

CPSC617: Category Theory for Computer Science

Fourth Exercise Sheet

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Do at least 10 of the questions below.

- (1) Prove that the pullback along any map of a retraction is itself a retraction. Show that the pullback of a section along any map is not necessarily a section (hint: can you find a counterexample in Sets?).
- (2) In any category with products prove that:

(a)

$$\begin{array}{ccc}
 C \times B & \xrightarrow{\pi_0} & C \\
 f \times 1_B \downarrow & & \downarrow f \\
 A \times B & \xrightarrow{\pi_0} & A
 \end{array}$$

is always a pullback.

(b)

$$A \xrightarrow{\Delta} A \times A \begin{array}{c} \xrightarrow{\pi_0} \\ \xrightarrow{\pi_1} \end{array} A$$

is always an equalizer.

- (3) An idempotent map $e : X \rightarrow X$ (with $ee = e$) is said to split if $e = rm$ where $r : X \rightarrow Y$ is epic and $m : Y \rightarrow X$ is monic. Prove that

(a) $mr = 1_Y$,

(b) In the following diagram

$$Y \xrightarrow{m} X \begin{array}{c} \xrightarrow{e} \\ \xrightarrow{1_X} \end{array} X \xrightarrow{r} Y$$

m is the equalizer of e and 1_X and r the coequalizer.

- (c) Suppose $r'm' = e$ also with $r' : X \rightarrow Y'$ epic and $m' : Y' \rightarrow X$ monic, then prove that there is a unique isomorphism $\alpha : Y \rightarrow Y'$ such that $m = \alpha m'$ and $e' = \alpha e$.

(d) An idempotent e splits if and only if the diagram

$$X \begin{array}{c} \xrightarrow{e} \\ \xrightarrow{1_X} \end{array} X$$

has either a limit or a colimit.

(e) If e is an idempotent in \mathbf{X} , show that any functor $F : \mathbf{X} \rightarrow \mathbf{Y}$ preserves the limit and colimit of the above diagram.

(4) Prove that if a functor $F : \mathbf{X} \rightarrow \mathbf{Y}$ preserves pullbacks and \mathbf{X} has products that F preserves equalizers (Hint: show that $F(\pi_0), F(\pi_1)$ are jointly monic).

(5) Prove that any category has limits for trees, that is acyclic graphs with a source. These are graphs with an object from which any other object can be reached (in a unique way) following the direction of the arrows.

(6) Prove that any category with equalizers has limits for finite graphs with a source (any other object can be reached from the source in a not-necessarily-unique way following the direction of the arrows).

(7) Prove that if a category has pullbacks it has limits for all finite connected acyclic graphs. These are the graphs which regarding each (directional) arrow as a two-way arrow leaves every object reachable from every other in a unique way.

(8) Prove that if a category has pullbacks and equalizers that every finite connected diagram has a limit. These are the diagrams which regarding each (directional) arrow as a two way arrow leaves every object reachable from every other (possibly in many ways).

(9) A parallel pair of arrows

$$A \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} B$$

is **contractible** in case there is a map $t : B \rightarrow A$ with $td_0 = 1_B$ and $d_1td_1 = d_0td_1$. A contractible pair is contractibly coequalized in case d coequalizes the pair

$$A \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} B \xrightarrow{d} C$$

and there is an $s : C \rightarrow B$ such that $sd = 1_C$ and $td_1 = ds$.

Prove that:

(a) The coequalizer (i.e the colimit) of any contractible pair necessarily contractibly coequalizes the pair.

(b) Any contractible coequalizing map of a contractible pair is the coequalizer (i.e. the colimit).

(c) Any functor preserves coequalizers of contractible pairs.

(10) Prove that for any small category \mathcal{C} the functor category $\mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ is complete and cocomplete.

- (11) Prove the category of matrices over (any field) \mathbb{R} (recall the objects are the natural numbers and the maps $\text{Mat}_{\mathbb{R}}(n, m)$ are $n \times m$ -matrices) is finitely complete and cocomplete. What famous algorithm is the calculation of equalizers?
- (12) Prove that any small category that is complete is a poset.