Please try to complete (at least) 10 questions from the problem set below:

1) The category \(2\) is

\[
\begin{array}{ccc}
A & \xrightarrow{a} & B \\
\downarrow & & \downarrow \\
A & \xleftarrow{1_A} & B
\end{array}
\]

What do the categories \(2 + 2\) and \(2 \times 2\) look like?

2) How many categories are there with 1, 2, and 3 arrows? What about 4, 5, and 6?

3) Prove that all maps in a preorder (regarded as a category) are bijic and that all sections and retractions are isomorphisms.

4) For any category \(C\), define \(\text{Sub}_C(A)\), the category of subobjects of \(A\), to be the category:

- **Objects:** monics \(m : A' \to A\);
- **Maps:** \(f : m_1 \to m_2\) maps in \(C\) such that \(f; m_2 = m_1\);
- **Identities:** \(1_{A'} : m \to m\) as in \(C\);
- **Composition:** As in \(C\).

Prove that \(\text{Sub}_C(A)\) is a preorder.

5) Here is an illustration of how two categories can have the same objects and maps but a completely different composition structure. Consider sets with relations but alter the composition to be:

\[
RS = \{(x, z) | \forall y. (x, y) \in R \lor (y, z) \in S\}.
\]

Prove that this forms a category (what are the identities?).

6) Show that in Sets

(a) a map \(f\) is monic if and only if it is **injective** \((f(x) = f(y) \text{ implies } x = y)\);

(b) a maps \(f\) is epic if and only if it is **surjective** (for every \(y\) in the codomain there is an \(x\) such that \(f(x) = y\));

(c) all epics are retractions;
(d) not all monics are sections;
(e) all bijics are isomorphisms.

Prove that the surjections and injections give a factorization system on \(\textbf{Sets}\).

(7) Prove that the inclusion \(\mathbb{Z} \to \mathbb{Q}\) is bijic in the category of (unital) commutative rings.

(8) (Harder) What are the monics in \(\textbf{Rel}\)? What are the epics in \(\textbf{Rel}\)? Are all bijics isomorphisms?

(9) Prove that if an idempotent is either epic or monic then it is the identity map. Prove that if \(rm = e\), where \(e\) is an idempotent, \(r\) is epic, and \(m\) is monic then the pair \((r, m)\) provides a splitting for the idempotent \(e\). Prove, further, that if \((r, m)\) and \((r', m')\) are any two splittings for \(e\) that there is a unique isomorphism \(\alpha\) such that \(r\alpha = r'\) and \(\alpha s = s'\).

(10) Given an example of two idempotents \(e_1\) and \(e_2\) such that neither \(e_1 e_2\) nor \(e_2 e_1\) are idempotents. Show that if \(e_1 e_2 = e_2 e_1\) (the idempotents commute) then the composite \(e_1 e_2\) is an idempotent.

(11) Show that in \(\textbf{Rel}\) equivalence relations are idempotents: does every equivalence relation split in \(\textbf{Rel}\).

(12) (Harder) Characterize the idempotents in \(\textbf{Rel}\): do all idempotents in \(\textbf{Rel}\) split? Either prove it or provide a counter example!

(13) Prove that \(\text{Mat}(\mathbb{R})\) is a category as defined and that transposition is a functor (actually a converse involution).

(14) Do all idempotents split in \(\text{Mat}(\mathbb{R})\)? Describe the epics: are all epics retractions?

(15) Give three examples of non-trivial (i.e. non-identity) idempotents in the category of tangles.

(16) (Harder) Do all idempotents split in the category of tangles?

(17) Prove that in any category \(\mathcal{F}\), all of whose hom-sets are finite (i.e. it is enriched over finite sets), that

(a) \(\mathcal{F}\) need \text{not} be a finite category (give an example!);
(b) Every monic endomorphism is an isomorphism;
(c) Every epic endomorphism is an isomorphism;
(d) (Harder) For every endomorphism \(g\) there is a (smallest) \(n \in \mathbb{N}^+\) such that \(g^n\) is an idempotent;
(e) (Harder) Each object has an idempotent \(e\) which is minimal, in the sense that for any other idempotent \(e'\) on that object such that \(ee' = e'e\) then \(ee' = e\);
(f) (Harder) An object is \textbf{fully retracted} in case its only idempotent is the identity. Show that if \(A\) and \(B\) are fully retracted objects and \(\mathcal{F}(A, B)\) and \(\mathcal{F}(B, A)\) are non-empty that \(A\) is isomorphic to \(B\);
(g) (Harder) When idempotents split every object has, up to isomorphism a unique full retraction.