Introduction to Semigroups for Restriction Category Theorists

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Whether they admit it or not

Cockett and Lack have invented **guarded semigroups**

(g.1**)** $\overline{x}x = x$

(g.2**)** $\overline{x}\,\overline{y} = \overline{y}\,\overline{x}$

(g.3**)** $\overline{\overline{x}y} = \overline{x}\,\overline{y}$

(g.4**)** $x\overline{y} = \overline{xy}\,x$

History of (g.4):

- Semigroups: Fountain 1979 (weaker form)
- Categories: Di Paola and Heller 1987

Let's start by learning something about semi-groups per se.

First of all, is there a spectral theory for semi-groups?

Let $E(S)$ be the set of idempotents of S.

(Learn these notations. They're in the HOMEWORK.)

True or false: Is every itempotent in some maximal subgroup? If so, in how many?

S is a **union of groups** if $\bigcup H_e = S$ so that the maximal subgroups partition S.

S is a **band** if $E(S) = S$. A commutative band is a semilattice.

Clearly a band is a union of groups.

Example: If G is a group and S is any band, $G \times S$ is a union of groups.

$E(S)$ is a poset with $e \leq f$ if $ef = f = fe$.

$a \in S$ is **regular** if $\exists\, x$ with $axa = a$. Such a, x are **inverse** if also $xax = x$. The semigroup S is **regular** if all its elements are.

- On $A \times B$ define $(a, b)(x, y) = (a, y)$. In this semigroup, any two elements are inverse.

- if $axa = a$ then a, xax are inverse.

- a, x are inverse with $ax = xa$ if and only if a, x are inverse in the group H_{ax} so, if such x exists for all a, S is a union of groups.

- Thus a regular commutative semigroup is a union of groups.

S is an **inverse semigroup** if each element has a unique inverse.

Every inverse semigroup is a guarded semigroup if

$$\overline{x} \;=\; xx^{-1}$$

Example: $\mathcal{I}(X) =$ injective partial functions $X \to X$ (f is **injective** means that if xf, yf are defined and equal then $x = y$) is an inverse semigroup with the usual functional inverses.

Contrasting Example: $Pfn(X) =$ all partial functions $X \to X$ with

$$x\overline{f} \;=\; \begin{cases} x & \text{if } x \in Dom(f) \\ \perp & \text{otherwise} \end{cases}$$

is a guarded semigroup.

Theorem. (Wagner, 1952; Preston, 1954) Every inverse semigroup is isomorphic to a subsemigroup of $\mathcal{I}(X)$ which is closed under inverses.

Proof construction: $\rho : S \to \mathcal{I}(S)$, $a \mapsto \rho_a$,

$$x\,\rho_a \;=\; \begin{cases} xa & \text{if } x \in Sa^{-1} \\ \perp & \text{otherwise} \end{cases}$$

But in an inverse semigroup,

$$Sa^{-1} = \{x : x\overline{a} = x\}$$

Using this form of the definition, the same proof construction establishes

Theorem Every guarded semigroup is isomorphic to a subsemigroup of $Pfn(X)$ which is closed under $\overline{f}$.

Consider the equation $\overline{fg}\,\overline{fh} = \overline{f\,\overline{g}\,h}$. Both correspond to $\{x : xfg, xfh$ are defined$\}$ in $Pfn(X)$. Therefore the equation must hold in all guarded semigroups.

Inverse Semigroups $\rightarrow$ *Guarded Semigroups* has two adjoints. The right adjoint (Cockett and Lack) sends S to the subsemigroup

$$I(S) = \{x : \exists a\; xax = x,\; xa = \overline{xa},\; ax = \overline{ax}\}$$

The left adjoint exists because Inverse semigroups are equationally definable with operations xy, x^{-1}

$$\begin{aligned} x(yz) &= (xy)z \\ xx^{-1}x &= x \\ (xy)^{-1} &= y^{-1}x^{-1} \\ xx^{-1}yy^{-1} &= yy^{-1}xx^{-1} \end{aligned}$$

Mark Lawson:

> "...although group theory is certainly concerned with symmetry, it is by no means the case that the converse is true"

He means that without **partial** symmetry, you can't tell the Cantor set from a line segment.

Richard Bellman:

> (from memory) "We all wear such intellectual blinders, it is a wonder that anything ever gets done "

Intellectual Blinder Theorem: S is an inverse semigroup if and only if S is regular and each two of its idempotents commute.

There's lots of study of regular semigroups and their interaction with semigroups in general. See Iséki's theorem in the homework. Here's another.

Theorem. Let S be a regular semigroup. Equivalent are

1. S is a group.

2. (Thierrin, 1951) S is cancellative.

3. S has one idempotent.

Extremal inverse semigroups:

- Semilattices $x = x^2$ (so that $x = x^{-1}$)

- Groups having an inverse means invertible, $xy = 1 = yx$

Both are full epireflective subcategories of inverse semigroups.

- Semilattice reflection add $x = x^2$ to the equational description.

- Group reflection is S/σ (Munn 1961) where $x\sigma y \;\Leftrightarrow\; \exists\, e^2 = e,\; ex = ey$.

Thus, the group reflection of $\mathcal{I}(X)$ is 1.

Extremal guarded semigroups:

- Semilattices $\overline{x} = x$.

- Monoids $\overline{x} = \overline{y}$.

Both are full epireflective subcategories of guarded semigroups.

- Semilattice reflection add the equation $\overline{x} = x$.

- Monoid reflection is S/σ where $x\sigma y \Leftrightarrow \exists\, \overline{e} = e,\ ex = ey$.

A semigroup homomorphism θ **separates idempotents** if $e^2 = e \neq f = f^2 \Rightarrow e\theta \neq f\theta$. Similarly, θ **separates guards** for guarded semigroups.

McAlister's Theorem (1974). Every inverse semigroup S is an idempotent-separating quotient of an inverse subsemigroup of a semidirect product of a group G with a semilattice L. If S is finite, G, L may be taken finite.

Theorem. Every guarded semigroup S is an guard-separating quotient of a guarded subsemigroup of a semidirect product of a monoid M with a semilattice L. If S is finite, M, L may be taken finite.

Green's preorders and equivalence relations in a semigroup.

$x \leq_{\mathcal{L}} y$ if $\exists z, zx = y$. (In N, $x \mid y$).

$x \leq_{\mathcal{R}} y$ if $\exists z, xz = y$.

$x \, \mathcal{L} \, y$ if $x \leq_{\mathcal{L}} y$, $y \leq_{\mathcal{L}} x$.

$x \, \mathcal{R} \, y$ if $x \leq_{\mathcal{R}} y$, $y \leq_{\mathcal{R}} x$.

$x \, \mathcal{H} \, y$ if $x \, \mathcal{L} \, y$, $x \, \mathcal{R} \, y$.

$\mathcal{D} = \mathcal{L}\mathcal{R} = \mathcal{L} \vee \mathcal{R} = \mathcal{R}\mathcal{L}$.

The Clifford-Preston "egg box" picture of a single $\mathcal{D}$-class. Rows are $\mathcal{R}$-classes. Columns are $\mathcal{L}$-classes. The cells are $\mathcal{H}$-classes.

		z		y
		x		w

Use notation L_x for the $\mathcal{L}$-class of x, etc. For $a\,\mathcal{R}\,b$, by definition there exists s, s' with $a = sb$, $b = s'a$. Then $\rho_s : L_a \to L_b$, $x \mapsto xs$ is bijective (with inverse $\rho_{s'}$) and maps a to b and $\mathcal{R}$-classes to $\mathcal{R}$-classes. In particular, all $\mathcal{H}$-classes have the same cardinality.

Proposition (Green, 1951). In any semigroup,

1. If x is regular and $x \mathcal{D} y$ then y is regular.

2. $L_x R_y \subset D_{xy}$.

3. a is regular $\Leftrightarrow$ L_a has an idempotent $\Leftrightarrow$ R_a has an idempotent.

4. If e is idempotent, H_e is a group –the same maximal subgroup as earlier.

5. If H is an $\mathcal{H}$-class, H is a group if and only if $H \cap H^2 \neq \emptyset$.

Example: In an inverse semigroup,

$$
\begin{aligned}
x \,\mathcal{L}\, y &\Leftrightarrow xx^{-1} = yy^{-1} \\
x \,\mathcal{R}\, y &\Leftrightarrow x^{-1}x = y^{-1}y
\end{aligned}
$$

Example: In the endomorphism semigroup X^X of all total functions $X \to X$,

$$
\begin{aligned}
f \,\mathcal{L}\, g &\Leftrightarrow Ran(f) = Ran(g) \\
f \,\mathcal{R}\, g &\Leftrightarrow \text{Ker pair } f = \text{Ker pair } g \\
f \,\mathcal{D}\, g &\Leftrightarrow |Ran(f)| = |Ran(g)|
\end{aligned}
$$

If $u^2 = u$, what is the group H_u? Let $A = \{x : xu = u\}$. The elements of H_u are all f which map A bijectively onto itself and which map X to A. The unique f^{-1} with $ff^{-1} = u = f^{-1}f$ is defined by $xf^{-1} =$unique $a \in A$ with $af = xu$.

In an inverse semigroup, $\overline{x} = xx^{-1}$ is the unique idempotent e with $x \mathcal{R} e$. Definition (Kilp, Fountain, Gomes, Gould 1977 on). A semigroup S is **left ample** if

- Every principal right ideal is projective as a right S-set.

- $\forall\, e^2 = e\ \forall a \ \ eaS = eS \cap aS$.

- Idempotents commute.

Say that $x\,\mathcal{R}^*\,y$ if and only if there exists an isomorphism $S^1x \to S^1y$ of left S-sets. In general, $\mathcal{R} \subset \mathcal{R}^*$.

Theorem. In a left ample semigroup, given x there exists a unique idempotent $x\,\mathcal{R}^*\,\overline{x}$

Left ample semigroups have been axiomatized as a quasivariety in xy, $\overline{x}$ in which $(g.1), (g.2)$ appear explicitly and a weakened form of $(g.4)$ also does.

Theorem. S is left ample $\Leftrightarrow$ S is isomorphic to a subsemigroup of $\mathcal{I}(X)$ closed under $\overline{f}$ $\Leftrightarrow$ S is a guarded semigroup satisfying the equational implication

$$x\overline{a} = x,\ y\overline{a} = y,\ xa = ya \ \Rightarrow\ x = y$$

Out of great kindness, I will spare you the original definition of **weakly left ample** semigroups. Not surprisingly, idempotents commute. You can use this for a definition:

Theorem. A semigroup is weakly left ample if and only if it is a guarded semigroup satisfying

$$e^2 = e \;\Rightarrow\; e = \overline{e}$$

As you can imagine there is a new equivalence relation $\widetilde{\mathcal{R}}$ with $\mathcal{R} \subset \mathcal{R}^* \subset \widetilde{\mathcal{R}}$ and a unique idempotent $x \widetilde{\mathcal{R}} \overline{x}$

I am hoping to get my paper published before somebody comes up with a new equivalence relation $\widehat{\mathcal{R}}_a^*$ and the concept of a **semi-feeble marginally ample** semigroup.

We have forgetful functors

$$\begin{aligned}
\textbf{Groups} &\longrightarrow \textbf{Inverse Semigroups} \\
&\longrightarrow \textbf{Left Ample Semigroups} \\
&\longrightarrow \textbf{Weakly Left Ample Semigroups} \\
&\longrightarrow \textbf{Guarded Semigroups} \\
&\longrightarrow \textbf{Semigroups} \\
&\longrightarrow \textbf{Sets}
\end{aligned}$$

all of which have a left adjoint, many unexplored so far.

Why haven't semigroup theorists discovered guarded semigroups yet?

My guess: A semigroup can be guarded in different ways. Any monoid can be guarded.

However note: A guarded semigroup has a natural partial order (Cockett and Lack)

$$x \leq y \;\Leftrightarrow\; \overline{x}\, y = x$$

and $x \leq y \;\Rightarrow\; axb \leq ayb$.

- The negative cone $N(S) = \{x : \forall a \;\; xa \leq a,\; ax \leq a\}$ is precisely $\{x : x = \overline{x}\}$.

- $\overline{x} = Min\{e \in N(S) : ex = x\}$.

Thus a partially-ordered semigroup can be guarded in at most one way.

A band is **right normal** if it also satisfies $xya = yxa$. In order to win acceptance in the semigroup community, I shall define

A **strongly right normal band** or $g\star$-**band**, is a semigroup with multiplication $x \star y$ and a unary operation $\overline{x}$ satisfying ($g.1, g.2, g.3$) as well as

$$x \star y \;=\; \overline{x} \star y$$

Indeed, a strongly right normal band is a right normal band. There is a forgetful functor from guarded semigroups to strongly right normal bands given by

$$x \star y \;=\; \overline{x}\, y$$

whose left adjoint is so far unexplored.

Alternate equational presentation for guarded semigroups with left unit u, $ux = x$.

- A semigroup xy with left unit u.

- A right normal band $x \star y$, also having u as left unit.

- Marriage equations

$$\begin{aligned} (x \star y)\, z &= x \star (yz) \\ x(y \star u) &= (xy) \star x \end{aligned}$$

The mutually-inverse functors are $\overline{x} = x \star u$; $x \star y = \overline{x}\, y$.

Fix an inf-semilattice L.

How can we construct a $g\star$ band B with $L = \{x \in B : x = \overline{x}\}$?

Let $\mathcal{C}$ be the category of such $g\star$-bands with homomorphisms that fix L.

For $\mathcal{E}$ a topos, let $\mathcal{E}_\star \to \mathcal{E}$ be the algebras of the exception monad $E \mapsto E+1$, i.e. all (E,τ) with $\tau : 1 \to E$ and morphisms that preserve the global elements.

Theorem $\mathcal{C} \cong (\mathbf{Set}^{L^{op}})_\star$.