CPSC617: Category Theory for Computer Science
Fourth Exercise Sheet

J.R.B. Cockett
Department of Computer Science

December 2, 2013

Do the first three questions and one other of your choice!

(1) Prove that the pullback along any map of a retraction is itself a retraction. Show that the pullback of a section along any map is not necessarily a section (hint: can you find a counter-example in Sets?).

(2) In any category with products prove that:

(a)

$$\begin{array}{ccc} C \times B & \xrightarrow{\pi_0} & C \\ {\scriptstyle f \times 1_B}\downarrow & & \downarrow{\scriptstyle f} \\ A \times B & \xrightarrow[\pi_0]{} & A \end{array}$$

is always a pullback.

(b)

$$A \xrightarrow{\Delta} A \times A \overset{\pi_0}{\underset{\pi_1}{\rightrightarrows}} A$$

is always an equalizer.

(3) An idempotent map $e : X \longrightarrow X$ (with $e; e = e$) is said to split if $e = r; m$ where $r : X \longrightarrow Y$ is epic and $m : Y \longrightarrow X$ is monic. Prove that

(a) $m; r = 1_Y$,

(b) In the following diagram

$$Y \xrightarrow{m} X \overset{e}{\underset{1_X}{\rightrightarrows}} X \xrightarrow{r} Y$$

m is the equalizer of e and 1_X and r the coequalizer.

(c) Suppose $r'; m' = e$ also with $r' : X \longrightarrow Y'$ epic and $m' : Y' \longrightarrow X$ monic, then prove that there is a unique isomorphism $\alpha : Y \longrightarrow Y'$ such that $m = \alpha m'$ and $r' = r\alpha$.

(d) An idempotent e splits if and only if the diagram

$$X \underset{1_X}{\overset{e}{\rightrightarrows}} X$$

has either a limit or a colimit.

(e) If e is an idempotent in $\mathbf{X}$, show that any functor $F : \mathbf{X} \longrightarrow \mathbf{Y}$ preserves the limit and colimit of the above diagram.

(4) (Harder) Prove that if a functor $F : \mathbf{X} \longrightarrow \mathbf{Y}$ preserves pullbacks and $\mathbf{X}$ has products that F preserves equalizers (Hint: show that $F(\pi_0), F(\pi_1)$ are jointly monic).

(5) (Harder) Here are some examples of special limits:

(a) Prove that every category has limits for trees, that is acyclic graphs with a source. These are graphs with an object from which any other object can be reached (in a unique way) following the direction of the arrows.

(b) Prove that any category with equalizers has limits for finite graphs with a source (any other object can be reached from the source in a not-necessarily-unique way following the direction of the arrows).

(c) Prove that if a category has pullbacks it has limits for all finite connected acyclic graphs. These are the graphs which regarding each (directional) arrow as a two-way arrow leaves every object reachable from every other in a unique way.

(d) Prove that if a category has pullbacks and equalizers that every finite connected diagram has a limit. These are the diagrams which regarding each (directional) arrow as a two way arrow leaves every object reachable from every other (possibly in many ways).

(6) A parallel pair of arrows

$$A \underset{d_1}{\overset{d_0}{\rightrightarrows}} B$$

is **contractible** in case there is a map $t : B \longrightarrow A$ with $td_0 = 1_B$ and $d_1 t d_1 = d_0 t d_1$. A contractible pair is contractably coequalized in case d coequalizes the pair

$$A \underset{d_1}{\overset{d_0}{\rightrightarrows}} B \xrightarrow{d} C$$

and there is an $s : C \longrightarrow B$ such that $sd = 1_C$ and $td_1 = ds$.

Prove that:

(a) The coequalizer (i.e the colimit) of any contractible pair necessarily contractibly coequalizes the pair.

(b) Any contractible coequalizing map of a contractible pair is the coequalizer (i.e. the colimit).

(c) Any functor preserves coequalizers of contractible pairs.

(Hint: beware my notes there is an typo in the condition!)

(7) Prove that for any small category $\mathcal{C}$ the functor category $\mathsf{Sets}^{\mathcal{C}^{\mathrm{op}}}$ is complete and cocomplete.

(8) Prove the category of matrices over (any field) $\mathbb{R}$ (recall the objects are the natural numbers and the maps $\mathsf{Mat}_{\mathbb{R}}(n, m)$ are $n \times m$-matrices) is finitely complete and cocomplete. What famous algorithm is the calculation of equalizers?

(9) Prove that any *small* category with arbitrary *small* products is a preorder.