Please try to complete (at least) 10 questions from the problem set below:

(1) The category $\mathbf{2}$ is

\[ 1_A \quad \begin{array}{c} a \end{array} \quad B \quad 1_B \]

What do the categories $\mathbf{2} + \mathbf{2}$ and $\mathbf{2} \times \mathbf{2}$ look like?

(2) How many categories are there with 1, 2, and 3 arrows? What about 4, 5, and 6?

(3) Prove that all maps in a preorder (regarded as a category) are bijic and that all sections and retractions are isomorphisms.

(4) For any category $\mathcal{C}$, define $\text{Sub}_{\mathcal{C}}(A)$, the category of subobjects of $A$, to be the category:

- **Objects:** monics $m : A' \to A$;
- **Maps:** $f : m_1 \to m_2$ maps in $\mathcal{C}$ such that $fm_2 = m_1$;
- **Identities:** $1_{A'} : m \to m$ as in $\mathcal{C}$;
- **Composition:** As in $\mathcal{C}$.

Prove that $\text{Sub}_{\mathcal{C}}(A)$ is a preorder.

(5) Here is an illustration of how two categories can have the same objects and maps but a completely different composition structure. Consider sets with relations but alter the composition to be:

\[ RS = \{(x, z) | \forall y. (x, y) \in R \lor (y, z) \in S\} \]

Prove that this forms a category (what are the identities?).

(6) Show that in $\textbf{Sets}$

(a) a map $f$ is monic if and only if it is **injective** ($f(x) = f(y)$ implies $x = y$);

(b) a map $f$ is epic if and only if it is **surjective** (for every $y$ in the codomain there is an $x$ such that $f(x) = y$);

(c) all epics are retractions;
(d) not all monics are sections;
(e) all bijics are isomorphisms.

Prove that the surjections and injections give a factorization system on $\text{Sets}$.

(7) Prove that the inclusion $\mathbb{Z} \to \mathbb{Q}$ is bijic in the category of (unital) commutative rings.

(8) (Harder) What are the monics in $\text{Rel}$? What are the epics in $\text{Rel}$? Are all bijics isomorphisms?

(9) Prove that if an idempotent is either epic or monic then it is the identity map. Prove that if $rm = e$, where $e$ is an idempotent, $r$ is epic, and $m$ is monic then the pair $(r, m)$ provides a splitting for the idempotent $e$. Prove, further, that if $(r, m)$ and $(r', m')$ are any two splittings for $e$ that there is a unique isomorphism $\alpha$ such that $r\alpha = r'$ and $\alpha m' = m$.

(10) Give an example of two idempotents $e_1$ and $e_2$ such that neither $e_1e_2$ nor $e_2e_1$ are idempotents. Show that if $e_1e_2 = e_2e_1$ (the idempotents commute) then the composite $e_1e_2$ is an idempotent.

(11) Show that in $\text{Rel}$ equivalence relations are idempotents: does every equivalence relation split in $\text{Rel}$.

(12) (Harder) Characterize the idempotents in $\text{Rel}$: do all idempotents in $\text{Rel}$ split? Either prove it or provide a counter example!

(13) Prove that $\text{Mat}(\mathbb{R})$ is a category as defined and that transposition is a functor (actually a converse involution).

(14) Do all idempotents split in $\text{Mat}(\mathbb{R})$? Describe the epics: are all epics retractions?

(15) Give three examples of non-trivial (i.e. non-identity) idempotents in the category of tangles.

(16) (Harder) Do all idempotents split in the category of tangles?

(17) Prove that in any category $\mathcal{F}$, all of whose hom-sets are finite (i.e. it is enriched over finite sets), that

(a) $\mathcal{F}$ need not be a finite category (give an example!);
(b) Every monic endomorphism is an isomorphism;
(c) Every epic endomorphism is an isomorphism;
(d) (Harder) For every endomorphism $g$ there is a (smallest) $n \in \mathbb{N}^+$ such that $g^n$ is an idempotent;
(e) (Harder) Each object has an idempotent $e$ which is minimal, in the sense that for any other idempotent $e'$ on that object such that $ee' = e'e$ then $ee' = e$;
(f) (Harder) An object is fully retracted in case its only idempotent is the identity. Show that if $A$ and $B$ are fully retracted objects and $\mathcal{F}(A, B)$ and $\mathcal{F}(B, A)$ are non-empty that $A$ is isomorphic to $B$;
(g) (Harder) When idempotents split every object has, up to isomorphism a unique full retraction.