1. For $U_i, U_j \subset M$ any two open sets of a manifold M, there exists an isomorphism

$$\mathcal{F}(U_i) \otimes \mathcal{F}(U_j) \xrightarrow{\cong} \mathcal{F}(U_i \,\dot{\cup}\, U_j)$$

2. For $\{V_i\}_i$ a Weiss cover of the open $U \subset M$,

$$\bigoplus_{i \neq j} \mathcal{F}(V_i \cap V_j) \to \bigoplus_i \mathcal{F}(V_i) \to \mathcal{F}(U) \to 0$$

is an exact sequence on the right and in the middle.

Given a factorization algebra $\mathcal{F}$ on M, its global sections define the *factorization homology* of $\mathcal{F}$ on M, usually denoted by $\int_M \mathcal{F}$.

Observables of a free scalar field theory

Consider a Riemannian manifold (M, g), fields are smooth functions $C^\infty(M)$ and the action functional is quadratic in the fields

$$S(\phi) = \frac{1}{2} \int_M \phi \left(\Delta_g + m^2 \right) \phi \mathrm{dvol}_g$$

Correlation functions are of the form

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \int_\phi d\phi \; \phi(x_1) \cdots \phi(x_n)$$

Therefore, there exists an observable

$$O(x_1, \dots, x_n) : \phi \mapsto \phi(x_1) \cdots \phi(x_n)$$

The divergence operator - 1/2

Observables are defined in terms of co-kernels of some divergence operator.

For it, let

$$\mathrm{Vect}'\left(C^{\infty}\left(U\right)\right) := \mathrm{Sym}\left(C_c^{\infty}\left(U\right)\right) \otimes C_c^{\infty}\left(U\right)$$

then an element of this space is a finite sum of monomials $f_1 \cdots f_n \frac{\partial}{\partial \phi}$ for $f_i, \phi \in C_c^{\infty}\left(U\right)$ acting on functions as

$$f_1 \cdots f_n \frac{\partial}{\partial \phi}\left(g_1 \cdots g_m\right) = f_1 \cdots f_n \sum_i g_1 \cdots \hat{g}_i \cdots g_m \int_U g_i\left(x\right) \phi\left(x\right) \mathrm{dvol}_g$$

Definition

The divergence operator associated to the action functional defined before is given by the linear map

$$\mathrm{Div'} : \mathrm{Vect'}_c\left(C^{\infty}\left(U\right)\right) \to \mathrm{Sym}\left(C_c^{\infty}\left(U\right)\right)$$

$$\mathrm{Div'}\left(f_1 \cdot f_n \frac{\partial}{\partial \phi}\right) = -f_1 \cdot f_n \left(\Delta + m^2\right) \phi + \sum_i f_1 \cdots \hat{f}_i \cdots f_n \int_U \phi\left(x\right) f_i\left(x\right) \mathrm{dvol}_g$$

The divergence operator - 2/2

(!) Since the compactly suported functions over an open set form a topological vector space, we switch to the completed version of the spaces.

[[CG1], Lemma 2.0.2] The divergence operator Div' extends continuously to a linear map

$$\mathrm{Div} : \bigoplus_{n\geq 0} C_c^{\infty}\left(U^{n+1}\right)_{S_n} \to \bigoplus_{n\geq 0} C_c^{\infty}\left(U^{n}\right)_{S_n}$$

This holds by virtue of the fact that Vect'$_c$ is a dense subspace of

$$\mathrm{Vect}_c\left(C_c^{\infty}(U)\right) := \bigoplus_{n\geq 0} C_c^{\infty}\left(U^{n+1}\right)_{S_n}$$

and, similarly, $\mathrm{Sym}_c^{\infty}(U)$ is a dense subspace of

$$P\left(C^{\infty}(U)\right) := \bigoplus_{n\geq 0} C_c^{\infty}\left(U^{n}\right)_{S_n}$$

Quantum observables - 1/2

Definition

[[CG1], Definition 2.0.3]The quantum observables of a free field theory are defined as

$$H^0\left(\mathrm{Obs}^q(U)\right) = \frac{P\left(C^\infty(U)\right)}{\mathrm{Im\ Div}}$$

for $U \subseteq M$ an open set.

(!) This extends to co-chain complexes.

For observables of a free scalar field theory, consider the Gaussian measure

$$\exp\left(-\frac{1}{\hbar}\int_M \phi\left(\Delta + m^2\right)\phi\right)d\phi$$

and define the divergence operator as follows

$$\begin{aligned} \mathrm{Div}_\hbar : \mathrm{Vect}_c\left(C^\infty(U)\right) &\to P\left(C^\infty(U)\right) \\ f_1\cdots f_n\frac{\partial}{\partial\phi} &\mapsto -\frac{1}{\hbar}f_1\cdots f_n\left(\Delta+m^2\right)\phi + \sum f_1\cdots\hat{f}_i\cdots\int_m f\phi \end{aligned}$$

Quantum observables - 2/2

Lemma

[[CG1], Lemma 4.0.1] There exists a prefactorization algebra $H^0\left(Obs^q_\hbar(U)\right)$ *over* $\mathbb{C}[\hbar]$ *such that*

$$H^0\left(Obs^q_\hbar(U)\right) = \begin{cases} H^0\left(Obs^q(U)\right) & \text{for } \hbar = 1 \\ H^0\left(Obs^{cl}(U)\right) & \text{for } \hbar = 0 \end{cases}$$

that assigns to each open set U *the co-kernel of the map*

$$\hbar Div_\hbar : Vect_c\left(C^\infty(U)\right)[\hbar] \to P\left(C^\infty(U)\right)[\hbar]$$

For M a compact Riemannian manifold and $m > 0$, $H^0\left(\mathrm{Obs}^q(M)\right) \cong \mathbb{R}$, and the correlator coincides with the pfa structure map

$$\langle - \rangle : H^0\left(\mathrm{Obs}^q(U_1)\right) \otimes \cdots \otimes H^0\left(\mathrm{Obs}^q(U_n)\right) \to H^0\left(\mathrm{Obs}^q(M)\right) \cong \mathbb{R}$$

for $U_i \subseteq M$ connected disjoint opens.

Sketches of an elephant - 1/2

For a free scalar field theory with action functional $\int_M \phi\Delta\phi$, the solutions of E-L equations are harmonic functions. Namely, the derived space of solutions is given by

$$\mathcal{E}(U) = \left(C^\infty(U) \overset{\Delta}{\to} C^\infty(U)[-1] \right)$$

$\mathcal{E}(M)$ is a sheaf of vector spaces on the site of smooth manifolds.

As before, we define polynomial functions as

$$P(\mathcal{E}(M)) = \oplus_n P_n(\mathcal{E}(M)) = \oplus_n \mathrm{Hom}_{DVS}\left(\mathcal{E}(M)^{\times n}, \mathbb{R}\right)_{S_n} = \oplus_n \mathcal{D}_c\left(M^n, \left(E^!\right)^{\boxtimes^n}\right)_{S_n}$$

Define the bundle $E^! := E^v \otimes \mathrm{Dens}_M$, then

$$\mathcal{E}^!(U) \cong (C_c^\infty(U)[1] \to C^\infty(U))$$

The classical observables are given by the following

$$\mathrm{Obs}^{cl}(U) = \mathrm{Sym}\left(\mathcal{E}_c^!(U)\right) = \mathrm{Sym}\left(C_c^\infty(U)[1] \overset{\Delta}{\to} C_c^\infty(U)\right)$$

Sketches of an elephant - 2/2

Define the folowing sheaf

$$\hat{\mathcal{E}}(U) := \mathcal{E}_c(U) \oplus \mathbb{R} \cdot \hbar$$

with Lie bracket given by

$$[\alpha, \beta] = \hbar \langle \alpha, \beta \rangle$$

then the quantum observables are given by the Chevalley-Eilenberg complex

$$\mathrm{Obs}^q(U) = C_\bullet\left(\hat{\mathcal{E}}(U)\right) = \left(\mathrm{Sym}\left(\hat{\mathcal{E}}(U)[1]\right), d\right) = \left(\mathrm{Obs}^{cl}(U)[\hbar], d\right)$$

1. As graded vector spaces

$$\mathrm{Obs}^{cl}(U)[\hbar] \cong \mathrm{Obs}^q(U)$$

2. The previous iso does not respect differentials!
3. The quantum observables constitute a PFA valued in BD algebras that quantize the P_0- algebras valued pfa of the classical observables. In other words, the Poisson bracket measure the failure for d to be a differential.

References

[CG1] Costello, K. and Gwilliam, O., Factorization Algebras in Quantum Field Theory, Volume 1, 2016, available online at: https://people.math.umass.edu/ gwilliam/vol1may8.pdf;

[CG2] Costello K. and Gwilliam O., Factorization Algebras in Quantum Field Theory, Volume 2, 2020, available at: https://people.math.umass.edu/ gwilliam/factorization2.pdf;

[CG23] Costello, K. and Gwilliam, O., Factorization Algebra, 2023, arXiv:2310.06137;

[PS] Pasqualone, F., Prefactorisation Algebras: Introduction and Examples, MSc Thesis, 2023, available at: https://github.com/Federica-Rike