An Exposition of Sheaves

Sarah Chisholm

January 12, 2009

1 Introduction

This survey serves to introduce sheaves and some basic properties of sheaves from a categorical perspective. The objective is that if one has a basic understanding of category theory that this approach to defining sheaves is more intuitive than the usual approach of a typical Algebraic Geometry text. We will begin by defining presheaves and sheaves. Following this, we will apply the definitions of sheaves to important structures found in Algebraic Geometry such as bundles, germs and stalks. We will follow with the definition of topoi and conclude by showing that the category of sheaves is a topos.

2 Sheaves

We will now build up the definition of a sheaf F on a topological space X. A sheaf enables one to describe a class of functions on X which are for instance continuous and or differentiable. With this notion we can show how a function which is defined on an open set U of X can in fact be restricted to functions defined on a subset V of U and then obtained once again by collating (joining together) such functions restricted to open subsets of U that cover U.

Note that the letter F is commonly used to denote a sheaf. This is likely due to the fact that early work done in this field of mathematics was by the french mathematician Jean-Pierre Serre and the french translation of sheaf is “faisceau.”

For a set X, a topology on X enables one to define continuous functions there. For an open subset U of X, we can determine whether or not a function

$f : U \longrightarrow \mathbb{R}$ is locally continuous. (It may be the case that $f : X \longrightarrow \mathbb{R}$ is not continuous on all of X but we can restrict to a smaller part of X, some subset U, where perhaps f may indeed be continuous, "locally.") That is, if f is continuous and V is an open subset of U, then f restricted to V is continuous, i.e. $f|_V : V \longrightarrow \mathbb{R}$ is continuous. Also, if U_i for $i \in I$, an indexed set, is any open covering of U and the functions $f_i : U_i \longrightarrow \mathbb{R}$ and continuous for all i then we have that there is at most one continuous function f with restrictions $f|_{U_i} = f_i$ for all i and $f_i(x) = f_j(x)$ for all $x \in U_i \cap U_j$ for all $i, j \in I$. This implies that continuous functions are uniquely collatable.

We can express these two properties with a function C. For each open set U of X we define

$$C(U) := \{f \mid f : U \longrightarrow \mathbb{R} \ \textit{continuous}\},$$

i.e. the set of continuous real valued functions on U. The elements f of $C(U)$ are called *sections* of C over U. Also, elements of $C(X)$ are called *global sections.* Now, consider a subset V of U. We can define a map

$$\begin{array}{rcl} \alpha : C(U) & \longrightarrow & C(V) \\ f & \mapsto & f|_V \end{array}$$

where $f|_V$ is defined as above. Also if W is a subset of V (therefore $W \subset V \subset U$) then $(f|_V)|_W = f|_W$, hence restriction is transitive. Thus we can define a contravariant functor C as follows.

$$\left\{ \begin{array}{l} \textit{Category} : \mathcal{O}(X) \\ \textit{Objects} : U \subset X \\ \textit{Maps} : \{V \subset U\} \end{array} \right\} \xrightarrow{C} \left\{ \begin{array}{l} \textit{Category} : \textbf{Sets} \\ \textit{Objects} : C(U) \\ \textit{Maps} : \{C(U) \to C(V); f \mapsto f|_V\} \end{array} \right\}$$

We denote $\mathcal{O}(X)$ as the category with objects as open subsets U of X and the maps $V \to U$ are given by the inclusions $V \subset U$.

At this stage it is possible to give the definition of a presheaf.

Definition 2.1. *A* presheaf *of sets P on a topological space X is a contravariant functor $P : \mathcal{O}(X) \longrightarrow$ **Sets** where the inclusion map $V \longrightarrow U$ of open sets $V \subset U$ gets mapped to $P(U) \longrightarrow P(V); t \mapsto t|V$. Also, if $W \subset V \subset U$, open sets, then $(t|_V)|_W = t|_W$.*

The open set U can be expressed as the union of the open sets U_i which cover U. Note that for $i \in I$, the functions $f_i : U_i \longrightarrow \mathbb{R}$ are in $\prod_i C(U_i)$.

Recall that $f|_{U_i} = f_i$. Define the maps β and γ by $\{f_i\} \mapsto \{f_i|_{(U_i \cap U_j)}\} = \{(f|_{U_i})|_{(U_i \cap U_j)}\}$ and $\{f_i\} \mapsto \{f_j|_{(U_i \cap U_j)}\} = \{(f|_{U_j})|_{(U_i \cap U_j)}\}$ respectively, define the map e by $f \mapsto \{f|_{U_i}\}$ and consider the following diagram.

$$C(U) \overset{e}{\dashrightarrow} \prod_i C(U_i) \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{i,j} C(U_i \cap U_j)$$

By the properties stated above, that $f_i(x) = f_j(x)$ for all $x \in U_i \cap U_j$, and that there is at most one continuous map $f : U \longrightarrow \mathbb{R}$ such that $f|_{U_i} = f_i$ for all i, the map e is the equalizer (the universal map) of β and γ since $e; \beta = e; \gamma$. This leads us to the definition of a sheaf.

Definition 2.2. *A* sheaf *of sets F on a topological space X is a contravariant functor $F : \mathcal{O}(X) \longrightarrow$ **Sets** such that for an open set U of X and an open covering $U = \bigcup_i U_i$, for $i \in I$, there exists an equalizer diagram*

$$F(U) \overset{e}{\dashrightarrow} \prod_i F(U_i) \underset{\gamma}{\overset{\beta}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$$

where for $f \in F(U)$, $e(f) = \{f|_{U_i}\}$ and for a family $f_i \in F(U_i)$, $\beta(f_i) = \{f_i|_{(U_i \cap U_j)}\}$ and $\gamma(f_i) = \{f_j|_{(U_i \cap U_j)}\}$.

Hence a sheaf is a presheaf with an additional condition. This condition is commonly referred to as the *sheaf axiom.*

Considering a more specific case where $X = \mathbb{R}^n$, Euclidean $n-$space, we can look at another situation of sheaves of functions. For an open set U in $\mathbb{R}^n$, define $C^k(U)$ to be the set of functions $f : U \longrightarrow \mathbb{R}$ such that f has continuous partial derivatives of all orders up to and including k. Again, we can define a contravariant functor $C^k : \mathcal{O}(X) \longrightarrow$ **Sets**. In the corresponding equalizer diagram, replacing C with C^k does not pose a problem as differentiability is local, and hence we again obtain an equalizer diagram. Therefore each C^k is a sheaf in $\mathbb{R}^n$ which gives

$$C^\infty \subset \cdots \subset C^k \subset C^{k-1} \subset \cdots \subset C^1 \subset C^0 = C$$

as a nested sequence of subsheaves on $\mathbb{R}^n$. Then we see that for every set U open in X the map $C(X) \longrightarrow C(U)$ is well defined, respects inclusion, and restricts the functions f. This is a contravariant functor and hence a presheaf. Also, with the functions f in $C(U_i)$, a function can be constructed in $C(\bigcup U_i)$ if it is clear how they restrict to each intersection $C(U_i \cap U_j)$, giving the sheaf condition.

We can also define $Sh(X)$, the *category of all sheaves* F of sets on X. The objects are sheaves and the maps are morphism $\alpha : F \Longrightarrow G$ which are natural transformations. Then for all maps (set inclusions) $\varphi : V \longrightarrow U \in \mathcal{O}(X)$ and sheaves F and G we have the following commutative diagram describing the maps in the category $Sh(X)$

$$\begin{array}{ccc} F(U) & \xrightarrow{F(\varphi)} & F(V) \\ {\scriptstyle \alpha_U}\downarrow & & \downarrow{\scriptstyle \alpha_V} \\ G(U) & \xrightarrow{G(\varphi)} & G(V). \end{array}$$

Likewise, the *category of all presheaves* P of sets on X, denoted as $\widehat{\mathcal{O}(X)} := \mathbf{Sets}^{\mathcal{O}(X)^{op}}$ (where the superscript *op* implies that the functors from $\mathcal{O}(X)$ to **Sets** are contravariant.) Again, objects are presheaves, and maps are morphisms $\beta : P \Longrightarrow H$ which are natural transformations. Then for all maps (set inclusions) $\varphi : V \longrightarrow U \in \mathcal{O}(X)$ and presheaves P and H we have the following commutative diagram describing the maps in the category $\widehat{\mathcal{O}(X)}$

$$\begin{array}{ccc} P(U) & \xrightarrow{P(\varphi)} & P(V) \\ {\scriptstyle \beta_U}\downarrow & & \downarrow{\scriptstyle \beta_V} \\ H(U) & \xrightarrow{H(\varphi)} & H(V). \end{array}$$

Then we see that $Sh(X)$ is a full subcategory of $\widehat{\mathcal{O}(X)}$.

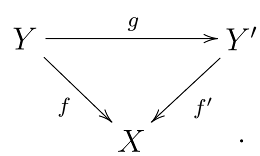
Consider now the following diagram

$$\begin{array}{ccccc} & & F(U_i) & \xrightarrow{F(U_i \cap U_j \subset U_i)} & F(U_i \cap U_j) \\ & \nearrow & \uparrow{\scriptstyle \pi_i} & & \uparrow{\scriptstyle \pi_{i,j}} \\ F(U) & \overset{e}{\dashrightarrow} & \prod_i F(U_i) & \underset{\gamma}{\overset{\beta}{\rightrightarrows}} & \prod_{i,j} F(U_i \cap U_j) \\ & \searrow & \downarrow{\scriptstyle \pi_j} & & \downarrow{\scriptstyle \pi_{i,j}} \\ & & F(U_j) & \xrightarrow[F(U_i \cap U_j \subset U_j)]{} & F(U_i \cap U_j). \end{array}$$

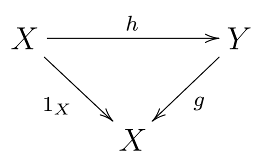
As maps into a product are determined by the composition with the projections of the product, then the maps e, β, and γ are the unique maps making this diagram commute for all $i, j \in I$. This description involving the equalizer diagram enables us to replace the category **Sets** in the definition of sheaves with any other suitable category with all small products. For instance we can describe sheaves of abelian groups, of rings, etc. which is desirable in the field of Algebraic Geometry.

3 Bundles, Germs and Stalks

Consider the slice category of the category of topological spaces and the space X, $\mathbf{Top}/X$. Objects are continuous maps from any space Y to the *base space* space X, i.e. $f : Y \longrightarrow X$, called *bundles* over X. Maps g are objects which for $Y, Y' \in \mathbf{Top}$, $g : Y \longrightarrow Y'$ such that $g; f' = f$, where

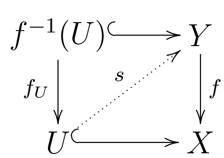


A map $h : X \longrightarrow Y$ such that $h; g = 1_X$ i.e. a map



in the category $\mathbf{Top}/X$, is called a $cross - section$ of a bundle. Also, for each $x \in X$, the inverse image $f^{-1}(x)$ is defined to be the *fibre* of Y over x. In this sense, one can regard a bundle as an indexed family of fibres $f^{-1}(x)$ "glued together" by the topology of Y.

If U is an open subset of X, then f restricts to a map $f_U : f^{-1}(U) \longrightarrow U$ which is again a bundle (over U). Also, the following diagram is a pullback in **Top**



where the continuous map s is a cross-section of the bundle f over U such that $s; f = i_U$. We denote $\Gamma_f(U) := \{s \mid s : U \longrightarrow Y,\ s; f = i_U,\ U \subset X\}$ as the set of all cross-sections s over U. For a subset $V \subseteq U$, $\Gamma_f(U) \longrightarrow \Gamma_f(V)$ is a restriction map and so $\Gamma_f : \mathcal{O}(X) \longrightarrow \mathbf{Sets}$ is a contravariant functor. Also, Γ_f is a sheaf of sets on X called the *sheaf of cross-sections* of the bundle f as a function s on U can be tested whether or not it is locally a cross-section.

If a set U is open in $\mathbb{C}^n$ (where $\mathbb{C}$ denotes the complex numbers) then a function $h : U \longrightarrow \mathbb{C}$ is called *holomorphic* on U if it has a convergent power series expansion in some neighbourhood of each point of U. Therefore, if h is holomorphic on U, $h|_V$ is holomorphic on V. We can say that the property that a function is holomorphic is a local property as h is holomorphic on U if and only if it is holomorphic on every set U_i of an open cover of U. Define $H(U) := \{h : U \to \mathbb{C} | h \textit{ holomorphic}\}$ These properties give us a sheaf H on $\mathbb{C}^n$:

$$\left\{\begin{array}{l} Category : \mathcal{O}(\mathbb{C}^n) \\ Objects : U \\ Maps : \{V \subset U\} \end{array}\right\} \xrightarrow{H} \left\{\begin{array}{l} Category : \mathbf{Sets} \\ Objects : H(U) \\ Maps : \{H(U) \to H(V); h \mapsto h|_V\} \end{array}\right\}.$$

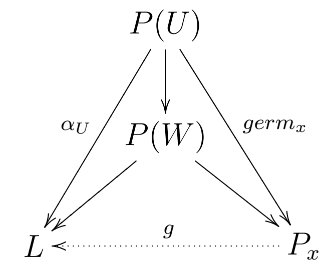
Let $g, h : U \longrightarrow \mathbb{C}$ be holomorphic functions. These functions have the same *germ* at a point $x \in U$ if their power series expansion about x are equal (implying that g and h agree on some neighbourhood of x). In some cases, the condition of a convergent power series expansion may not exist, for instance if the functions map to $\mathbb{R}$, but it may still be the case that the functions can agree in some neighbourhood about a point x. Let $germ_x(h)$ denote the germ of a function h at a point x. Then $germ_x(h) = germ_x(g) \Longrightarrow h(x) = g(x)$ but the converse need not be true.

Let P be a presheaf on a space X, $x \in X$ a point, U and V open neighbourhoods of x and let $g \in P(U)$, $h \in P(V)$. Then $germ_x(g) = germ_x(h)$ if there exists an open set $W \subset U \cap V$ such that $x \in W$ and $g|_W = h|_W \in P(W)$. Two functions having the same germ at a point gives an equivalence relation. The equivalence class of such a function h is denoted by $germ_x(h)$. Denote

$$P_x := \{germ_x(h) \mid h \in P(U),\ x \in U \ open\ X\}$$

as the set of all germs at x. This set is called the *stalk* of P at x. The functor restricting P to the open neighbourhoods of x is denoted by $P^{(x)}$. The functions $germ_x : P(U) \longrightarrow P_x$ form a cone on $P^{(x)}$ as $germ_x(h) =$

$germ_x(h|_W)$ if $x \in W \subset U$ and $h \in P(U)$.



If the set of maps $\{\alpha_U : P(U) \longrightarrow L,\ x \in U\}$ is another cone over $P^{(x)}$, the equivalence relation of germs implies that there exists a unique function $g : P_x \longrightarrow L$ such that $germ_x; g = \alpha$. Thus we have that P_x is the colimit

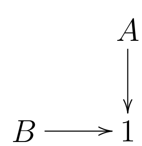
$$P_x = \varinjlim_{x \in U} P(U).$$

4 Topoi

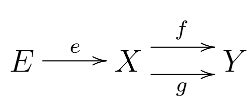
A *topos* is a category $\mathbb{X}$ such that

1. $\mathbb{X}$ has all finite limits and colimits,
2. $\mathbb{X}$ has exponentials,
3. $\mathbb{X}$ has a subobject classifier $1 \longrightarrow \Omega$.

More specifically, a *finite limit* in a category $\mathbb{X}$ means a limit of a functor $G : \mathbb{Y} \longrightarrow \mathbb{X}$ where $\mathbb{Y}$ is a finite category. If $\mathbb{X}$ has a terminal object $\mathbf{1}$ and has all pullbacks, then it has all finite limits. Note that $\mathbb{X}$ has binary products since we can construct $A \times B$ as the pullback of



and it has products of no factors, i.e. of the terminal object. Therefore it has all finite products. Also, the equalizer e of the pair f, g :



can be constructed as a pullback from the maps $(f, g) : f \longrightarrow Y \times Y$ and the diagonal $\Delta : Y \longrightarrow Y \times Y$. Then we have the following commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{e;f=e;g} & Y \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow{(f,g)} & Y \times Y. \end{array}$$

Hence, we see that a category with a terminal object and pullbacks has all finite limits as they can be constructed from finite products and equalizers. A category $\mathbb{X}$ has all finite colimits if it has an initial object $\mathbf{0}$ and pushouts. This is the dual notation of finite limits.

Let $\mathbb{X}$ be a category with products. An *exponential* of objects X and Y of $\mathbb{X}$ consists of an object Y^X and a map $\alpha : Y^X \times X \longrightarrow Y$ such that for any object Z and map $\beta : Z \times X \longrightarrow Y$ there exists a unique map $\gamma : Z \longrightarrow Y^X$ such that $(\gamma \times 1_X); \alpha = \beta$.

$$\begin{array}{ccc} Z \times X & \xrightarrow{\gamma \times 1_X} & Y^X \times X \\ & {\scriptstyle \beta}\searrow \quad \swarrow{\scriptstyle \alpha} & \\ & Y & \end{array} .$$

Consider the category **Sets** and a subset S of an object X. Let $\alpha : S \hookrightarrow X$ be the usual inclusion map. Define $\varphi_S : X \longrightarrow \mathbf{2}$ where $\varphi_S(x) = 0$ if $x \in S$ and $\varphi_S(x) = 1$ if $x \notin S$. This function is commonly known as the characteristic function. We can regard the set $\mathbf{2} = \{0, 1\}$ as the set of truth values. (In this case, we assign 0 as true and 1 as false.) We can also define the map *true* : $\{0\} \hookrightarrow \{0, 1\}$ where $true(0) = 0$. Let β be the unique map from S to the final object. With the above maps defined we can construct a pullback diagram:

$$\begin{array}{ccc} S & \xrightarrow{\beta} & \{0\} \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle true} \\ X & \xrightarrow{\varphi_S} & \{0,1\}. \end{array}$$

Note that for $x \in S$, $\beta(x) = 0$ and $true(\beta(x)) = true(0) = 0$. Also, $\alpha(x) = x$ and $\varphi_S(\alpha(x)) = \varphi(x) = 0$ as $x \in S$. Therefore the diagram commutes.

In a typical category, (i.e. a topos) we define the object Ω to be the corresponding truth values. This leads to the following definition. In a category

$\mathbb{X}$ with finite limits, a *subobject classifier* is the monic map $true : \mathbf{1} \longrightarrow \Omega$ such that for all monics $S \hookrightarrow X$ in $\mathbb{X}$ there exists a unique map φ which, with the given monic, forms a pullback square

$$\begin{array}{ccc} S & \longrightarrow & \mathbf{1} \\ \big\downarrow & & \big\downarrow{\scriptstyle true} \\ X & \overset{\varphi}{\dashrightarrow} & \Omega. \end{array}$$

Each Topos is a cartesian closed category (CCC) as it has all finite products, i.e. a terminal object and products. Also, all objects are exponential, and so the conditions for a CCC are satisfied.

As an example of a topos, consider the category **Sets**. For a category to have all finite limits, we know that it suffices that it has a terminal object and all pullbacks. The terminal object is $\mathbf{1} = \{0\}$. Given sets A, B, C and D such that

$$\begin{array}{ccc} D & \overset{e}{\dashrightarrow} & A \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & \overset{g}{\longrightarrow} & C, \end{array}$$

the pullback of $A \longrightarrow C \longleftarrow B$ is simply the set of ordered pairs $D = \{(a, b) \mid a \in A, b \in B, f(a) = g(b) \in C\}$. Therefore, **Sets** has all finite limits. It also has all finite colimits by duality. The exponential Y^X in **Sets** is simply the set of all functions from the set X to the set Y. As we have seen previously, the subobject classifier in **Sets** is the map $true : \{0\} \longrightarrow \{0, 1\}$. Thus, the category **Sets** is a topos.

5 Sheaves as Topoi

Before showing that the category of sheaves is a topos we will define the notions of a sieve and a subfunctor. Let $\mathbb{X}$ be small category. A *subfunctor* of the contravariant functor $G : \mathbb{X} \longrightarrow$ **Sets** is a functor $H : \mathbb{X} \longrightarrow$ **Sets** such that for $C \in \mathbb{X}$, $H(C) \subset G(C)$ and for $f : D \longrightarrow D \in \mathbb{X}$, $H(f) : S(D) \longrightarrow S(C)$ is a restriction of $g(f)$, for a subset S of X. A *subsheaf* of a sheaf F on X is a subfunctor of F which is also a sheaf. A *sieve* S on U is defined to be a subfunctorof $Hom(\_, U)$. Also, a sieve S on U is a *covering sieve* for U if U is the union of all open sets V in S. Now the first condition that we require to show that $Sh(X)$ is a topos is stated in the proposition below.

Let X be any space. Then for each open set U in X there is a presheaf $\mathbf{y}(U) := Hom(\_, U)$ defined for each open set V such that $Hom(V, U) = \mathbf{1}$ if $V \subset U$ and $Hom(V, U) = \emptyset$ if $V \not\subset U$.

Proposition 5.1. *For any space X, the category $Sh(X)$ has all small limits and the inclusion of sheaves into presheaves preserves the limits.*

Proof. Consider two maps $F \rightrightarrows G$ of sheaves and take their equalizer $E \longrightarrow F \rightrightarrows G$ as presheaves. Let P denote a presheaf. As the set of functors $Hom(P, \_)$ preserve limits, it also preserves equalizers. Let S be a covering sieve for U and consider the following commutative diagram.

$$\begin{array}{ccccc} Hom(\mathbf{y}(U), E) & \longrightarrow & Hom(\mathbf{y}(U), F) & \rightrightarrows & Hom(\mathbf{y}(U), G) \\ \cong \big\downarrow \alpha & & \cong \big\downarrow \beta & & \cong \big\downarrow \gamma \\ Hom(S, E) & \longrightarrow & Hom(S, F) & \rightrightarrows & Hom(S, G). \end{array}$$

The rows are equalizers in **Sets** and the vertical maps are induced by the inclusion map $S \hookrightarrow \mathbf{y}(U)$. The maps β and γ are isomorphisms because F and G are sheaves, which gives two left exact sequences of modules. Therefore α is also an isomorphism and so E is a sheaf. As E is the equalizer in presheaves, it is clear that it is also the equalizer in sheaves. $\square$

Corollary 5.2. *A subobject of a sheaf F in the category $Sh(X)$ is isomorphic to a subsheaf of F.*

Proof. Let G be the subobject of F with the injective map $g : G \rightarrowtail F$ in $Sh(X)$. Recall that a map g is monic if and only if it satisfies the pullback diagram

$$\begin{array}{ccc} G & \xrightarrow{1_G} & G \\ 1_G \big\downarrow & & \big\downarrow g \\ G & \underset{g}{\rightarrowtail} & F \end{array}$$

in $Sh(X)$. By Proposition 5.1, this is a pullback in the category of presheaves on X, $\widehat{\mathcal{O}(X)}$, as well. Recall also that in $\widehat{\mathcal{O}(X)}$ pullbacks are computed pointwise. Therefore the map g is computed pointwise. This implies that each set $G(U) \cong S(U)$ for a subset $S(U) \subset F(U)$, where S is a subfunctor of F, and so $G \cong S$. Because G is a sheaf, we have that S is a sheaf as well. $\square$

Proposition 5.3. *If F is a sheaf and P is a presheaf both of sets on the space X, then the presheaf exponential F^P is a sheaf.*

Proof. Note that the sheaf F and the presheaf P are functors from the category $\mathcal{O}(X)$ to the category **Sets**. The definition of F^P applied to the open set U is

$$F^P(U) := Hom(\mathbf{y}(U) \times P, F)$$

where the hom-sets are natural transformations and $\mathbf{y}(U)$ is the representable presheaf $Hom(\_, U)$ given by U. By the definition of $\mathbf{y}(U)$, $\mathbf{y}(U)(V)$ is $\mathbf{1}$ or $\emptyset$ depending upon whether or not V is a subset of U respectively. This allows us to describe $F^P(U)$ in an alternative manner. $F^P(U) \cong Hom(P|_U, F|_U)$ for contrvariant functors P and F restricted to $\mathcal{O}(U)$.

Note that $F^P(U)$ is a functor where for $V \subset U$ then the natural transformation $\alpha : P|_U \Longrightarrow F_U$ restricts to $\alpha|_V : P|_V \Longrightarrow F|_V$ and hence $F^P(U)$ is also a presheaf. To see that F^P also satisfies the sheaf axiom, let $U = \bigcup_i U_i$ be a covering and $\beta_i : P|_{U_i} \Longrightarrow F|_{U_i}$ a natural transformation for each i. The maps β_i can be collated together to form $\beta : P|_U \Longrightarrow F|_U$ by collating their values which lie in the sheaf F. Hence F^P is also a sheaf. □

For the property concerning subobject classifiers, define Ω to be a presheaf on X where for each open subset $U \subset X$,

$$\Omega(U) = \{W \mid W \subset U,\ W \text{ open in } X\}.$$

The maps are given by intersections as follows:

$$\left\{ \begin{array}{l} Category : \mathcal{O}(X) \\ Objects : U \\ Maps : \{V \subset U\} \end{array} \right\} \xrightarrow{\Omega} \left\{ \begin{array}{l} Category : \mathbf{Sets} \\ Objects : \Omega(U) \\ Maps : \{\Omega(U) \to \Omega(V); W \mapsto W \cap V\} \end{array} \right\}.$$

Theorem 5.4. *For any topological space X, the presheaf Ω is a sheaf on X and is a subobject classifier for $Sh(X)$.*

Proof. Let $U = \bigcup_i U_i$ be an open covering of U. For open sets $V_i \subset U_i$ for every i such that $V_i \cap U_j = V_j \cap U_i \subset U_i \cap U_j$ for all i and j, there exists a unique open set $V = \bigcup_i V_i$ such that $V \cap U_i = V_i$ for all i. This gives the required sheaf axiom and so Ω is a sheaf.

Let $S \subset F$ be a subobject of a sheaf F. By Corollary 5.2, S is a subsheaf of F. Then each object $S(U)$ is a subset of $F(U)$. Consider the natural transformation $\alpha : F \Longrightarrow \Omega$ and define $\alpha_U : F(U) \Longrightarrow \Omega(U)$ as follows. If $x \in F(U)$

then $\alpha_U(x) = W = \bigcup_i W_i$ where each $W_i \subset U$ such that $x|_{W_i} \in S(W_i)$. As S is a subsheaf, then $x|_W \in S(W)$. It is clear that the map α_U is a natural transformation in U. Now consider the following pullback diagrams

$$\begin{array}{ccc} P & \dashrightarrow & \mathbf{1} \\ \vdots & & \downarrow{\scriptstyle true} \\ F & \xrightarrow[\alpha]{} & \Omega. \end{array}$$

$$\begin{array}{ccc} P(U) & \dashrightarrow & \mathbf{1} \\ \vdots & & \downarrow \\ F(U) & \xrightarrow[\alpha_U]{} & \Omega(U). \end{array}$$

where *true* is defined to be the map which for each U, takes the point of $\mathbf{1}$ to the maximum element $U \in \Omega(U)$. Note that the pullback is taken pointwise. Then for every open set U, $P(U)$ is produced as the subset of $F(U)$ consisting of points $x \in F(U)$ such that $\alpha_U(x) = U$. Therefore the pullback P is the subsheaf S. Also, if S is the pullback of the map *true* along any map from the sheaf F to the sheaf Ω, then α is unique. Thus, the sheaf Ω together with the map *true* is a subobject classifier. □

As $Sh(X)$ contains all small limits, it has all small colimits by duality. Therefore, the category of sheaves $Sh(X)$ is a topos.

References

[1] Steve Awodey. *Category Theory.* Clarendon Press, Oxford, 2006.

[2] David Dummit & Richard Foote. *Abstract Algebra* Second Ed. Prentice Hall, NJ, 1991.

[3] Robin Hartshorne. *Algebraic Geometry* Graduate Texts in Mathematics. Springer, USA, 1977.

[4] Saunders Mac Lane. *Categories for the Working Mathematician.* Springer-Verlag, NY, 1971.

[5] Saunders Mac Lane & Ieke Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* Springer-Verlag, NY, 1992.

[6] Davis Mumford. *The Red Book of Varieties and Schemes.* Lecture Notes in Mathematics. Second Edition. Springer, Germany, 1999.