Axioms for the category of finite-dimensional Hilbert spaces and linear contractions

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THE UNIVERSITY of EDINBURGH

Hilbert spaces

- *Hilbert spaces* are vector spaces with geometry (via an *inner product*)

$$\|x\| = \sqrt{\langle x|x\rangle} \qquad\qquad \cos\theta = \frac{\langle x|y\rangle}{\|x\|\,\|y\|}$$

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- **Con** is the category of Hilbert spaces and linear contractions.
- **FCon** is the full subcategory of finite-dimensional Hilbert spaces.

Dagger categories

- A *dagger category* is a category equipped with a choice of $f^\dagger : Y \to X$ for each $f : X \to Y$, such that

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- Examples include **Con** and **FCon** where the dagger is the ***adjoint***.

Characterisation of Con

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Theorem (Heunen, Kornell and van der Schaaf)

A dagger rig category $(\mathbf{D}, \otimes, I, \oplus, O)$ *is equivalent to* **Con** *if and only if*

1. *O is initial,*
2. $i_1 = (I \cong I \oplus O \xrightarrow{1 \oplus 0} I \oplus I)$ *and* $i_2 = (I \cong O \oplus I \xrightarrow{0 \oplus 1} I \oplus I)$ *are jointly epic,*

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8. *every dagger monomorphism is a kernel,*
9. *every directed diagram has a colimit.*

Goal for today:

A similar characterisation of FCon

Scalars and vectors from contractions

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$$X = \left\{ \tfrac{1}{a} \cdot x \;\middle|\; x \in X, \|x\| \leqslant 1, a \in \mathbb{C}, |a| \leqslant 1, a \neq 0 \right\}$$

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- Construct $\mathbf{C}$ from $\mathbf{D}$ by "formally inverting" the elements of $\mathbf{D}(I, I) \backslash \{0\}$ (like the field of fractions)
- $\mathbf{C}(I, I)$ is an ***involutive field***
- $\mathbf{C}(I, X)$ is an ***orthomodular space*** over $\mathbf{C}(I, I)$ with $\langle x|y\rangle = x^{\dagger}y$.

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Apply to $\mathbf{C}(I, I^{\oplus\mathbb{N}})$ to show that $\mathbf{C}(I, I)$ is $\mathbb{R}$ or $\mathbb{C}$.

SUBGOAL:

A conceptual proof that
the field of scalars of C is $\mathbb{R}$ or $\mathbb{C}$
that does not use infinite dimensionality

Con has *sequential colimits*.

Sequential colimits of contractions

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$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

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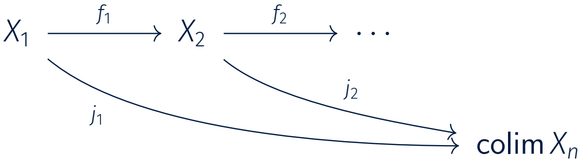

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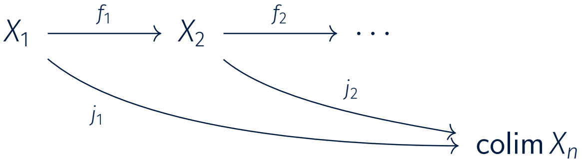

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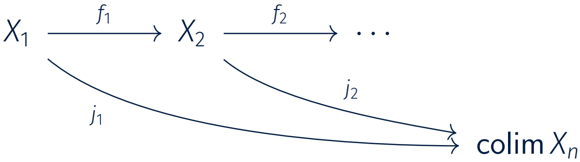

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$$\|j_1 x\| = \inf_{n \in \mathbb{N}} \|f_n \dots f_2 f_1 x\| = \lim_{n \to \infty} \|f_n \dots f_2 f_1 x\|$$

BIG IDEA:

Turn these observations about Con into definitions about D and C.

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$$x^\dagger x \geqslant y^\dagger y \qquad \Longleftrightarrow \qquad y = fx \text{ for some } f \in \mathbf{D}(X, Y)$$

Infima from sequential colimits

$$x_1{}^\dagger x_1 \quad \geqslant \quad x_2{}^\dagger x_2 \quad \geqslant \quad \cdots$$

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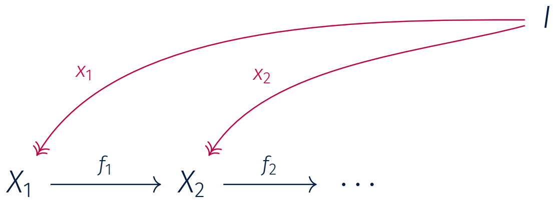

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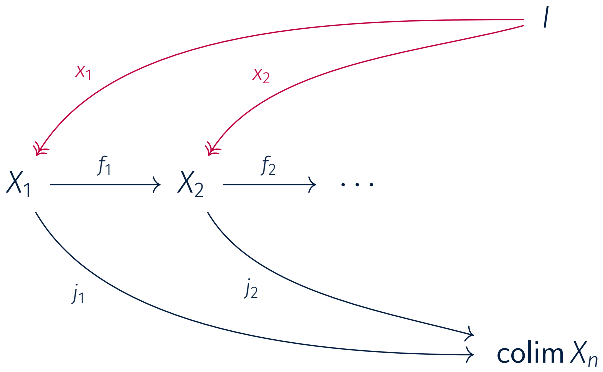

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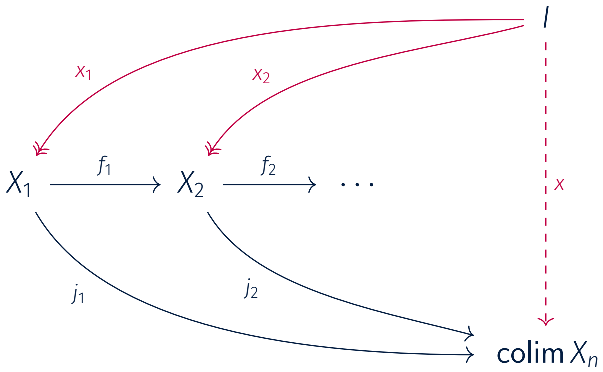
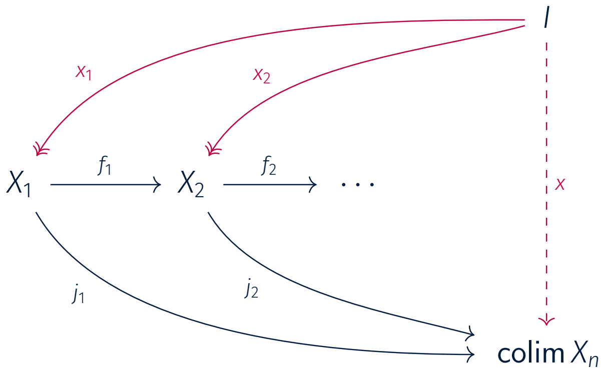


Infima from sequential colimits

$$x_1{}^\dagger x_1 \quad \geqslant \quad x_2{}^\dagger x_2 \quad \geqslant \quad \cdots \quad \geqslant \quad x^\dagger x = \inf_{n\in\mathbb{N}} x_n{}^\dagger x_n$$



Identifying the real or complex numbers

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*Every partially ordered **field** that has infima of bounded decreasing sequences is isomorphic to* $\mathbb{R}$.

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The most challenging part of our work was bridging the gap.

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8. *every dagger monomorphism is a kernel,*
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Finite dimensionality

- The sequential diagram

$$\mathbb{C} \xrightarrow{x_1 \mapsto (x_1, 0)} \mathbb{C}^2 \xrightarrow{(x_1, x_2) \mapsto (x_1, x_2, 0)} \mathbb{C}^3 \xrightarrow{(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, 0)} \cdots$$

  does not have a colimit in **FCon**.

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- An object in **Con** is dagger finite if and only if it is finite dimensional.

Characterisation of FCon

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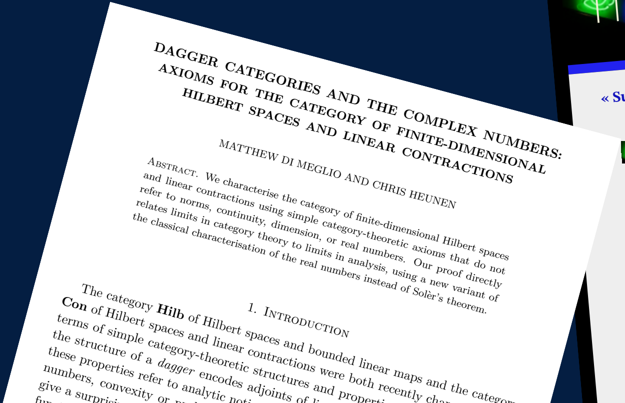
Contact me at m.dimeglio@ed.ac.uk



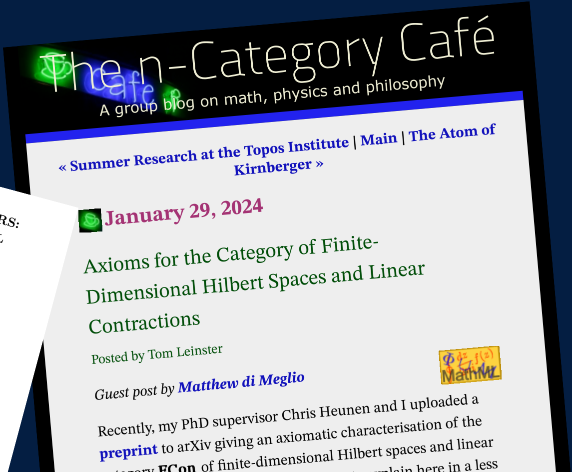
DAGGER CATEGORIES AND THE COMPLEX NUMBERS: AXIOMS FOR THE CATEGORY OF FINITE-DIMENSIONAL HILBERT SPACES AND LINEAR CONTRACTIONS

MATTHEW DI MEGLIO AND CHRIS HEUNEN

ABSTRACT. We characterise the category of finite-dimensional Hilbert spaces and linear contractions using simple category-theoretic axioms that do not refer to norms, continuity, dimension, or real numbers. Our proof directly relates limits in category theory to limits in analysis, using a new variant of the classical characterisation of the real numbers instead of Solèr's theorem.

1. INTRODUCTION

The category **Hilb** of Hilbert spaces and bounded linear maps and the category **Con** of Hilbert spaces and linear contractions were both recently characterised in terms of simple category-theoretic structures and properties. The structure of a *dagger* encodes adjoints of linear maps. These properties refer to analytic notions



All axioms for FCon

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3. ${i_1}^\dagger d \neq 0 \neq {i_2}^\dagger d$ for some $d \colon I \to I \oplus I$,
4. I is dagger simple,
5. I is a monoidal separator,

All axioms for FCon

6. if $x\colon A \to X$ and $y\colon A \to Y$ are epic, then $x^\dagger x = y^\dagger y$ if and only if $y = fx$ for some isomorphism $f\colon X \to Y$,
7. every parallel pair has a dagger equaliser,
8. every dagger monomorphism is a kernel,
9. every bounded sequential diagram has a colimit,
10. every object is dagger finite.

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- The right-shift map $R\colon \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ satisfies

$$R(x_1, x_2, \dots) = (0, x_1, \dots) \qquad \text{and} \qquad R^\dagger(x_1, x_2, \dots) = (x_2, x_3, \dots)$$

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- $\ell_2(\mathbb{N})$ embeds isometrically in all infinite-dimensional Hilbert spaces, so no infinite-dimensional Hilbert space is dagger finite