CPSC617: Category Theory for Computer Science
First Exercise Sheet

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Please try to complete (at least) 10 questions from the problem set below:

(1) The category $\mathbf{2}$ is

$$1_A \circlearrowright A \xrightarrow{a} B \circlearrowleft 1_B$$

What do the categories $\mathbf{2}+\mathbf{2}$ and $\mathbf{2}\times\mathbf{2}$ look like?

(2) How many categories are there with 1,2, and 3 arrows? What about 4,5, and 6?

(3) Prove that all maps in a preorder (regarded as a category) are bijic and that all sections and retractions are isomorphisms.

(4) For any category $\mathcal{C}$, define $\mathsf{Sub}_{\mathcal{C}}(A)$, the category of subobjects of A, to be the category:

Objects: monics $m : A' \to A$;

Maps: $f : m_1 \to m_2$ maps in $\mathcal{C}$ such that $f; m_2 = m_1$;

Identities: $1_{A'} : m \to m$ as in $\mathcal{C}$;

Composition: As in $\mathcal{C}$.

Prove that $\mathsf{Sub}_{\mathcal{C}}(A)$ is a preorder.

(5) Here is an illustration of how two categories can have the same objects and maps but a completely different composition structure. Consider sets with relations but alter the composition to be:

$$RS = \{(x,z) | \forall y.(x,y) \in R \vee (y,z) \in S\}.$$

Prove that this forms a category (what are the identities?).

(6) Show that in Sets

(a) a map f is monic if an only if it is **injective** ($f(x) = f(y)$ implies $x = y$);

(b) a maps f is epic if and only if it is **surjective** (for every y in the codomain there is an x such that $f(x) = y$);

(c) all epics are retractions;

(d) not all monics are sections;

(e) all bijics are isomorphisms.

Prove that the surjections and injections give a factorization system on Sets.

(7) Prove that the inclusion $\mathbb{Z} \to \mathbb{Q}$ is bijic in the category of (unital) commutative rings.

(8) (Harder) What are the monics in Rel? What are the epics in Rel? Are all bijics isomorphisms?

(9) Prove that if an idempotent is either epic or monic then it is the identity map. Prove that if $rm = e$, where e is an idempotent, r is epic, and m is monic then the pair (r, m) provides a splitting for the idempotent e. Prove, further, that if (r, m) and (r', m') are any two splittings for e that there is a unique isomorphism α such that $r\alpha = r'$ and $\alpha s = s'$.

(10) Given an example of two idempotents e_1 and e_2 such that neither $e_1 e_2$ nor $e_2 e_1$ are idempotents. Show that if $e_1 e_2 = e_2 e_1$ (the idempotents commute) then the composite $e_1 e_2$ is an idempotent.

(11) Show that in Rel equivalence relations are idempotents: does every equivalence relation split in Rel.

(12) (Harder) Characterize the idempotents in Rel: do all idempotents in Rel split? Either prove it or provide a counter example!

(13) Prove that $\mathsf{Mat}(\mathbb{R})$ is a category as defined and that transposition is a functor (actually a converse involution).

(14) Do all idempotents split in $\mathsf{Mat}(\mathbb{R})$? Describe the epics: are all epics retractions?

(15) Give three examples of non-trivial (i.e. non-identity) idempotents in the category of tangles.

(16) (Harder) Do all idempotents split in the category of tangles?

(17) Prove that in any category $\mathcal{F}$, all of whose hom-sets are finite (i.e. it is enriched over finite sets), that

(a) $\mathcal{F}$ need *not* be a finite category (give an example!);

(b) Every monic endomorphism is an isomorphism;

(c) Every epic endomorphism is an isomorphism;

(d) (Harder) For every endomorphism g there is a (smallest) $n \in \mathbb{N}^+$ such that g^n is an idempotent;

(e) (Harder) Each object has an idempotent e which is minimal, in the sense that for any other idempotent e' on that object such that $ee' = e'e$ then $ee' = e$;

(f) (Harder) An object is **fully retracted** in case its only idempotent is the identity. Show that if A and B are fully retracted objects and $\mathcal{F}(A, B)$ and $\mathcal{F}(B, A)$ are non-empty that A is isomorphic to B;

(g) (Harder) When idempotents split every object has, up to isomorphism a unique full retraction.