The Basics of Triangulated Categories

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1 Introduction

To provide something in the way of motivation, an example (following the paper by Redondo and Solotar [10]).

Consider a complex of abelian groups

$$\cdots \xrightarrow{d^{-2}} A^{-1} \xrightarrow{d^{-1}} A^{0} \xrightarrow{d^{0}} A^{1} \xrightarrow{d^{1}} \cdots .$$

The requirement that $d^{n-1}d^{n} = 0$ ensures $\operatorname{im}(d^n) \subseteq \ker\left(d^{n-1}\right)$, so the n^{th} cohomology group

$$H^n(A) := \ker(d^n)/\operatorname{im}\left(d^{n-1}\right)$$

is well-defined. A morphism of complexes is a family of homomorphisms $f^n : A^n \to B^n$ such that

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{n-1} & \longrightarrow & A^{n} & \longrightarrow & A^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^{n}} & & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & B^{n-1} & \longrightarrow & B^{n} & \longrightarrow & B^{n+1} & \longrightarrow & \cdots
\end{array}$$

commutes. Any such homomorphism induces a homomorphism between the quotient groups,

$$f_*^n : H^n(A) \to H^n(B).$$

If all of these induced maps are isomorphisms, the overall morphism between complexes is called a quasiisomorphism.

The natural context for all of this is not just the category of abelian groups, but any abelian category (which, thankfully for the sake of terminological consistency, abelian groups is an example of), and since homological algebra is really only concerned with complexes up to quasiisomorphism, an even better category would be one where quasiisomorphisms are in fact isomorphisms. The derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$, obtained by inverting quasiisomorphisms (*i.e.*, by localizing the category of complexes over $\mathcal{A}$ by the class of quasiisomorphisms), is just such a category.

However preferable, the move to the derived category creates some difficulties. The derived category of an abelian category need not be abelian; in particular, the notion of short exact sequences breaks down. Take a short exact sequence of abelian groups

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} 0$$

(so, f is injective, g surjective, and $\ker(g) = \operatorname{im}(f)$). Such a sequence gives rise to a short exact sequence of complexes with A, B, and C concentrated in degree zero

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

But now consider the morphism of complexes

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C & \longrightarrow & 0 & \longrightarrow & \cdots,
\end{array}$$

which induces the following maps on homology groups

$$\begin{array}{cccc}
0 & 0 & B/\operatorname{im}(f) = B/\ker(g) & 0 \\
\downarrow & \downarrow & \downarrow{\scriptstyle \sim} & \downarrow \\
0 & 0 & C & 0;
\end{array}$$

i.e., the above complexes are quasiisomorphic, and hence isomorphic in the derived category. But unfortunately

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

is not exact. An alternate sequence

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{f} & B & \xrightarrow{1} & B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots,
\end{array}
$$

which could be written suggestively as $A \to B \to C \to TA$, is the substitute for short exact sequences appropriate to the context of derived categories arrived at by Jean-Louis Verdier, a student of Grothendieck, in his thesis. The 'triangles' of triangulated categories are sequences of this form.

2 Additive Categories

This section briefly recalls some facts about additive categories as preparation for the definition of triangulated categories.

Definition 2.1. *A* **preadditive category** $\mathcal{A}$ *is one enriched in the category of abelian groups; that is, for* $X, Y \in \mathcal{A}$, $\mathrm{Hom}(X, Y)$ *has the structure of an additive abelian group and composition of morphisms is bilinear.*

Example 2.2. *$Mod(R)$ and $Mat(R)$ for any ring R.*

As all hom-sets are abelian groups, each pair of objects X, Y has a zero morphism between them corresponding to the identity of the group, which must satisfy $f0 = 0 = 0g$ for any $f \in \mathrm{Hom}(X', X)$ and $g \in \mathrm{Hom}(Y, Y')$. With the structure of full-blown additive categories available it is possible to be more specific about the nature of this map as well as the sum of maps.

Proposition 2.3. *Let $\mathcal{A}$ be a preadditive category and $X_1, X_2 \in \mathcal{A}$.*

1. *If $X_1 \times X_2$ exists, let $\pi_k : X_1 \times X_2 \to X_k$, $k = 1, 2$ be the canonical projections and $\iota_k : X_k \to X_1 \times X_2$ be the morphisms defined by*

$$\iota_k \pi_j = \begin{cases} 1_k & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

   Then $\pi_1\iota_1 + \pi_2\iota_2 = 1_{X_1 \times X_2}$.

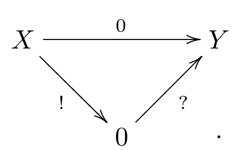
2. *The product $X_1 \times X_2$, equipped with the maps above, is a coproduct of X_1 and X_2 with ι_1 and ι_2 as coprojections.*

These common products and coproducts are called direct sums or biproducts, and denoted $X_1 \oplus X_2$.

Definition 2.4. *A functor $f : \mathcal{A} \to \mathcal{B}$ between pre-additive categories is* **additive** *if $F(f + g) = F(f) + F(g)$ for all $f, g \in \mathrm{Hom}(X, Y)$, $X, Y \in \mathcal{A}$.*

Definition 2.5. *An* **additive category** *is a preadditive category that admits finite direct sums.*

Because they have finite direct sums, additive categories have empty products and coproducts, and so therefore have final objects and initial objects. Since products and coproducts coincide, so do final and terminal objects. These objects are called zero objects. The zero morphism between objects is, in an additive category, the map that factors through the zero object



Proposition 2.6. *Let $\mathcal{A}$ be preadditive, $X, Y \in \mathcal{A}$, and $f, g \in \mathrm{Hom}(X, Y)$. If $X \oplus X$ and $Y \oplus Y$ exist, then $f + g$ coincides with the composite*

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y,$$

where Δ is the diagonal and ∇ the codiagonal map.

Proposition 2.7. *If $\mathcal{A}$ and $\mathcal{B}$ are additive categories and $F : \mathcal{A} \to \mathcal{B}$ is a functor, F is additive if and only if it commutes with finite direct sums.*

3 Triangulated Categories

3.1 Definition

Definition 3.1. *1. An* **additive category with translation** *is an additive category $\mathcal{A}$ together with an additive automorphism called the* **translation of** $\mathcal{A}$*,* i.*e., endofunctors $T, T^{-1} : \mathcal{A} \to \mathcal{A}$ such that $TT^{-1} \cong 1_{\mathcal{A}} \cong T^{-1}T$.*

2. A functor of additive categories with translation $F : (\mathcal{A}, T) \to (\mathcal{B}, T')$ is an additive functor with an isomorphism

$$TF \cong FT'.$$

A natural transformation θ of functors of additive categories with translation $F, F' : (\mathcal{A}, T) \to (\mathcal{B}, T')$ is a natural transformation in the usual sense that additionally makes the following diagram commute

$$\begin{array}{ccc} TF & \xrightarrow{T(\theta)} & TF' \\ \downarrow\sim & & \downarrow\sim \\ FT' & \xrightarrow[T'(\theta)]{} & F'T'. \end{array}$$

3. A subcategory of an additive category with translation $(\mathcal{A}, T)$ is a subcategory of $\mathcal{A}$ whose translation is the restriction of T.

Definition 3.2. *A* **triangle** *in an additive category with translation is a diagram*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \ .$$

A **morphism of triangles** *is a commuting diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ f\downarrow & & g\downarrow & & h\downarrow & & \downarrow T(f) \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & TX'. \end{array}$$

The **rotation** *of a triangle* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ *is the triangle*

$$T^{-1}Z \xrightarrow{-T^{-1}(w)} X \xrightarrow{u} Y \xrightarrow{v} Z \ .$$

The terminology comes from the convention of sometimes writing the map $Z \to TX$ as $Z \overset{+1}{\to} X$, so any triangle can alternately be written

$$\begin{array}{ccc} & Z & \\ {}^{+1}\swarrow & & \nwarrow^{g} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

Note also that, given a morphism of triangles as above, rotating the morphism gives a morphism of rotated triangles,

$$\begin{array}{ccccccc}
T^{-1}Z & \xrightarrow{-T^{-1}(w)} & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\
\downarrow{\scriptstyle T^{-1}(h)} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
T^{-1}Z' & \xrightarrow[-T^{-1}(w')]{} & X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z'.
\end{array}$$

Definition 3.3. *A* **triangulated category** *is an additive category with translation* $(\mathcal{A}, T)$ *with a class of* **distinguished triangles** *satisfying the following axioms:*

TR1 (Rotation):

- *(a) A triangle is distinguished if and only if its rotated triangle is as well.*
- *(b) Triangles isomorphic to distinguished triangles are distinguished.*

TR2 (Existence of Cones):

- *(a) Any morphism* $u : X \to Y$ *in* $\mathcal{A}$ *can be completed in a not necessarily unique way to a distinguished triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX \; .$$

  Any Z *satisfying this property is called a* **mapping cone for** u*.*

- *(b) The triangle* $X \xrightarrow{1} X \to 0 \to TX$ *is distinguished for any* $X \in \mathcal{A}$*.*

TR3 (Morphisms):

Any commutative square

$$\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
X' & \xrightarrow[u']{} & Y'
\end{array}$$

can be completed to a morphism of given distinguished triangles in a not necessarily unique way

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots & & \downarrow{\scriptstyle T(f)} \\
X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & TX'.
\end{array}$$

TR4 (Octahedral Axiom; Verdier's Axiom)

Given $u : X \to Y$ *and* $v : Y \to Z$ *and any choice of mapping cones (provided by TR2),*

$$X \longrightarrow Y \longrightarrow C_u \longrightarrow TX$$

$$Y \longrightarrow Z \longrightarrow C_v \longrightarrow TY$$

$$X \longrightarrow Z \longrightarrow C_{uv} \longrightarrow TX,$$

there is a distinguished triangle $C_u \xrightarrow{\alpha} C_{uv} \xrightarrow{\beta} C_v \xrightarrow{\gamma} TC_u$ *that makes*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \longrightarrow & C_u & \longrightarrow & TX \\
\downarrow & & \downarrow v & & \downarrow \alpha & & \downarrow \\
X & \xrightarrow[uv]{} & Z & \longrightarrow & C_{uv} & \longrightarrow & TX \\
\downarrow & & \downarrow & & \downarrow \beta & & \downarrow \\
Y & \xrightarrow[v]{} & Z & \longrightarrow & C_v & \longrightarrow & TY \\
\downarrow & & \downarrow & & \| & & \downarrow \\
C_u & \xrightarrow[\alpha]{} & C_{uv} & \xrightarrow[\beta]{} & C_v & \xrightarrow[\gamma]{} & TC_u
\end{array}$$

commute.

Puppe produced a similar definition in his work on stable homotopy [9] that excludes the final axiom, which was added by Verdier. That axiom's other name is explained by the fact that, using the notational convention for maps to translated objects described above, it is possible to write TR4 on an octahedron. This is described in Hartshorne's notes [3] or the book by Kashiwara and Schapira [4]. The non-uniqueness of the map in TR3 is apparently a source of difficulty in some applications, and there have been efforts (by, for example, Neeman [6]) to re-axiomatize in a way that ensures uniqueness.

There is also some redundancy in the axioms. For instance, TR3 can be derived from the other axioms by taking a commuting square

$$\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
f \downarrow & & \downarrow g \\
X' & \xrightarrow[u']{} & Y',
\end{array}$$

choosing mapping cones, C_u, $C_{u'}$, and using TR4 to obtain two commuting squares

$$
\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \longrightarrow & C_u & \longrightarrow & TX \\
\| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \alpha} & & \downarrow \\
X & \xrightarrow[k=ug]{} & Y' & \longrightarrow & C_k & \longrightarrow & TX \\
{\scriptstyle u}\downarrow & & \| & & \downarrow{\scriptstyle \gamma} & & \downarrow \\
Y & \xrightarrow[g]{} & Y' & \longrightarrow & C_g & \longrightarrow & TY \\
\downarrow & & \downarrow & & \| & & \downarrow \\
C_u & \xrightarrow[\alpha]{} & C_k & \xrightarrow[\gamma]{} & C_g & \longrightarrow & TC_u
\end{array}
\qquad
\begin{array}{ccccccc}
X & \xrightarrow[f]{} & X' & \longrightarrow & C_f & \longrightarrow & TX \\
\| & & {\scriptstyle u'}\downarrow & & {\scriptstyle \gamma'}\downarrow & & \downarrow \\
X & \xrightarrow[k=fu]{} & Y' & \longrightarrow & C_k & \longrightarrow & TX \\
{\scriptstyle f}\downarrow & & \| & & {\scriptstyle \beta}\downarrow & & \downarrow \\
X' & \xrightarrow[u']{} & Y' & \longrightarrow & C_{u'} & \longrightarrow & TX' \\
\downarrow & & \downarrow & & \| & & \downarrow \\
C_f & \xrightarrow[\gamma']{} & C_k & \xrightarrow[\beta]{} & C_{u'} & \longrightarrow & TC_f.
\end{array}
$$

The composite $\alpha\beta : C_u \to C_{u'}$ then completes a morphism of the chosen distinguished triangles.

3.2 Elementary Properties

Because of TR1, not only is the "backward" rotation of a distinguished triangle distinguished, but its "forward" rotation as well: all three of

$$X \xrightarrow{\quad u \quad} Y \xrightarrow{\quad v \quad} Z \xrightarrow{\quad w \quad} TX$$

$$T^{-1}Z \xrightarrow{\ -T^{-1}(w)\ } X \xrightarrow{\quad u \quad} Y \xrightarrow{\quad v \quad} Z$$

$$Y \xrightarrow{\quad v \quad} Z \xrightarrow{\quad w \quad} TX \xrightarrow{\ -T(u)\ } TY$$

are distinguished. It's also sometimes possible (for example, after two rotations in either direction) to remove the negatives by taking an isomorphic triangle.

Proposition 3.4. *For any distinguished triangle* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ *in a triangulated category* $\mathcal{T}$, $uv = 0$, $vw = 0$, *and* $wT(u) = 0$.

Proof. Consider

$$
\begin{array}{ccccccc}
X & \xrightarrow{1} & X & \longrightarrow & 0 & \longrightarrow & TX \\
{\scriptstyle 1}\downarrow & & {\scriptstyle u}\downarrow & & \vdots & & \downarrow{\scriptstyle 1} \\
X & \xrightarrow[u]{} & Y & \xrightarrow[v]{} & Z & \xrightarrow[w]{} & TX.
\end{array}
$$

Since the left square commutes, there is, by TR3, a map $0 \to Z$ that completes the morphism of triangles, which proves $uv = 0$. The other equations follow similarly by looking at the diagrams

$$\begin{array}{ccccccc} Y & \xrightarrow{1} & Y & \longrightarrow & 0 & \longrightarrow & TY \\ {\scriptstyle 1}\downarrow & & {\scriptstyle v}\downarrow & & \vdots & & \downarrow{\scriptstyle 1} \\ Y & \xrightarrow[v]{} & Z & \xrightarrow[w]{} & TX & \xrightarrow[-T(u)]{} & TY \end{array}$$

and

$$\begin{array}{ccccccc} Z & \xrightarrow{1} & Z & \longrightarrow & 0 & \longrightarrow & TZ \\ {\scriptstyle 1}\downarrow & & {\scriptstyle v}\downarrow & & \vdots & & \downarrow{\scriptstyle 1} \\ Z & \xrightarrow[w]{} & TX & \xrightarrow[T(u)]{} & TY & \xrightarrow[T(v)]{} & TZ. \end{array}$$

□

Definition 3.5. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $F : \mathcal{T} \to \mathcal{A}$ is* **cohomological** *if, for any distinguished triangle $X \to Y \to Z \to TX$, the sequence*

$$F(X) \to F(Y) \to F(Z)$$

is exact in $\mathcal{A}$.

Abelian categories have gone undefined and will remain so (see, for example the book by Kashiwara and Schapira [4] or Freyd [1]). For the purposes here, it is sufficient to work in the category $\mathcal{A}\mathbf{b}$ of abelian groups.

To prepare for the next proposition, observe that it is possible to rotate the axiom TR3. In the diagram below, consisting of two distinguished triangles extended by rotation

$$\begin{array}{ccccccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX & \xrightarrow{-T(u)} & TY & \xrightarrow{-T(v)} & TZ \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle T(f)} & & \downarrow{\scriptstyle T(g)} & & \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & TX' & \xrightarrow[-T(u')]{} & TY' & \xrightarrow[-T(v')]{} & TZ' \end{array}$$

each square commutes if and only if the other does, and the existence of a morphism that completes a morphism of triangles between the triangles starting on the left is equivalent to the existence of a morphism completing a morphism between the triangles ending on the right via rotation. So TR3 could equivalently state that any commuting square between distinguished triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & TX \end{array}$$

can be completed nonuniquely to a morphism of triangles.

Proposition 3.6. *In any triangulated category $\mathcal{T}$ and for any $A \in \mathcal{T}$, the functors* $\operatorname{Hom}(A, -)$ *and* $\operatorname{Hom}(-, A)$ *are cohomological.*

Proof. Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ be a distinguished triangle and $g : A \to Y$ a morphism with $gv = 0$ (one in the kernel of $\operatorname{Hom}(A, Y) \xrightarrow{v_*} \operatorname{Hom}(A, Z)$). Because the middle square in

$$\begin{array}{ccccccc}
A & \xrightarrow{1} & A & \longrightarrow & 0 & \longrightarrow & TA \\
{\scriptstyle f}\,\vdots & & {\scriptstyle g}\downarrow & & \downarrow & & \vdots \\
X & \xrightarrow[u]{} & Y & \xrightarrow[v]{} & Z & \xrightarrow[w]{} & TX
\end{array}$$

commutes, there is an f such that $fu = g$ by the observation made above. Thus,

$$\operatorname{Hom}(A, X) \xrightarrow{u_*} \operatorname{Hom}(A, Y) \xrightarrow{v_*} \operatorname{Hom}(A, Z)$$

is exact.

Now let $f : Y \to A$ satisfy $uf = 0$. Rotate the upper triangle in the diagram above and by TR3, there is a map $g : Z \to A$ that completes

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{1} & A & \longrightarrow & 0 \\
\uparrow & & \uparrow{\scriptstyle f} & & & & \\
X & \xrightarrow[u]{} & Y & \xrightarrow[v]{} & Z & \xrightarrow[w]{} & TX.
\end{array}$$

Hence

$$\operatorname{Hom}(X, A) \xleftarrow{u^*} \operatorname{Hom}(Y, A) \xleftarrow{v^*} \operatorname{Hom}(Z, A)$$

is also exact. □

In fact, rotating the bottom triangles and arguing the same way as above allows the exact sequences to be extended to long exact sequences:

$$\cdots \xrightarrow{T^{n-1}(w)_*} \operatorname{Hom}(A, T^nX) \xrightarrow{T^n(u)_*} \operatorname{Hom}(A, T^nY) \xrightarrow{T^n(v)_*} \operatorname{Hom}(A, T^nZ) \xrightarrow{T^n(w)_*} \cdots$$

$$\cdots \xleftarrow[T^{n+1}(w)^*]{} \operatorname{Hom}(T^nX, A) \xleftarrow{T^n(u)^*} \operatorname{Hom}(T^nY, A) \xleftarrow{T^n(v)^*} \operatorname{Hom}(T^nZ, A) \xleftarrow[T^n(w)^*]{} \cdots.$$

This one proposition is the source of many other basic properties.

Corollary 3.7. *Given a morphism of distinguished triangles*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle T(f)} \\
X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & TX',
\end{array}$$

if f and g are both isomorphisms, then h is as well.

Proof. Follows from the Five Lemma for abelian groups, comparing the exact sequences obtained from the functor $\mathrm{Hom}(A, -)$. □

Corollary 3.8. *In a triangulated category, any two mapping cones associated to a morphism $u : X \to Y$ are isomorphic. Moreover, any two distinguished triangles associated to a single mapping cone C_u*

$$X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} TX$$

$$X \xrightarrow{u'} Y \xrightarrow{v'} C_u \xrightarrow{w'} TX$$

have $v' = vh$ and $w' = h^{-1}w$ for some isomorphism h of C_u.

Proof. The first statement follows from the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & C_u & \xrightarrow{w} & TX \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & \vdots & & \downarrow{\scriptstyle 1} \\ X & \xrightarrow[u]{} & Y & \xrightarrow[v']{} & C'_u & \xrightarrow[w']{} & TX. \end{array}$$

and TR3, together with the previous corollary.

The second statement follows by observing that any object Z isomorphic via a map h to a mapping cone for u is also a cone since

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v'} & Z & \xrightarrow{w'} & TX \\ {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 1} \\ X & \xrightarrow[u]{} & Y & \xrightarrow[v]{} & C_u & \xrightarrow[w]{} & TX \end{array}$$

is an isomorphism. □

Proposition 3.9. *For any distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$, the following are equivalent.*

1. *u is monic*
2. *$w = 0$*
3. *v is a retraction*
4. *v is epic*
5. *u is a section*
6. *$Y \cong X \oplus Z$.*

Proof.

(1 ⇒ 2): By Proposition 3.4, $wT(u) = 0 = 0T(u)$, so $w = 0$ since u is monic and T is an equivalence.

(2 ⇒ 3): Since $\mathrm{Hom}(A, -)$ is cohomological,

$$\mathrm{Hom}(Z, Y) \overset{w_*}{\to} \mathrm{Hom}(Z, Z) \overset{0_*}{\to} \mathrm{Hom}(Z, TX)$$

is exact, hence there is an $s : Z \to Y$ with $sw = 1$.

(3 ⇒ 4): Obviously.

(4 ⇒ 5): Since v is epic $vw = 0 = v0$ implies $w = 0$. The exactness of

$$\mathrm{Hom}(Y, X) \overset{u^*}{\to} \mathrm{Hom}(X, X) \overset{0^*}{\to} \mathrm{Hom}(T^{-1}Z, X)$$

provides a morphism $t : Y \to X$ with $ut = 1$.

(5 ⇒ 1): Is clear.

(2 ⇒ 6): A twice rotated version of TR3 obtains from the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ {\scriptstyle 1}\downarrow & & \vdots & & \downarrow & & \downarrow{\scriptstyle 1} \\ X & \xrightarrow[1]{} & X & \longrightarrow & 0 & \longrightarrow & TX \end{array}$$

a map $\pi : Y \to X$ satisfying $u\pi = 1_X$. TR3 also yields, from the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ {\scriptstyle 0}\downarrow & & {\scriptstyle 1_y - \pi u}\downarrow & & \vdots & & \downarrow{\scriptstyle 0} \\ 0 & \xrightarrow[0]{} & Y & \xrightarrow[1]{} & Y & \longrightarrow & 0, \end{array}$$

a map $\iota : Z \to Y$ with $v\iota = 1_Y = \pi u \Leftrightarrow \pi u + v\iota = 1_Y$. Then, the exact sequence

$$\mathrm{Hom}(Y, Z) \overset{v^*}{\leftarrow} \mathrm{Hom}(Z, Z) \overset{w^*=0}{\longleftarrow} \mathrm{Hom}(TX, Z)$$

gives $\iota v = 1_Z$, since $v^*(\iota v) = v\iota v = (1 = \pi u)v = v - \pi uv = v$ since $uv = 0$ by Prop. 3.4. Finally, $\iota\pi = 0$ by the exact sequence

$$\mathrm{Hom}(Z, T^{-1}Z) \overset{0}{\to} \mathrm{Hom}(Z, X) \overset{u_*}{\to} \mathrm{Hom}(Z, Y)$$

since $u_*(\iota\pi) = \iota\pi u = \iota(1 - v\iota) = 0$ because $\iota v = 1_Z$. Then by Proposition 2.3, Y is a direct sum of X and Z, with π the projection onto X, ι the coprojection from Z, and u and v the remaining coprojection and projection, respectively. This final observation immediately gives the implication 6 ⇒ 1 (or 6 ⇒ 4), completing the proof.

□

Corollary 3.10. *In any triangulated category, all monics and epics are split, and a morphism $f : A \to B$ is an isomorphism if and only if it is both monic and epic.*

Definition 3.11. 1. *A functor $F : \mathcal{T} \to \mathcal{T}'$ between triangulated categories is* **triangulated** *if it is a functor of additive categories with translation and maps distinguished triangles to distinguished triangles. A natural transformation of triangulated functors is a natural transformation of functors of additive catgories with translation in the sense of Definition 3.1.*

2. *A subcategory $(\mathcal{T}', T')$ of a triangulated category $(\mathcal{T}, T)$ is subcategory of additive categories with translation that is itself triangulated and such that the inclusion functor is triangulated (every triangle distinguished in $\mathcal{T}'$ is also distinguished in $\mathcal{T}$).*

Proposition 3.12. *Let $\mathcal{T}'$ be a full triangulated subcategory of $\mathcal{T}$. Then,*

1. *If $\mathcal{T}'$ contains the triangle $X \xrightarrow{f} Y \to Z \to TX$ and it is distinguished in $\mathcal{T}$, then it is distinguished in $\mathcal{T}'$.*

2. *If $X \to Y \to Z \to TX$ is distinguished in $\mathcal{T}$ and $X, Y \in \mathcal{T}'$, then Z is isomorphic to an object in $\mathcal{T}'$.*

Proof. Both assertions follow from the same proof. Since $\mathcal{T}'$ is triangulated, $X \xrightarrow{f} Y$ completes to a distinguished triangle

$$X \xrightarrow{f} Y \to Z' \to TX$$

in $\mathcal{T}'$, which is also distinguished in $\mathcal{T}$. Therefore, by TR3,

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & TX \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle 1} \\ X & \xrightarrow[f]{} & Y & \longrightarrow & Z & \longrightarrow & TX \end{array}$$

completes to a morphism of triangles in $\mathcal{T}$, and the map $Z' \to Z$ must be an isomorphism by Corollary 3.7. □

Proposition 3.13. *In the situation of TR3,*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \vdots{\scriptstyle \gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & TX', \end{array}$$

if $\mathrm{Hom}(Y, X') = 0$ *and* $\mathrm{Hom}(TX, Y') = 0$, *then the map γ exists uniquely.*

Proof. First, note that γ is nonunique if and only if there exist γ_1, γ_2 such that $g\gamma_1 = \beta g' = g\gamma_2 \Leftrightarrow g(\gamma_1 = \gamma_2) = 0$, with $\gamma_1 - \gamma_2 \neq 0$, which is true if and only if γ is nonunique with $\beta = 0$. Thus, it is sufficient to consider only

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\
{\scriptstyle 0}\downarrow & & {\scriptstyle 0}\downarrow & & {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle 0} \\
X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & TX'
\end{array}$$

and show $\gamma = 0$.

Because $\gamma h' = 0$ and the sequence

$$\mathrm{Hom}(Z, Y') \stackrel{g'_*}{\to} \mathrm{Hom}(Z, Z') \stackrel{h'_*}{\to} \mathrm{Hom}(Z, TX')$$

is exact, there must exist a morphism $u : Z \to Y'$ such that $ug' = \gamma$. Likewise, $g\gamma = 0$ and the exactness of

$$\mathrm{Hom}(Y, Z') \stackrel{g^*}{\leftarrow} \mathrm{Hom}(Z, Z') \stackrel{h^*}{\leftarrow} \mathrm{Hom}(TX, Z')$$

gives a map $v : TX \to Z$ such that $hv = \gamma$. There is, then, a morphism $w : TY \to TX'$ that completes the morphism of triangles

$$\begin{array}{ccccccc}
Z & \xrightarrow{h} & TX & \xrightarrow{-T(f)} & TY & \xrightarrow{-T(g)} & TZ \\
{\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle w}\vdots & & \downarrow{\scriptstyle T(u)} \\
Y' & \xrightarrow[g']{} & Z' & \xrightarrow[h']{} & TX' & \xrightarrow{-T(f)} & TY'.
\end{array}$$

Since $\mathrm{Hom}(Y, X') = 0$, $w = 0$, which forces $vh' = 0$. A reprise of the argument above with vh' in place of $\gamma h'$ shows that v factors through Y'. But $\mathrm{Hom}(TX, Y') = 0$, so $v = 0$, giving finally $\gamma = hv = 0$. □

4 An Example: The Homotopy Category

As was noted in the introduction, the motivating example of a triangulated category is the derived category of an abelian category. This section only goes halfway to that example, describing the homotopy category of an additive category. The derived category can be obtained from the homotopy category of an abelian category by localizing by the class of all quasiisomorphisms. The intermediate step of taking the homotopy category is not strictly necessary, but is advantageous since it allows for a calculus of fractions when working in the derived category. Information on this, and on the derived category generally can be found in the notes by Verdier [11], or those by Hartshorne [3].

Let $\mathcal{A}$ be an additive category. The **category of complexes in** $\mathcal{A}$, $C(\mathcal{A})$, can be defined as an additive catgory with translation as follows.

Objects: Collections of objects of $\mathcal{A}$, $\{A^i \mid i \in \mathbb{Z}\}$ such that for every $i \in \mathbb{Z}$ there are morphisms $d^i : A^i \to A^{i+1}$ (called **differentials**) satisfying $d^i d^{i+1} = 0$. Such collections will be denoted $A^\bullet$.

Maps: $f : A^\bullet \to B^\bullet$ are collections of maps $f^i : A^i \to B^i$ such that

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_A^{i-2}} & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \xrightarrow{d_A^{i+1}} & \cdots \\
 & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\
\cdots & \xrightarrow[d_A^{i-2}]{} & B^{i-1} & \xrightarrow{d_B^{i-1}} & B^i & \xrightarrow{d_B^i} & B^{i+1} & \xrightarrow{d_B^{i+1}} & \cdots
\end{array}$$

commutes.

Translation: $T : C(\mathcal{A}) \to C(\mathcal{A})$ defined by

$$\begin{array}{ll}
T(A^\bullet)^i = A^{i+1}, & d_A^{i+1} = -T(d_A^i) \\
T(f)^i = f^{i+1} & \text{on maps otherwise.}
\end{array}$$

The inverse functor T^{-1} is the obvious reversal of this.

An alternate phrasing of this would be that each object of $C(\mathcal{A})$ has a map $d_A : A^\bullet \to TA^\bullet$ satisfying $d_{TA} = -Td_A$ and every morphism of complexes f is a collection of maps that makes

$$\begin{array}{ccc}
A & \xrightarrow{d_A} & TA \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle T(f)} \\
B & \xrightarrow[d_B]{} & TB
\end{array}$$

commute.

The homotopy category $K(\mathcal{A})$ is just $C(\mathcal{A})$ modulo homotopic maps.

Definition 4.1. *A map $f : A^\bullet \to B^\bullet$ in $C(\mathcal{A})$ is* **homotopic to** 0 *if there exists a morphism $u : A^\bullet \to T^{-1}B^\bullet$ such that $f = d_A T(u) + uT^{-1}(d_B)$. Two maps f and g are homotopic if $f - g$ is homotopic to 0.*

A noncommuting diagram visualizing this:

$$\begin{array}{ccccc}
T^{-1}A^\bullet & \xrightarrow{d_{T^{-1}A}} & A^\bullet & \xrightarrow{d_A} & TA^\bullet \\
\downarrow{\scriptstyle T^{-1}(f)} & \overset{u}{\swarrow} & \downarrow{\scriptstyle f} & \overset{T(u)}{\swarrow} & \downarrow{\scriptstyle T(f)} \\
T^{-1}B^\bullet & \xrightarrow[d_{T^{-1}B}]{} & B^\bullet & \xrightarrow[d_B]{} & TB^\bullet;
\end{array}$$

or, within complexes,

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f^{n-1}} & \overset{u^n}{\swarrow} & \downarrow{\scriptstyle f} & \overset{n+1}{\swarrow} & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & B^{n-1} & \longrightarrow & B^n & \longrightarrow & B^{n+1} & \longrightarrow & \cdots .
\end{array}$$

Definition 4.2. *The* **homotopy category** *of an additive category $\mathcal{A}$ is*

Objects: *Objects of $C(\mathcal{A})$.*

Maps: *For $A^\bullet, B^\bullet \in C(\mathcal{A})$,*

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet) / \mathrm{Ht}(A^\bullet, B^\bullet),$$

where $\mathrm{Ht}(A^\bullet, B^\bullet) := \{f \in \mathrm{Hom}_{C(\mathcal{A})}(A^\bullet, B^\bullet) \mid f \textit{ is homotopic to } 0\}$.

Composition in this category is well-defined courtesy of the following.

Lemma 4.3. *Let $f : A^\bullet \to B^\bullet$ and $g : B^\bullet \to C^\bullet$ be maps in $C(\mathcal{A})$. If f or g is homotopic to 0, then fg is as well.*

Proof. Suppose it is f that is homotopic to 0, so

$$f = d_A T(u) + uT^{-1}(d_B)$$

for some $u : A^\bullet \to T^{-1}B^\bullet$. Then

$$\begin{aligned} fg &= (d_A T(u) + uT^{-1}(d_B))g \\ &= d_A T(u)g + uT^{-1}(d_B)g \\ &= d_A T(u)g + uT^{-1}(g)T^{-1}(d_C) \\ &= d_A T(uT^{-1}(g)) + (uT^{-1}(g))T^{-1}(d_C), \end{aligned}$$

hence fg is homotopic to 0 with cross map $uT^{-1}(g)$. When g is homotopic to 0 the proof is similar. □

The quotient functor $C(\mathcal{A}) \to K(\mathcal{A})$ is clearly additive, so the induced action of the translation of $C(\mathcal{A})$ makes $K(\mathcal{A}$ into an additive category with translation. Triangulating $K(\mathcal{A})$ requires one more definition.

Definition 4.4. *Let $f : A^\bullet \to B^\bullet$ be a morphism in $C(\mathcal{A})$. The* **mapping cone of** *f is the complex $\mathrm{Mc}(f)$ with*

$$\mathrm{Mc}(f)^n := A^{n+1} \oplus B^n$$

and differential

$$d^n_{\mathrm{Mc}(f)} := \begin{pmatrix} d^n_{TA} & T(f)^n \\ 0 & d^n_B \end{pmatrix}.$$

The mapping cone comes equipped with maps

$$\begin{aligned} &\alpha(f) : B^\bullet \to \mathrm{Mc}(f) \\ &\alpha(f)^n : B^n \overset{(0 1_{Y^n})}{\to} \mathrm{Mc}(f)^n = A^{n+1} \oplus B^n \end{aligned}$$

and

$$\begin{aligned} &\beta(f) : \mathrm{Mc}(f) \to TA^\bullet \\ &\beta(f)^n : A^{n+1} \oplus B^n \overset{\begin{pmatrix} 1_{X^{n+1}} \\ 0 \end{pmatrix}}{\Longrightarrow} A^{n+1}. \end{aligned}$$

Therefore, the mapping cone gives a triangle in $C(\mathcal{A})$,

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\alpha} \mathrm{Mc}(f) \xrightarrow{\beta} TA^\bullet.$$

The homotopy catgeory is made into a triangulated category by declaring any triangle isomorphic (in $K(\mathcal{A})$) to a mapping cone triangle a distinguished triangle. The proof that this is indeed a valid triangulated structure is lengthy but straightforward, consisting mostly of calculations, and can be found in the book by Kashiwara and Schapira [4]. However, it does require the taking of homotopy first, as some of the required diagrams commute only up to homotopy.

5 Further Theory

This final section offers a brief summary, without proofs, of some aspects of the theory discussed at greater length in Neeman's book [7].

Freyd proved the following in the context of stable homotopy [2].

Theorem 5.1. *Let $\mathcal{T}$ be a triangulated category. There exists an abelian category $A(\mathcal{T})$ and a cohomological functor $\mathcal{T} \to A(\mathcal{T})$ such that any other cohomological functor $\mathcal{T} \to \mathcal{A}$ ($\mathcal{A}$ an abelian category) factors as $\mathcal{T} \to A(\mathcal{T}) \to \mathcal{A}$. The functor $A(\mathcal{T}) \to \mathcal{A}$ is exact and is unique up to canonical equivalence; moreover, any natural transformation of cohomological functors $\mathcal{T} \to \mathcal{A}$ factors uniquely through a natural transformation of the functor $A(\mathcal{T}) \to \mathcal{A}$.*

In fact, $A(-)$ is a functorial association between the category of triangulated categories and triangulated functors and abelian categories and exact functors. Also, the cohomological functor $\mathcal{T} \to A(\mathcal{T})$ is a fully faithful embedding.

Proposition 5.2. *Let $F : \mathcal{S} \to \mathcal{T}$ be a triangulated functor. If $G : \mathcal{T} \to \mathcal{S}$ is a right adjoint to F, then it is also triangulated, and $\mathcal{A}(G) : A(\mathcal{S}) \to A(\mathcal{T})$ is right adjoint to $A(F)$. Further, if all idempotents in $\mathcal{S}$ split, F has a right adjoint if and only if $A(F)$ has a right adjoint. In this case, if G is right adjoint to F and G' is right adjoint to $A(F)$, then $A(G)$ and G' are naturally isomorphic.*

Although interesting, this proposition is not particularly useful on its own because the abelian categories obtained from the association are, in Neeman's word, "terrible." For one thing, they are not well-generated, and Neeman writes that he does not know of a single instance of someone proving the existence of an adjunction between them. Pushing a little further provides a more usable criterion.

Theorem 5.3 (Brown Representability)**.** *Let $\mathcal{T}$ be a well-generated triangulated category and $H : \mathcal{T}^{\mathrm{op}} \to \mathcal{A}\mathbf{b}$ a contravariant functor to the category of abelian groups. Then H is representable if and only if it is cohomological and carries coproducts in $\mathcal{T}$ to products in $\mathcal{A}\mathbf{b}$.*

Definition 5.4. *Let $\mathcal{T}$ be a triangulated category admitting coproducts indexed by sets of any infinite cardinality. Such a category* **satisfies the representability theorem** *if its contravariant representable functors are exactly the cohomological functors from $\mathcal{T}^{\mathrm{op}}$ to abelian groups that send coproducts to products.*

Proposition 5.5. *Let $F : \mathcal{S} \to \mathcal{T}$ be a triangulated functor. If $\mathcal{S}$ satisfies the representability theorem, F has a right adjoint if and only if it respects coproducts.*

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