Complete at least ten of the following questions.

(1) Describe how the following are natural transformations between functors on \textbf{Sets}:

(a) Flattening a list of lists to a list,
(b) Appending two lists together,
(c) The first projection from the cartesian product,
(d) The diagonal map for the cartesian product,

(2) The \textbf{center} of a category consists of the natural endo-transformations on the identity functor. Prove that the center of any category is a commutative monoid. Is this a functor from categories to commutative monoids \textbf{Cat} \to \textbf{CMon} (think about the situation for groups)?

(3) Prove carefully that \textbf{Poset} can be viewed as a \textbf{Poset}-enriched category.

(4) Prove carefully that the underlying functor from the category of categories to the category of directed graphs has a left adjoint.

(5) Let $\mathbb{R}$ be the real numbers viewed as a category by using the usual ordering; the integers may also be regarded as a category, $\mathbb{Z}$, by using the usual ordering and the usual inclusion of the integers into the reals is a functor. Prove that this functor has a left and right adjoint.

(6) A \textbf{Galois connection} is a contravariant adjunction between posets. That is $f \dashv g : P_1^{\text{op}} \to P_2$.

(a) Show that $g(f(g(x))) = g(x)$ and $f(g(f(y))) = f(y)$ for any Galois connection and that the full subposets determined by $\{x | x = g(f(x))\} \subseteq P_1$ and $\{y | y = f(g(y))\} \subseteq P_2$ are (contra-)isomorphic.

(b) Show that any relation between sets $R \subseteq X \times Y$ induces a Galois connection $f_R \dashv g_R : P(X) \to P(Y)^{\text{op}}$ by

\[
  f_R(X') = \{y \in Y | \forall x \in X'.xRy\} \quad g_R(Y') = \{x \in X | \forall y \in Y'.xRy\}
\]
(c) The original Galois connection works as follows: let $F$ be a field extension of some field $K \subseteq F$ consider the automorphism group, $G$, of field isomorphisms $\alpha$ such that

$$
\begin{array}{c}
K \\
\downarrow \\
F \\
\alpha \\
\uparrow \\
F
\end{array}
$$

then consider the relation $R \subseteq F \times G$ where $R(x, g) \iff x = g(x)$: that is a field element is related to a group element if the group element fixes the field element. Show that the group elements that fix a set of field elements is a subgroup and the field elements fixed by a set of group elements are a subfield ... discuss what this Galois correspondence tells one!

(7) Show that for any adjunction $(\eta, \epsilon) : F \dashv G : X \to Y$ the full subcategories $X_\eta$, with objects $X$ for which $\eta_X$ is an isomorphism, $Y_\epsilon$, with objects $Y$ for which $\epsilon_Y$ is an isomorphism are equivalent.

(8) Show that in any adjunction the following are equivalent

- $\eta_{G(F(X))}$ is an isomorphism,
- $G(F(\eta_X))$ is an isomorphism,
- $\eta_G(Y)$ is an isomorphism,
- $\epsilon_{F(G(Y))}$ is an isomorphism,
- $F(G(\epsilon_Y))$ is an isomorphism,
- $\epsilon_F(X)$ is an isomorphism.

Call an adjunction satisfying any one of these conditions a “Galois adjunction” and conclude that the full subcategories of objects $\{G(F(X))|x \in X\}$ and $\{F(G(Y))|Y \in Y\}$ are equivalent.

(9) Let $f : X \to Y$ be any map of sets then prove that this induces a chain of adjoints $\exists_f \dashv f^* \dashv \forall_f : \mathcal{P}(X) \to \mathcal{P}(Y)$ where

- $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are the powersets (set of all subsets) of $X$ and $Y$ respectively,
- $\exists_f : \mathcal{P}(X) \to \mathcal{P}(Y); S \mapsto \{y \in Y|\exists x \in X \cdot f(x) = y \land x \in S\}$,
- $f^* : \mathcal{P}(Y) \to \mathcal{P}(X); T \mapsto \{x \in X|f(x) \in T\}$,
- $\forall_f : \mathcal{P}(X) \to \mathcal{P}(Y); S \mapsto \{y \in Y|\forall x \in X \cdot f(x) = y \Rightarrow x \in S\}$.

(10) Prove that the functor $A \times _{-} : \text{Sets} \to \text{Sets}$ has a right adjoint (hint: think hom-set).

(11) The category of posets has an obvious inclusion into the category of preorders. Prove that this is a reflection (hint: how do you turn a preorder into an order?).

(12) Prove that the category of finite sets and relations is equivalent to the category of Boolean matrices:

**Objects:** Natural numbers $n \in \mathbb{N}$;
Maps: Boolean matrices \([b_{ij}]_{i=1,...,n} : n \to m;\)

Identity: The diagonal matrix;

Composition: Matrix multiplications with \(\land\) as multiplication and \(\lor\) as addition.

(13) Show that Rel, the category of sets and relations with the ordinary composition, is a poset enriched category (with \(R \leq S\) iff \(R \subseteq S\)). This means it is a 2-category (whose hom objects are posets) and thus we may talk of adjoints. Prove that a relation is a left adjoint in Rel if and only if it is a function.