# Explorations in the category of combinatorial games

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**Abstract**

ABSTRACT

## 1 Introduction

In combinatorial game theory, we consider games of the following form.

1. Play alternates between two players, *Left* and *Right*.
2. There is no hidden information.
3. Any play of the game terminates after a finite number of moves.
4. The player whose turn it is to move when no move is possible, is said to *lose* the game. The other player has *won*.

A game may have options available to both Left and Right. A game $x$ may be identified with its sets of left and right options. The Conway notation is to write the game as $\{ x^L \mid x^R \}$ where one can think of $L$ as indexing over the Left options, and $R$ as indexing over the Right. An important concept which led to considerably greater understanding of these games is the *disjunctive sum*, written as $x + y$. When playing the sum of games, players choose whether to move in the component $x$ or the component $y$. In Conway's notation, this becomes

$$x + y = \left\{ x^L + y, x + y^L \mid x^R + y, x + y^R \right\}.$$

This definition is recursive; it relies on the fact that since plays terminate in a finite number of moves, when unfolding the definition the left or right options of $x$ and $y$ are eventually empty. The game with empty option sets on both sides, $\{ \mid \}$, is usually written 0, since $0 + x = x$ for any $x$. Also of importance is the concept of the *negation* of a game, where we will write

$$-x = \left\{ -x^R \mid -x^L \right\}.$$

Thus the negative of a game is obtained by switching the roles of the players, Left and Right. Note that $x - x$ is not identical to 0 as you might expect, but there is an important sense in which it is isomorphic to 0. If you are playing second in the game $x - x$, and your opponent makes some move in a component, you will always be able to respond by making the "mirror image" move in the opposite component. That is, if he takes $x^R$ in $+x$, you can respond with $-x^R$ in $-x$, leaving the game $x^R - x^R$. By induction, you can guarantee for yourself the last move in this game, and so you will win[1]. Likewise, the second player wins $0 = \{ \mid \}$ since the first player, immediately now, has no options to play.

Although the games came logically first, an important subclass, the *surreal numbers* appeared earlier in print. They were given that name by Knuth[1][2] who was working from an earlier University of Calgary research paper by Conway[3]. The full combinatorial theory of games eventually appeared in *Winning Ways*[4], a seminal work whose multi-decade development caused it to be beaten to print by *On Numbers and Games*[5].

## 1.1 Joyal's category

It was after reading this last book by Conway that André Joyal realized[6] that combinatorial games formed a category. The objects of the category are simply games, and the arrows between them are strategies. Some explanation is in order. A strategy $s : x \to y$ can be thought of as a winning strategy for Left, playing second, in the game $y - x$. The identity $1_x : x \to x$ is simply the copycat strategy mentioned earlier, since this clearly exists and gives a second-player win for any game $x$.

[1] In different cultures, this is known as the *copycat* strategy, the *mirroring* strategy, or the *Tweedledum-Tweedledee* strategy.

[2] Knuth was then writing in distraction from his great work, *The Art of Computer Programming*[2]. In a slightly later distraction, he would get fed up with the state of computer typesetting and develop TEX. Since this author first heard of surreal numbers via a Martin Gardner article describing Knuth's work, this paper would be doubly nonexistent if not for Knuth's diversions.

It may not, however, be completely obvious that strategies can *compose* in any meaningful sense. However, they do, in a rather nice way. If we have strategies $s : x \to y$ and $t : y \to z$, we will (here, informally) create the strategy $st$[3] from $x$ to $z$ as follows.

Suppose Right begins by selecting an option $z^R$ of $z$. Since we have a map $y \to z$, the only thing we know how to do is to give a response in the game $z^R - y$. If this response is in the $z$ component, we are fine, but if it is in $-y$ then we are in trouble, since the component $-y$ does not appear in the game we are actually playing. So what we will do is imagine that we are playing the game

$$z - y + y - x,$$

where we keep in mind that the $y$ components are actually "virtual". If our strategy calls for us to make a move in $-y$, we can just pretend to make a move there. Now we will pretend that our opponent makes the copycat move in the component $+y$. But now we have a strategy on $y - x$, which tells us what move to make next. If this is in $-x$, then we can happily make that move on the real game. If not, then we again pretend to make a move in the $y$ component, pretend our opponent plays the mirror image across to the $-y$ component, and now use our strategy in $z - y$. We may have to bounce back and forth across these virtual games several times before making a real move, but this must eventually happen, since eventually there will be nothing left of the virtual components, anyway.

Using the strategy $s$ guarantees there are an even total of moves in $y - x$, and using the strategy $t$ guarantees an even number of moves in $z - y$. Pretending that our opponenet always used the copycat strategy, $1_y$ across the virtual components guarantees an even number of moves in $-y + y$ as well. Thus, the total number of moves in $z - x$ is even; since we went second, we have the last move and thus win the game.

The proof that this composition is associative is notoriously difficult, and we leave it for *!—-TODO—-!*!

[3]We use the convention that function application goes from left to right, so the map $st$ is the map often written as $t \circ s$. This is often simpler to read, particularly when comparing map compositions to diagrams.

## 2 A game logic

As Andreas Blass has argued[7] that a two player game semantics is "necessary" for understanding linear logic, we may equally consider what logic underlies a game theory. The concept of a categorical *module* (or bifunctor) has proven helpful in the context of other kinds of games[8], and has a nice interpretation in this category. A module can be thought of as a collection of arrows between different categories, which composes with arrows on the left and the right. Because of the structure of combinatorial games, we are interested primarily in an *endomodule*, a module from a category to itself. In this case it is important to note that the arrows of the module (also called *cross-arrows* or *cross-maps*) do not compose with each other.

In our context, a cross-arrow $x \nrightarrow y$ can be considered a strategy for Left to win the game $y - x$, playing *first*.

The first rule in our logic will be the identity, since we clearly always have a map from an object to itself.

$$\frac{}{1_x : x \to x} \quad \text{identity}$$

In order to create a cross-map $x \nrightarrow y$, we will need a good first option, either $x^R$ or $y^L$, and a (plain) map $x^{R_1} \to y$ or $x \to y^{L_1}$. This makes sense because to have a winning strategy playing first is to have a winning strategy playing second after a good first move. This gives two rules, known as injection and projection.

$$\frac{s : x \to y^{L_1}}{\overrightarrow{y}^{L_1} \cdot s : x \nrightarrow \{\, y^L \mid y^R \,\}} \quad \text{injection}$$

$$\frac{s : x^{R_1} \to y}{\overleftarrow{x}^{R_1} \cdot s : \{\, x^L \mid x^R \,\} \nrightarrow y} \quad \text{projection}$$

Symbols such as $s$ and $\overleftarrow{x}^{R_1}$ can be considered terms in the logic. The left arrow above the $\overleftarrow{x}$ is to emphasize that the move is made in the component on the left-hand side; without this notational cruft we may see some redundancy when describing moves in $x \to x$.

The last primary logic rule is bitupling, which creates normal maps. Since $s : x \to y$ is a strategy on $y - x$, we must have a good response to any left option of $x$, and any right option of $y$. A good response here means a winning strategy playing first; thus, the rule is

$$\frac{(s^L : x^L \nrightarrow \{\, y^L \mid y^R \,\})_{x^L} \, (s^R : \{\, x^L \mid x^R \,\} \nrightarrow y^R)_{y^R}}{s = \{\, s^L \mid s^R \,\} : \{\, x^L \mid x^R \,\} \to \{\, y^L \mid y^R \,\}} \quad \text{bituple.}$$

We also need cut rules to show that arrows compose as expected. These take the form

$$\frac{s{:}x\to y \quad t{:}y\to z}{st{:}x\to z},$$

$$\frac{s{:}x\to y \quad t{:}y\nrightarrow z}{st{:}x\nrightarrow z},$$

and

$$\frac{s{:}x\nrightarrow y \quad t{:}y\to z}{st{:}x\nrightarrow z}.$$

However, we can use cut-elimination to remove these. The plain maps compose as follows.

$$\frac{\dfrac{\left(s^L{:}x^L\nrightarrow y\right)_{x^L}\left(s^R{:}x\nrightarrow y^R\right)_{y^R}}{s{:}x\to y} \qquad \dfrac{\left(t^L{:}y^L\nrightarrow z\right)_{y^L}\left(t^R{:}y\nrightarrow z^R\right)_{z^R}}{s{:}y\to z}}{st{=}\left\{\, s^L t \;\middle|\; st^R \,\right\}{:}x\to z}$$

We need two derivations for each of the cross-cuts.

$$\frac{s{:}x\to y \qquad \dfrac{t{:}y\to z^{L_1}}{\overrightarrow{z}^{L_1}\cdot t{:}y\nrightarrow z}}{s;\overrightarrow{z}^{L_1}\cdot t{=}\overrightarrow{z}^{L_1}\cdot st{:}x\nrightarrow z} \quad or \quad \frac{\dfrac{\left(s^L{:}x^L\nrightarrow y\right)_{x^L}\left(s^R{:}x\nrightarrow y^R\right)_{y^R}}{s{:}x\to y} \qquad \dfrac{t{:}y^{R_1}\to z}{\overleftarrow{y}^{R_1}\cdot t{:}y\nrightarrow z}}{s;\overleftarrow{y}^{R_1}\cdot t{=}s^{R_1}t{=}s(\overrightarrow{y}^{R_1})t{:}x\nrightarrow z}.$$

The derivations for the other cross-cuts are dual to these.

## 2.1 Examples

One thing these logic rules give us is a way to effectively construct new game categories from arbitrary categories. The simplest category is $\emptyset$, the category with no objects. Applying our rules once to this, we get $\mathfrak{G}(\emptyset)$, the category of games with options drawn from $\emptyset$. Since there are no options in $\emptyset$, that gives us just one game, $\{\ |\ \}$. Let's call this game 0. We can easily derive $0 \to 0$ from the bitupling rule, since trivially there are neither left nor right options in 0.

Applying the rules another time gives us $\mathfrak{G}^2(\emptyset)$. The collection of objects in this category is effectively $\{0, \{0\mid\ \}, \{\ \mid 0\}, \{0\mid 0\}\}$[4]. There are standard names given for these new games:

$$\begin{aligned} \{0\mid\ \} &= 1, \\ \{\ \mid 0\} &= -1, \\ \{0\mid 0\} &= *. \end{aligned}$$

[4]Technically, in the logic as written we might distinguish between $\{0\mid\ \}$ and $\{0,0\mid\ \}$, giving us infinitely many objects. However, to keep things simple we will simply note that the difference between these games is not significant from a game theoretical perspective.

Without naming all the strategies yet, the relationships among these objects are as:

$$-1 \overset{\rightarrow}{\nrightarrow} 0 \overset{\rightarrow}{\nrightarrow} 1$$
$$-1 \overset{\rightarrow}{\nrightarrow} * \overset{\rightarrow}{\nrightarrow} 1$$
$$0 \nrightarrow * \nrightarrow 0$$

with the compositions as well. We can easily derive, for an example, $0 \nrightarrow *$:

$$\frac{\emptyset : 0 \to 0}{\vec{0} \cdot \emptyset : 0 \nrightarrow *} \quad \text{injection},$$

and we can derive $* \nrightarrow 0$ from the projection rule as well. So, Left should win the games $* - 0$ and $0 - *$ playing first. Because we cannot derive $0 \to *$ or $* \to 0$ (we would need some cross-map $0 \nrightarrow *$ or $* \nrightarrow 0$) Left does not win these games going second.

In fact, Right wins (playing first) if Left plays second. We can think of $\mathfrak{G}^2(\emptyset)$ as a *polarized category* which is effectively a module $\mathfrak{G}^2(\emptyset) \nrightarrow \mathfrak{G}^2(\emptyset)$. The *polarized dual* of this is $\mathfrak{G}^2(\emptyset)^* \nrightarrow^* \mathfrak{G}^2(\emptyset)^*$. Since we can easily identify $x^*$ with $-x$ for any $x$ in this game category, we see that $0 \nrightarrow *$ becomes $0 \nrightarrow^* *$ in the dual, which is the module from Right's perspective. Therefore Right, going first, wins.

Likewise, we can easily derive $0 \to 1$, and then take the polarized dual to see $0^* \to^* 1^*$, or $0 \to^* -1$, as we might expect. So the polarized dual really is just the module from the other player's perspective.

If we continue this construction from the logic for all ordinals $\Omega$, we get $\mathfrak{G}^\Omega()$, the standard category of combinatorial games with the added module structure. What happens if we start with a different structure?

It's an unfortunate fact that this logic, as written, does not allow us to find out much about the left or right options of an object, unless that object was itself created using this game construction[5]. So if we have an object in a category which was not constructed this way, we consider it *atomic*. We can think of an atom as a kind of game where we are ignorant of its structure,

[5]In particular, we would be interested in growing the theory of *loopy* games with this logic. Loopy games may not be well-founded, "containing" themselves as left or right options. Of particular importance are the games on $= \{\text{on} \mid \}$, off $= \{ \mid \text{off}\}$, and dud $= \{\text{dud} \mid \text{dud}\}$.

and therefore cannot derive all the strategies we might otherwise have been able to.

Let's begin with a category $\{A\}$, with one atom, $A$, and one arrow, $1_A : A \to A$. This looks similar to the category $\mathfrak{G}(\emptyset)$, but as we will see, $\mathfrak{G}(A)$ is quite different from $\mathfrak{G}^2(\emptyset)$. The objects in $\mathfrak{G}(A)$ are $\{A, 0, \{\ \mid A\}, \{A \mid\ \}, \{A \mid A\}\}$, and there are strategies as follows.

$$\begin{aligned} \{\ \mid A\} &\to 0 \to \{A \mid\ \}, \\ \{\ \mid A\} &\to \{A \mid A\} \to \{A \mid\ \}, \\ \{\ \mid A\} &\nrightarrow A \nrightarrow \{A \mid\ \}, \\ A &\nrightarrow \{A \mid A\} \nrightarrow A, \end{aligned}$$

and strategies from composition. This module is considerably poorer, because of our ignorance of the "structure" of $A$.

## 3 Idempotents

Suppose we have an idempotent strategy $e : x \to x$; that is, a strategy such that $ee = e$. What kind of moves might $e$ call for? We know it must have a good response for all moves $\overleftarrow{x}^L$ and $\overrightarrow{x}^R$ available to Right. The trivial possible responses are copycat moves, that is

$$\begin{aligned} e &: \overleftarrow{x}^L \mapsto \overrightarrow{x}^L \cdot e_{x^L}, \qquad \text{or} \\ e &: \overrightarrow{x}^R \mapsto \overleftarrow{x}^R \cdot e_{x^R} \end{aligned}$$

for some Right moves $\overleftarrow{x}^L$ and $\overrightarrow{x}^R$. Here the maps $e_{x^L}$ and $e_{x^R}$ must be idempotents on $x^L$ and $x^R$ respectively. These kinds of moves compose in an idempotent way, since we map across as

$$\overleftarrow{x}^L \overset{e}{\mapsto} \overrightarrow{x}^L \overset{1_x}{\mapsto} \overleftarrow{x}^L \overset{e}{\mapsto} \overrightarrow{x}^L,$$

and leave the strategy $e_{x^L}^2 = e_{x^L}$.
With that out of the way, we have two types of non-copycat responses, either in the same component, or the opposite one. Responses to the opposites component will take the forms

$$\begin{aligned} e &: \overleftarrow{x}^L \mapsto \overrightarrow{x}^{L'} \cdot s, \qquad \text{or} \\ e &: \overrightarrow{x}^R \mapsto \overleftarrow{x}^{R'} \cdot t. \end{aligned}$$

Here the map $s$ can be any map $x^L \to x^{L'}$ if $e$ takes $\overleftarrow{x}^{L'}$ to $\overrightarrow{x}^{L'} \cdot e_{x^{L'}}$, with

$$se_{x^{L'}} = s.$$

This function $e_{x^{L'}}$ must be an idempotent, since we are saying that $e$ must use a copycat response for $x^{L'}$. Fortunately, if have any $s' : x^L \to x^{L'}$ and idempotent $e_{x^{L'}}$, we can create $s = s'e_{x^{L'}}$, in which case we clearly have $se_{x^{L'}} = s$.
Likewise, $t$ must be a map $x^{R'} \to x^R$, with $e$ taking $\overrightarrow{x}^{R'} \mapsto \overleftarrow{x}^{R'} \cdot e_{x^{R'}}$ with $e_{x^{R'}}t = t$.
These options are called *dominated.* If we have any map $x \to y$, we can think of $y$ being at least as good as $x$. So if Right makes some move $x^R$ which is dominated by our response $x^{R'}$, Right would have done as well for himself if he had played this option $x^{R'}$. So in some sense, $x^R$ is actually a "bad" choice for a move.

The other possibility is that we respond in the same component as our opponent. Then we will have some responses

$$\begin{aligned} e &: \overleftarrow{x}^L \mapsto \overleftarrow{x}^{LR} \cdot s, \qquad \text{or} \\ e &: \overrightarrow{x}^R \mapsto \overleftarrow{x}^{RL} \cdot t. \end{aligned}$$

Here we must have $se = s$ and $et = t$. However if we have any maps

$$\begin{aligned} s' &: x^{LR} \to x, \qquad \text{and} \\ t' &: x \to x^{RL}, \end{aligned}$$

we can obviously just set $s = s'e$ and $t = et'$ to guarantee $s = se$ and $t = et$. Moves of this type, which can be "beaten" by another move in the same component, are called *reversible.* It is as though we have "reversed" our opponent's move, yielding a position at least as favorable for us as before the opponent moved.
Thus, any idempotent strategy is made up of *copycat* moves, *dominating* moves, and *reversing* moves. For any $x^L$, we will choose one of the following types.

1. Perform a copycat move, facilitated by an idempotent $e_{x^L}$.
2. Perform a dominating move, facilitated by any map $d : x^L \to x^{L'}$, with our response to $x^{L'}$ of type 1. In this case, our response will be $\overrightarrow{x}^{L'} \cdot de_{x^{L'}}$.

3. Perform a reversing move, facilitated by any map $r : x^{LR} \to x$. In this case, our response will be $\overleftarrow{x}^{LR} \cdot re$.

Likewise we have the three types of responses for moves $x^R$, and the above material shows we can construct a strategy using these maps into an idempotent on $x$. These concepts are important in combinatorial game theory; if a game has only the trivial idempotent $1_x$, it is said to be in *canonical form.* A game $x$ is said to *have* a canonical form $c$ if there are arrows to and from $c$.

### 3.1 Idempotents split

It turns out that idempotents *split*, meaning that given any idempotent $e : x \to x$, there exists some $y$ and maps

$$x \xrightarrow{f} y \xrightarrow{g} x$$

so that $fg = e$ and $gf = 1_y$. Recall that there is some set of left options of $x$ which are responded to using a copycat move, and then some idempotent $e_{x^L}$. Inductively, these idempotents split to

$$x^L \xrightarrow{f_{x^L}} y^L \xrightarrow{g_{x^L}} x^L$$

for some $y^L$. Let $C^L(x)$ be the set of all $y^L$ corresponding to these copycat moves.

We also will need $R^L(x)$, the set of *fully reversed* left options of $x$. Recall that a reversible left option is one that we respond to in the same component, taking for example $x^L$ to $x^{LR}$. If we would then respond to $x^{LRL}$ in the opposite component,$x^{LRL}$ has been fully reversed. However, if we would respond to this in the same component *again*, we may need to continue to $x^{LRLRL}$, $x^{LRLRLRL}$, etc., to reach a fully reversed move $x^{L\cdots L}$. Since our games are well-founded, we must eventually reach this set of fully reversed moves. Defining $C^R(x)$ and $R^R(x)$ in the dual manner, we will set

$$y = \left\{ C^L(x) \cup R^L(x) \;\middle|\; C^R(x) \cup R^R(x) \right\}.$$

Now our map $f : x \to y$ will take the form

$$
\begin{array}{rcll}
\overleftarrow{x}^L & \mapsto & \overrightarrow{y}^L \cdot f_{x^L} & \text{for copycat moves} \\
\overleftarrow{x}^L & \mapsto & \overrightarrow{y}^{L'} \cdot sf_{x^{L'}} & \text{for dominated moves} \\
\overleftarrow{x}^L & \mapsto & \overleftarrow{x}^{LR} \cdot sf & \text{for reversible moves} \\
\overleftarrow{x}^R \cdot f_{x^R} & \mapsfrom & \overrightarrow{y}^R & \text{for copycat moves} \\
\overleftarrow{x}^R \cdot s^{L\cdots R} & \mapsfrom & \overrightarrow{y}^{RL\cdots R} & \text{for fully reversed moves}
\end{array}
$$

Here $s^{L\cdots R}$ is the strategy such that

$$s^{LRL\cdots R} : \overleftarrow{x}^{RL} \mapsto \overleftarrow{x}^{RLR} \cdot s^{L\cdots R}, \ldots s^{LR} : \overleftarrow{x}^{R\cdots L} \mapsto \overleftarrow{x}^{R\cdots R} \cdot 1_{x^{R\cdots R}}, \ldots$$

and so on. That is, the strategy keeps the responses headed towards $x^{L\cdots L}$ as long as possible; once there, it uses the identity strategy. It turns out that what the strategy does for other options is irrelevant.

The map $g : y \to x$ will be defined dually:

$$
\begin{array}{rcll}
\overleftarrow{y}^L & \mapsto & \overrightarrow{x}^L \cdot g_{x^L} & \text{for copycat moves} \\
\overleftarrow{y}^{LR\cdots L} & \mapsto & \overrightarrow{x}^L \cdot s^{R\cdots L} & \text{for fully reversed moves} \\
\overleftarrow{y}^R \cdot g_{x^R} & \mapsfrom & \overrightarrow{x}^R & \text{for copycat moves} \\
\overleftarrow{y}^{R'} \cdot sg_{x^{R'}} & \mapsfrom & \overrightarrow{x}^{R'} & \text{for dominated moves} \\
\overrightarrow{x}^{RL} \cdot sg & \mapsfrom & \overrightarrow{x}^R & \text{for reversible moves}
\end{array}
$$

We will first see that $e = fg$.

1. For a copycat option $\overleftarrow{x}^L$, we have
$$
\begin{aligned}
(fg)(\overleftarrow{x}^L) &= g(\overrightarrow{y}^L \cdot f_{x^L}) \\
&= \overrightarrow{x}^L \cdot f_{x^L} g_{x^L} \\
&= \overrightarrow{x}^L \cdot e_{x^L} \\
&= e(\overleftarrow{x}^L).
\end{aligned}
$$

2. For a dominated option $\overleftarrow{x}^L$, we have
$$
\begin{aligned}
(fg)(\overleftarrow{x}^L) &= g(\overrightarrow{y}^{L'} \cdot sf_{x^{L'}}) \\
&= \overrightarrow{x}^{L'} \cdot sf_{x^{L'}} g_{x^{L'}} \\
&= \overrightarrow{x}^{L'} \cdot se_{x^{L'}} \\
&= e(\overleftarrow{x}^L).
\end{aligned}
$$

3. Finally, for a reversible option $\overleftarrow{x}^L$ we have

$$\begin{aligned}(fg)(\overleftarrow{x}^L) &= g(\overleftarrow{x}^{LR} \cdot sef)\\ &= \overleftarrow{x}^{LR} \cdot sfg\\ &= \overleftarrow{x}^{LR} \cdot se\\ &= e(\overleftarrow{x}^L).\end{aligned}$$

This shows that the responses to all $x^L$ are the same for $e$ as for $fg$. Dually, we see that the responses to all $x^R$ are the same, and so $e = fg$.
Now, we need to show that $gf = 1_y$. There are essentially only two cases here.

1. For a copycat option $\overleftarrow{y}^L$ we have

$$\begin{aligned}(gf)(\overleftarrow{y}^L) &= f(\overrightarrow{x}^L \cdot g_{x^L})\\ &= \overrightarrow{y}^L \cdot g_{x^L} f_{x^L}\\ &= \overrightarrow{y}^L \cdot 1_{y^L}.\end{aligned}$$

2. For a fully reversed option $\overleftarrow{y}^{LRL\cdots}$ we have

$$\begin{aligned}(gf)(\overleftarrow{y}^{L\cdots L}) &= f(\overrightarrow{x}^L \cdot s^{R\cdots L})\\ &= s^{R\cdots L}(\overleftarrow{x}^{LR} \cdot s_1 f)\\ &= (s_1 f)(\overrightarrow{x}^{LRL} \cdot s^{R\cdots L})\\ &= s^{R\cdots L}(\overleftarrow{x}^{LRLR} \cdot s_1 f)\\ &= \dots\\ &= s^{RL}(\overleftarrow{x}^{L\cdots R} \cdot s_n f)\\ &= (s_n f)(\overrightarrow{x}^{L\cdots L} \cdot 1_{x^{L\cdots L}}\\ &= \overrightarrow{y}^{L\cdots L} \cdot 1_{y^{L\cdots L}}.\end{aligned}$$

Again the situation is dual, so having shown the map functions properly in one irection, it must also work in the other. Therefore $gf = 1_y$. It is now clear that if a game has a canonical form, it has an idempotent which splits over its canonical form.

# 4 Fragments

There is considerably more to be learned about the category of combinatorial games and its construction. As mentioned earlier, a disappointing feature of the logic developed in this paper is that it does not seem to model left and right membership in a nice way. However, there are hints that such a construction may be useful. For instance, let $\mathcal{L}$ be a powerset of the left options available to our games, and $\mathcal{R}^*$ be the powerset of right options. For some $L \in \mathcal{L}$, we will first use $x \rightarrowtail L$ to indicate that $L$ is the left set of $x$. We can use $L \rightarrowtail y$ to be a map from all the elements $l \in L$ to cross-maps $l \twoheadrightarrow y$.

But we will have arrows in the category $\mathcal{L}$, as well; the reasonable way to do this is have $L \to L'$ be a map that takes $l \in L$ to $l' \in L'$ with $l \to l'$, or else to some $l^R$ with a map from the left options of $l^R$ to $L'$. (This is essentially the maps between the games $\{L \mid \} \to \{L' \mid \}$.) If we have a map $L' \rightarrowtail y$ and $L \to L'$, then there are good responses to all elements in $L'$, which is at least as "strong" (if not stronger) than $L$, the set of left options of $x$. So it is reasonable to write $x \rightarrowtail L$ to mean that the left options of $x$ are *no stronger* than $L$. In fact this gives us modules $X \rightarrowtail \mathcal{L}$ and $\mathcal{L} \rightarrowtail X$.

Likewise, we can write $R \twoheadrightarrow y$ to indicate that $y$'s right options are no stronger than $R \in \mathcal{R}^*$. However, the arrows in $\mathcal{R}^*$ must now somehow be the *opposite* as though defined in $\mathcal{L}$; we have $R \to R'$ if every $r' \in R'$ has a good response $r \in R$ and $r \to r'$, or else $r'^L$ with a map from $R$ to the right options of $r'^L$. This allows us to interpret this as modules $\mathcal{R}^* \twoheadrightarrow X$ and $X \twoheadrightarrow \mathcal{R}^*$. We can now rewrite the "bitupling" rule as

$$\frac{x \rightarrowtail L \quad L \rightarrowtail y \qquad x \twoheadrightarrow R \quad R \twoheadrightarrow y}{x \to y}.$$

Here we need not know exactly what $x$'s left options are, nor $y$'s right options. But we do know that $x$'s left options are no stronger than $L$, and $y$'s right options no stronger than $R$. We can perhaps simplify our notation by noting that we have a kind of biproduct $\mathcal{L}|\mathcal{R}^*$ with arrows to and from $X$ which can be defined as

$$\frac{x \rightarrowtail L \quad x \twoheadrightarrow R}{\overline{\phantom{x}} \; x \to L|R}$$

and

$$\frac{L \rightarrowtail y \quad R \twoheadrightarrow y}{\overline{\phantom{x}} \; L|R \to y}.$$

Then we can rewrite the bitupling as the trivial-looking

$$\frac{x \to L|R \quad L|R \to y}{x \to y}.$$

The process of constructing games then becomes that of "graduating" objects $L|R$ to games $\{L \mid R\}$. Note this category $\mathcal{L}|\mathcal{R}^*$ is a biproduct since we clearly have projections

$$L \mapsfrom L|R \overset{*}{\mapsto} R$$

and injections $L \mapsto L|\emptyset$ and $\emptyset|R \overset{*}{\mapsfrom} R$. It's possible that building up a logic based on this will allow us to model loopy games and other interesting generalizations cleanly.[6]

## References


[1] D. E. Knuth. *Surreal Numbers.* Addison-Wesley, 1974.

[2] D. E. Knuth. *The Art of Computer Programming.* Addison-Wesley, 1997.

[3] J. H. Conway. All numbers great and small. *Univ. of Calgary Math Dept. Research Paper No. 149*, February 1972.

[4] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning Ways For Your Mathematical Plays.* A K Peters, Ltd., second edition, 2001.

[5] J. H. Conway. *On Numbers and Games.* A K Peters, second edition edition, 2001.

[6] André Joyal. Remarques sur la théorie des jeux à deux personnes. *Gazette des Sciences Mathématiques du Québec I*, 4:46–52, 1977. Available online in translation by Robin Houston as "Remarks on the theory of two-player games" at `http://www.maths.manchester.ac.uk/ rhouston/Joyal-games.ps`.

[7] A. Blass. Is game semantics necessary? In E. Börger, Y. Gurevich, and K. Meinke, editors, *Computer Science Logic (CSL'93)*, pages 66–77, Swansea, UK, 1993.

[8] J. R. B. Cockett and R. A. G. Seely. Polarized category theory, modules, and game semantics. *Theory and Applications of Categories*, 18(2):4–101, 2007.


[6] It may also be a good idea to allow for the left and right options to be drawn from different categories for generality. For instance, the Abramsky opponent games are built using the empty category for left options, and player games for right options; while player games are built using opponent games for left options, and the empty category for right ones.