CPSC617: Category Theory for Computer Science
Third Exercise Sheet

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You should attempt questions 1, 3, 4, 9, 10 and two others of your choice.

(1) (a) Prove that there is a monad $\mathbb{P} = (\mathcal{P}, \eta, \mu)$ on **Sets** given by the functor which takes a set X to the set of all of its subsets, $\mathcal{P}(X)$, and a map $f : X \longrightarrow Y$ to

$$\mathcal{P}(f) : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y); \{x | x \in S \subset X\} \mapsto \{f(x) | x \in S \subset X\}$$

which has unit the map which associates the singleton set with each element and multiplication the maps which takes the union of the set of subsets.

(b) Describe explicitly the maps and composition in the Kleisli category. Prove that this category is isomorphic to the category of relations.

(c) Prove that the inclusion functor of maps into relations has a right adjoint.

(d) Show that the Eilenberg-Moore category, $\mathsf{Set}^{\mathbb{P}}$ is the category of complete semi-Lattices (these are partially ordered set with a "join" operation, $\bigvee_{i \in I} x_i$, such that for all index sets I (including the empty one!)

$$\frac{(x_i \leq z)_{i \in I}}{\bigvee_{i \in I} x_i \leq z.}$$

(2) Prove there is a monad on sets, $\mathbb{E} = (\_ + 1, \eta, \mu)$, given by the functor which adds a single point to each set where the unit inserts the original elements and the multiplication amalgamates the added points (this is the "exception" monad).

Describe the Kleisli category, $\mathsf{Set}_{\mathbb{E}}$, for this monad.

(3) Prove there is a monad on sets, $\mathbb{L} = (L, \eta, \mu)$, given by the list functor together with the unit which inserts singleton lists and the multiplication which flattens the lists (this is the "list" monad).

Describe the Eilenberg-Moore category, $\mathsf{Set}^{\mathbb{L}}$ for this monad.

(4) The Kleisli triple form of a monad (due to Ernie Manes!) is as follows:

Let $\mathbb{X}$ be a category. A Kleisli triple consists of

- An object function T,

- A family of maps for each $A \in \mathbb{X}$, $\eta_A : A \longrightarrow T(A)$,
- A family of functions for each $A, B \in \mathbb{X}$:, $\#(\_) : \mathbf{X}(A, T(B)) \longrightarrow \mathbf{X}(T(A), T(B))$ so that

$$\frac{f : A \longrightarrow T(B)}{\#(f) : T(A) \longrightarrow T(B)}$$

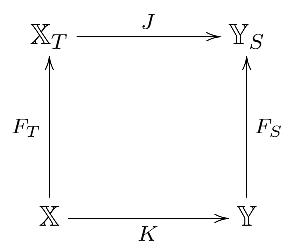
such that the following there identities hold:

- $\eta_A \#(f) = f$
- $\#(\eta_A) = 1_{T(A)}$
- $\#(f)\#(g) = \#(f; \#g))$

Prove carefully that:

(a) T is a functor with $T(f)$ defined to be $\#(f\eta)$,
(b) $\eta : 1_{\mathbb{X}}$ is a natural transformation,
(c) $\mu_A = \#(1_{T(A)})T^2(A) \longrightarrow T(A)$ is a natural transformation,
(d) (T, η, μ) is a monad.

(5) (Harder) Prove that for monads $\mathbb{T} = (T, \eta^T, \mu^T)$ and $\mathbb{S} = (S, \eta^S, \mu^S)$ and their Kleisli categories the following square of functors commute



if and only if there is a "distributive law" that is a natural transformation

$$\alpha : TK \longrightarrow KS$$

(so that on objects $K(T(A)) \xrightarrow{\alpha_A} S(K(A)))$ such that

$$(\eta^T K)\alpha = K\eta^S \qquad (\mu^T K)\alpha = (T\alpha)(\alpha S)(K\mu^S).$$

(6) Prove carefully that the bag monad, $\mathbb{B} = (B, \eta, \mu)$, is really a monad! What are the Eilenberg-Moore algebras for the bag monad?

(7) Prove that the state monad, $\mathbb{S} = (\mathsf{St}, \eta, \mu)$, on Set is a monad. On objects it is defined as $\mathsf{St}(X) = (S \times X)^S$ where

$$\eta : X \longrightarrow (S \times X)^S; x \mapsto \lambda s.(s, x)$$

and

$$\mu : (S \times (S \times X)^S)^S \longrightarrow (S \times X)^S; f \mapsto \lambda s.(\lambda(s', f').f's')(fs)$$

Give an (alternative) description of the Kleisli category. (Harder) what is an Eilenberg-Moore algebra for the state monad?

(8) (Harder) The **filter monad**, $\mathbb{F} = (\mathcal{F}, \eta, \mu)$, on sets is defined as

$$\mathcal{F}(X) = \{U \subseteq \mathcal{P}(X) | X \in U, \forall u, v \in U. \Rightarrow u \cap v \in U, \forall u \in U, v \in \mathcal{P}(X).u \subseteq v \Rightarrow v \in U\}$$

that is $\mathcal{F}(X)$ is the set of **filters** in the powerset of X. A filter is a set of subsets which is upward closed, that is contains all supersets of its members, and contains the intersection of any finite set of its members. In particular this means a filter must contain the full set as it must contain the intersection of the empty set of its members (which is the full set). The set of all subsets – including the empty set – is clearly a filter (a filter is said to be *proper* if it is not this one). The unit of the monad is

$$\eta : X \longrightarrow \mathcal{F}(X); x \mapsto \{X' \subset X | x \in X'\}$$

takes an element x to the principal filter generated by $\{x\}$. Given a map $f : X \longrightarrow \mathcal{F}(Y)$ we may construct a map $f^\sharp : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$ where $f^\sharp(U) = \{Y' \subseteq Y | f^{-1}(\Box Y') \in U\}$ where $\Box Y' = \{V \in \mathcal{F}(Y) | Y' \in V\}$.

Prove that this defines a monad.

(Harder) Prove that the algebras of this monad are precisely continuous lattices! A continuous lattice has all meets (infima) and has joins (suprema) of *directed* sets (recall a subset of a partially ordered set is *directed* in case it is nonempty and every pair of elements u and v from the set is dominated, that is there is a z in the set with $u \leq z$ and $v \leq z$). The morphisms must clearly preserve this structure. Given a continuous lattice X notice that there is a canonical structure map defined by:

$$\nu : \mathcal{F}(X) \longrightarrow X; U \mapsto \bigvee_{u \in U} \bigwedge_{x \in u} x$$

(9) (Harder but even more fun!) The **ultra-filter monad** on sets is defined as

$$\mathcal{U}(X) = \{U \subseteq \mathcal{P}(X) | U \in \mathcal{F}, \forall X' \subseteq X.X' \in U \vee X \backslash X' \in U\}$$

that is $\mathcal{U}(X)$ is the set of ultra filters on X. An ultra-filter is a proper filter which for each subset contains either it or its complement. Clearly this implies that the filter is a maximal proper filter (and actually this characterizes maximal proper filters). As for the filter monad the unit on x picks out the principal filter containing $\{x\}$. The lifting map is defined in the same manner as for filters.

Prove this is a monad. (Harder) Prove that the algebras of this monad are precisely compact Hausdorff spaces! Toward this end it is useful to realize that each compact Hausdorff space comes with with a canonical ultra-filter structure map as each ultra-filter converges on such a space to a unique point. Conversely, the convergence properties of such a space determine it.

(10) Prove that the nonempty list functor $\mathbb{L}^+ = (L^+, \epsilon, \delta)$ is a comonad where the counit takes the first element of the list while the comultiplication takes all the tails of the list. What is the coEilenberg-Moore category, $\mathsf{Set}^{\mathbb{L}^+}$? (Hint: in a forest every element has a unique finite path down to the root of the tree in which it sits.) Describe the adjunction induced by the comonad $F \dashv G : \mathsf{Set}^{\mathbb{L}^+} \longrightarrow \mathsf{Set}$.