Restriction Categories and $\mathcal{M}$-Categories

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Abstract

This paper gives an exposition of the relationship between restriction categories and $\mathcal{M}$-categories, both of which are formulations of partial maps. The categories of restriction categories and $\mathcal{M}$-categories are both developed as 2-categories. The 2-equivalence between the 2-category of $\mathcal{M}$-categories and the full sub-2-category of split restriction categories is described in detail.

1 Introduction

A rigorous treatment of partiality is beneficial to mathematics, as partial functions arise everywhere. One area where this treatment is of obvious significance is computability theory, since results in computability theory require partiality [5]. For example, the standard way to found computability theory is via recursive functions and minimalization, the latter being partial. The aim of this paper is to investigate an equivalence between two theories that account for partiality.

An early attempt to define and classify partiality categorically is given by partial map categories. An easy textbook presentation of partial map categories is given in chapter 2 of [1]. Partial map categories are interesting and intuitively useful in that the partiality of a map, f, is given by spans. A span, $X \to Y$, is a pair of maps (m, f), $m : X' \to X$ and $f : X' \to Y$. Spans can account for partiality if one takes X' to be a domain of definition for f, and m to be monic. The drawback to partial map categories is that composition relies on forming pullbacks, and so reasoning based on this formalization of partiality can be cumbersome.

Cockett and Lack investigated a simpler approach to handling partiality using restriction categories [3]. In a restriction category, the partiality of a map, f, is determined through an idempotent map assigned to f. These idempotents capture the partiality of maps in an algebraic manner. Restriction categories have the advantage that all reasoning flows directly from these algebraic axioms, and simplifies reasoning about partiality. The drawback to restriction categories is that some of the axioms have an unobvious interpretation. In the work of Cockett and Lack, an equivalence is developed between partial map categories and restriction categories where these idempotents split. A more detailed history of this account of partiality in light of semigroup theory is made by Cockett and Manes in [4]. Restriction categories have implication outside of just discussing partiality of maps alone. Manes highlights some extra applications of restriction categories to semigroup theory and topology in [6].

2 Restriction Categories

Definition 2.1. *Given a category,* **C***, a restriction structure on* **C** *gives for each,* $A \xrightarrow{f} B$*, a restriction arrow,* $A \xrightarrow{\overline{f}} A$*, that satisfies four axioms*

R1 $f\overline{f} = f$

R2 *If* $dom(f) = dom(g)$ *then* $\overline{g}\,\overline{f} = \overline{f}\,\overline{g}$

R3 *If* $dom(f) = dom(g)$ *then* $\overline{g\overline{f}} = \overline{g}\,\overline{f}$

R4 *If* $dom(g) = cod(f)$ *then* $\overline{g}f = f\overline{gf}$

Then a category, **C***, with a specified restriction structure is a* restriction category

Example 2.1. *The trivial restriction*

The restriction structure given by $\overline{f} = 1_{dom(f)}$ always expresses a restriction structure, called the trivial restriction, and a category with only the trivial restriction is called a trivial restriction category.

Example 2.2. *Sets and partial functions*

Take the category of sets and partial functions. One can define a restriction for $A \xrightarrow{f} B$ here by $\overline{f}x = \begin{cases} x & f(x) \text{ defined} \\ \uparrow & \text{otherwise} \end{cases}$

R1 $f(\overline{f}x) = fx$ whether f(x) is defined or not.

R2 If both g and f are defined at x, then $\overline{g}\,\overline{f}x = x = \overline{f}\,\overline{g}x$. If both g and f are undefined at x, then $\overline{g}(\overline{f}x) = \uparrow = \overline{f}(\overline{g}x)$. Then WLOG, assume f is defined and g is not (at x). Then $\overline{f}(\overline{g}x) = \overline{f}\uparrow = \uparrow = \overline{g}x = \overline{g}(\overline{f}x)$.

R3 First if f is defined at x, then $g\overline{f}x = gx$, so that if g is defined at x, $\overline{g\overline{f}}x = x = \overline{g}\,\overline{f}x$, and if g is undefined at x, then $\overline{g\overline{f}}x = \uparrow = \overline{g}\,\overline{f}x$. On the other hand, if f is undefined at x, then $g\overline{f}x$ is undefined at x, so $\overline{g\overline{f}}x = \uparrow = \overline{g}\uparrow = \overline{g}\,\overline{f}x$.

R4 In the case that f is defined at x, and g is defined at, fx, then $\overline{g}fx = fx$. On the other hand, gfx is defined so $f\overline{gf}x = fx$. If g were not defined at fx, then $\overline{g}fx = \uparrow = f\uparrow = f\overline{gf}x$. In the case that f is undefined at x, then g is necessarily undefined at fx, and $\overline{g}fx = \uparrow = f\uparrow = f\overline{gf}x$.

Thus sets and partial functions form a restriction category. From this restriction category, one can see that the restriction structure defines the partiality of given functions in the usual sense. Indeed, consider the restriction morphism $\overline{f} : A \longrightarrow A$ (for any morphism f); then $\overline{f}A$ gives precisely a domain on which f is total.

Example 2.3. *Free restriction category on a graph*

Another restriction category, the free restriction category on a graph can be given explicitly, and is a nice example of a restriction category that presents a different interpretation of what partiality means. First, we will give the construction then show the restriction axioms hold.

Take a directed multigraph, G, then form a category, the free restriction category [2], where

Obj: Nodes of G

Arr: $A \overset{(s,S)}{\longrightarrow} B$ where S is a prefix-closed set of paths out of A, and $s \in S$ is a path from $A \to B$ (called the trunk). Prefix closed is the property that if tr is a path in S, then r is a path in S.

Comp: Given, $A \overset{(s,S)}{\longrightarrow} B$ and $B \overset{(t,T)}{\longrightarrow} C$ take the composite to be, $(t,T)(s,S) : A \longrightarrow C = (ts, S \cup Ts) : A \longrightarrow C$.

Id: $([], \{[]\}) : A \longrightarrow A$

The restriction on this category is given by $\overline{(s,S)} = ([], S)$. The restriction axioms are particularly easy to check.

R1

$$(s,S)([],S) = (s, S \cup S) = (s,S)$$

R2

$$\begin{aligned}([],S)([],T) &= ([], T \cup S)\\ &= ([], S \cup T)\\ &= ([],T)([],S)\end{aligned}$$

R3

$$\begin{aligned}\overline{(s,S)\overline{(t,T)}} &= \overline{(s,S)([],T)}\\ &= \overline{(s, T \cup S)}\\ &= ([], T \cup S)\\ &= ([],S)([],T)\end{aligned}$$

R4

$$\begin{aligned}\overline{(s,S)}(t,T) &= ([],S)(t,T)\\ &= (t, T \cup St)\\ &= (t,T)([], T \cup St)\\ &= (t,T)\overline{(st, T \cup St)}\\ &= (t,T)\overline{(s,S)(t,T)}\end{aligned}$$

If the trunk, $A \to B$, is thought of as a choice of a particular path out of A, then the restriction giving the nil trunk can be thought of as reflecting the choice to leave any other path out of A still open. This partiality then is a sort of partiality on the determination of paths out of A.

The following lemma shows that restriction morphisms are idempotent, and that the restriction operator is idempotent.

Lemma 2.1. *Let,* $\mathbf{C}$*, be any restriction category, and let* $f, g \in \mathbf{C}$ *with* $dom(g) = cod(f)$*.*

1. $\overline{f}\,\overline{f} = \overline{f}$
2. $\overline{f}\,\overline{gf} = \overline{gf}$
3. $\overline{\overline{g}f} = \overline{gf}$
4. $\overline{\overline{f}} = \overline{f}$

Proof.

1.

$$\begin{aligned} \overline{f}\,\overline{f} &= \overline{\overline{f}f} && \text{R3} \\ &= \overline{f} && \text{R1} \end{aligned}$$

2.

$$\begin{aligned} \overline{f}\,\overline{gf} &= \overline{gf}\,\overline{f} && \text{R2} \\ &= \overline{gf\overline{f}} && \text{R3} \\ &= \overline{gf} && \text{R1} \end{aligned}$$

3.

$$\begin{aligned} \overline{\overline{g}f} &= \overline{f\overline{gf}} && \text{R4} \\ &= \overline{f}\,\overline{gf} && \text{R3} \\ &= \overline{gf} && (2) \end{aligned}$$

4.

$$\begin{aligned} \overline{f} &= \overline{f1} && \\ &= \overline{\overline{f}1} && (3) \\ &= \overline{\overline{f}} && \end{aligned}$$

□

Since restriction maps are idempotent, we can consider the case where restriction idempotents split.

Definition 2.2. *A restriction idempotent,* e*, is a* split restriction *if there exists an* m, r *such that* $e = mr, rm = 1$*. In the splitting of* e*,* m *is a* restriction monic*.*

Given a restriction monic, then there is a unique restriction idempotent which it splits. This can be seen in that given a restriction monic, the retraction is determined. Further, given the retraction, the monic is determined.

Lemma 2.2. *Let* $\mathbf{X}$ *be a restriction category. Then*

1. *Let both* $rm, sm = 1$*, and let* $mr = \overline{r}$ *and* $ms = \overline{s}$*. Then* $r = s$
2. *Let both* $rm, rn = 1$*, and let* $mr = \overline{r}$ *or* $nr = \overline{r}$*. Then* $m = n$*.*

Proof.

1.

$$\begin{aligned}
r &= r\overline{r} && \text{R1} \\
&= rmr && \text{assumption} \\
&= rm(sm)r = r(ms)(mr) && \text{assumption sm = 1} \\
&= r\overline{s}\,\overline{r} && \text{assumption} \\
&= r\overline{r}\,\overline{s} && \text{R2} \\
&= r(mr)(ms) = (rm)(rm)s && \text{assumption} \\
&= s && \text{assumption rm = 1}
\end{aligned}$$

2. WLOG, suppose $mr = \overline{r}$,

$$\begin{aligned}
mr = m(rn)r = (mr)nr & && \text{assumption rn = 1} \\
= \overline{r}\,\overline{nr} & && \text{assumption,lemma 2.1.4} \\
= \overline{nr} & && \text{lemma 2.1.2} \\
= nr & && \text{lemma 2.1.4}
\end{aligned}$$

Since, r is epic, m = r.

□

Definition 2.3. *A* split restriction category *is a restriction category in which all restriction idempotents split.*

Restriction categories present a clean way to define what a total map is.

Definition 2.4. *In a restriction category, a* total map*,* $A \xrightarrow{f} B$*, satisfies the property that* $\overline{f} = 1_A$*.*

Recalling the sets and partial functions example from above a map, f, is total, i.e. the restriction is the identity on dom(f), when $\forall x.\overline{f}x = x$. This means though that f is defined at all x, giving the typical meaning of totality. To consider the other example, a map in the free restriction category is total when $\overline{(s, S)} = ([], \{[]\})$. But this means $S = \{[]\}$ to begin with; thus, the only total maps in this category are the identity maps.

Lemma 2.3. *In any restriction category,*

1. *All monics are total*

2. *If f and g are total and composable, then gf is total*
3. *If gf is total, then f is total*

Proof.

1. Assuming f is monic then by R1, $f\overline{f} = f = f1$ implies $\overline{f} = 1$

2. Assuming f,g are total and composable, then

$$\begin{aligned} \overline{gf} &= \overline{\overline{g}f} && \text{lemma 2.1.3} \\ &= \overline{f} && \text{by assumption} \\ &= 1 && \text{by assumption} \end{aligned}$$

3. Assuming $\overline{gf}$ is total then

$$\begin{aligned} \overline{f} &= \overline{f}\,\overline{gf} && \text{by assumption} \\ &= \overline{gf} && \text{lemma 2.1.2} \end{aligned}$$

□

Knowing that all monics are total, we can see that split restriction categories have a special significance. They are categories in which given any restriction idempotent, we can factor out a total map. This fact essentially gives a measure of partiality for any map in a split restriction category.

Further, considering that identities are monic, we get an immediate corollary:

Corollary 2.1. *The total maps in a restriction category,* $\mathbf{C}$*, denoted* **Total**$(\mathbf{C})$*, form a subcategory of* $\mathbf{C}$*.*

Definition 2.5. *Given two restriction categories,* $\mathbf{C}$ *and* $\mathbf{D}$*, a* restriction functor,$F : \mathbf{C} \to \mathbf{D}$*, is a functor which preserves the restriction, i.e.* $F(\overline{f}) = \overline{F(f)}$*.*

Lemma 2.4. *For all restriction functors,*

1. *The composite of restriction functors is a restriction functor.*
2. *Restriction functors preserve totality.*
3. *Restriction functors preserve the splitting of a restriction idempotent.*

Proof.

1. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be restriction categories and $\mathbf{A} \xrightarrow{\mathbf{F}} \mathbf{B}, \mathbf{B} \xrightarrow{\mathbf{G}} \mathbf{C}$ be restriction functors. Then we have $GF(\overline{f}) = G(\overline{F(f)}) = \overline{GF(f)}$.

2. Assume $X \xrightarrow{f} Y$ is total. Then $\overline{F(f)} = F(\overline{f}) = F(1) = 1$.

3. Let $\overline{f} = mr : X \to X$ be a split restriction for $f : X \to Y$, and let $F : \mathbf{C} \to \mathbf{D}$ be a restriction functor. Then since, $\overline{F(f)} = F(\overline{f}) = F(mr) = F(m)F(r)$, and $F(r)F(m) = F(rm) = F(1) = 1$, we have that $F(mr)$ splits $\overline{F(f)}$.

□

The above lemma gives the notion of a category of restriction categories, $\mathbf{rCat_0}$ where the objects are restriction categories and the arrows are restriction functors. We will shortly see an important example of a restriction functor.

Definition 2.6. *A* restriction natural transformation *between restriction functors, $F, G : \mathbf{C} \to \mathbf{D}$, is a natural transformation $\eta : F \Rightarrow G$ where η_c is total for each component c.*

Lemma 2.5. *The vertical and horizontal composites of restriction transformations is a restriction transformation.*

Proof. The fact that vertical composition preserves totality guaranteed by lemma 2.3.2. To see horizontal composition preserves totality, consider the situation where α, β are restriction natural transformations. Then, one has the following situation, given $\mathbf{A}, \mathbf{B}, \mathbf{C}$ restriction categories, and F_i, G_i restriction functors.:

$$\begin{tikzcd}
\mathbf{A} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{B} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbf{C}
\end{tikzcd}$$

consider $\alpha\beta = (F_1\beta)(\alpha G_2)$. now, $(\alpha G_2)_x = G_2(\alpha_x)$ is total for any x, since α_x is total by assumption, and G_2 is a restriction functor, hence preserves totalness. also $(F_1\beta)_y = \beta_{F_1(y)}$ is total by assumption. by lemma 2.3.2, we have that $(F_1\beta)(\alpha G_2)$ is total.

□

Proposition 2.1. *The category,* $\mathbf{rcat}$*, equipped with objects as restrictions categories, arrows as restriction functors, and 2-cells as restriction natural transformations is a 2-category.*

Proof. Lemma 2.4 gives the 1-cells for ordinary category of restriction categories, and lemma 2.5 extends the category with restriction natural transformations as an appropriate 2-cell structure for a 2-category of restriction categories. □

Lemma 2.4.3 says that given any two split restriction categories, $\mathbf{C}, \mathbf{D}$ that any map $F : \mathbf{C} \longrightarrow \mathbf{D} \in \mathbf{rCat}$ preserves splittings. Thus we have a full subcategory $\mathbf{rCat_s}$ of split restriction categories.

3 $\mathcal{M}$-Categories

Starting with a desire to understand partiality, restriction categories give a clean way to discuss partial maps. This is clear from the example of the category of sets and partial functions, that the restriction structure captures neatly both partiality through idempotent maps as well as totality. Another way to understand partiality comes from a development of $\mathcal{M}$-Categories, which will be defined below. As above, $\mathcal{M}$-Categories form a 2-category, and this development will be made below.

Definition 3.1. *Let $\mathbf{C}$ be a category, and $\mathcal{M}$ a class of monics in $\mathbf{C}$. $\mathcal{M}$ is a* stable system of monics *in case*

SSM1 *All isomorphisms are in $\mathcal{M}$*

SSM2 *$\mathcal{M}$ is closed to composition*

SSM3 *For any $m : B' \to B \in \mathcal{M}$, $f : A \to B \in \mathbf{C}$ the following pullback, called an $\mathcal{M}$-pullback, exists and $m' \in \mathcal{M}$:*

$$\begin{tikzcd}
A' \arrow[r, "f"] \arrow[d, "m'"'] & B' \arrow[d, "m"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Definition 3.2. *An $\mathcal{M}$-*Category *is a pair $(\mathbf{C}, \mathcal{M})$ where $\mathbf{C}$ is a category with a specified system of stable monics $\mathcal{M}$.*

Example 3.1. *Trivial system of monics*

Give any category, take $\mathcal{M}$ to be the collection of isos. This is called the trivial system of monics.

Example 3.2. *Categories with pullbacks and all monics*

Given a category with pullbacks, the set of all monics form a stable system of monics. Isomorphisms are monic, and the composite of monics is monic. Further, pullbacks preserve monicity, since if we have $f : A \to B, m : B' \to B$, the pullback exists by assumption.

$$\begin{tikzcd}
A' \arrow[r, "f'"] \arrow[d, "d"'] & B' \arrow[d, "m"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

Suppose there are two maps $h, k : Q \to A'$ s.t. $dh = dk$. Then we have also that $mf'k = fdk = fdh = mf'h$, thus from monicity of m, $f'k = f'h$. This means that both triangles, (1) and (2) in the following diagram commute,

$$\begin{tikzcd}
Q \arrow[dr, shift left, "h"] \arrow[dr, shift right, "k"'] \arrow[drr, bend left, "f'k=f'h"] \arrow[ddr, bend right, "dk=dh"'] & (2) & \\
(1) & A' \arrow[r, "f'"'] \arrow[d, "d"] & B' \arrow[d, "m"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

But the universal property of pullbacks states that there is a unique arrow $Q \to A'$ such that both triangles commute, so $h = k$, proving d is monic.

In particular, (**Sets**,Monics) is an $\mathcal{M}$-Category.

Lemma 3.1. *Given any split restriction category, $\mathbf{C}$, the collection of restriction monics M_C forms a stable system of monics in the subcategory* **Total**$(\mathbf{C})$.

Proof.

SSM1 $1 = \overline{1}$ is a restriction idempotent, so given any isomorphism ϕ, $1 = \phi\phi^{-1}$, so ϕ is the monic part of the splitting of a restriction idempotent.

SSM2 It suffices to show that $m'm$ is the monic part of the splitting for $m'mrr'$ The situation is expressed by the following diagram where $e' = m'r', e = mr$, $r'm' = 1_B, rm = 1_C$

$$\begin{tikzcd}
A \arrow[loop above, "e'"] \arrow[d, bend right, "r'"'] \\
B \arrow[u, bend right, "m'"'] \arrow[d, bend right, "r"'] \arrow[loop below, "e"] \\
C \arrow[u, bend right, "m"']
\end{tikzcd}$$

First, $rr'm'm = rm = 1$ by assumption that $r'm' = 1$ and $rm = 1$.
Consider,

$$\begin{aligned}
m'mrr' &= m'\overline{mr}r' && \text{assumption} \\
&= m'r'\overline{mrr'} && \text{R4} \\
&= \overline{m'r'}\,\overline{mrr'} && \text{assumption} \\
&= \overline{m'r'\overline{mrr'}} && \text{R3}
\end{aligned}$$

So $m'm$ is the monic of the splitting of $\overline{m'r'\overline{mrr'}} = m'mrr'$. One can also note that since $m'mrr'$ is a restriction idempotent thus by lemma 2.1.4, $m'mrr' = \overline{m'mrr'}$. Thus $m'm$ is the restriction monic of $m'mrr'$.

SSM3 The proof that M_C has $\mathcal{M}$-pullbacks is a bit more involved. It involves constructing pullbacks in any split restriction category, using only the splittings and composition. This construction will be referred to as $(P_{M_X}))$. Suppose that $m_e : B' \to B \in M_C, A \to B \in \mathbf{Total(C)}$, the fact that we are in a split restriction category immediately gives the following data.

$$\begin{tikzcd}
& B' \arrow[d, "m_e"] \\
A \arrow[r, "f"'] \arrow[loop below, "\overline{ef}"] & B \arrow[u, bend right, "r_e"'] \arrow[loop below, "\overline{e}"]
\end{tikzcd}$$

Next, we can form a square by splitting $\overline{ef} = mr$.

$$\begin{tikzcd}
A' \arrow[r, "r_e f m"] \arrow[d, "m"'] & B' \arrow[d, "m_e"] \\
A \arrow[u, bend left, "r"] \arrow[r, "f"'] \arrow[loop below, "\overline{ef}"] & B \arrow[u, bend right, "r_e"'] \arrow[loop below, "\overline{e}"]
\end{tikzcd}$$

To show that the square commutes, i.e. $fm = m_e r_e f m$, note that

$$\begin{aligned} m_e r_e f m &= \overline{e} f m && \text{assumption, } m_e \in M_C \\ &= f\overline{ef}m && \text{R4} \\ &= fmrm && \text{assumption, } mr = \overline{ef} \\ &= fm && \text{splitting, } rm = 1 \end{aligned}$$

On a technical point, we are in **Total**(**C**), so we must show that $r_e f m$ is a total map. But, f is assumed to be total. m, m_e are total, since all monics are total by lemma 2.3.1, and so fm is total. Since $fm = m_e r_e f m$, $r_e f m$ is total by 2.3.3. The splitting of restrictions guaranteed that m is monic, so it remains to show that we indeed have a pullback defined by this square. Suppose we have an object Q with maps $k : Q \longrightarrow B', h :\longrightarrow A \in$ **Total**(**C**) that form a commuting square.

$$\begin{tikzcd} Q \arrow[rrd, bend left, "k"] \arrow[rdd, bend right, "h"'] \arrow[rd, dotted, "q"] \arrow[rrd, phantom, "(b)"] \arrow[rdd, phantom, "(a)"] & & \\ & A' \arrow[r, "r_e f m"'] \arrow[d, "m"] & B' \arrow[d, "m_e"] \\ & A \arrow[r, "f"'] & B \end{tikzcd}$$

Now, we know there is a map from $q : Q \longrightarrow A'$, namely rh. Now, to see that (a) commutes,

$$\begin{aligned} h &= h\overline{m_e k} && \text{by assumption } m_e k \text{ is total} \\ &= h\overline{m_e r_e m_e k} && r_e m_e = 1 \text{ per splittings} \\ &= h\overline{e m_e k} && m_e r_e = e \text{ per splittings} \\ &= \overline{ef}h && \text{R4} \\ &= mrh && \overline{ef} = mr \end{aligned}$$

(b) commutes since m_e is monic,

$$\begin{aligned} m_e r_e f m q &= m_e r_e f m r h \\ &= m_e r_e f g && \text{h = mrh by above} \\ &= m_e r_e m_e k && \text{commutativity} \\ &= m_e k && r_e m_e = 1 \end{aligned}$$

Since m_e is monic, $r_e f m q = k$, and (b) does commute. Finally, the fact that m is monic ensures that q is unique.

□

Thus for any split restriction the above lemma gives another example of an $\mathcal{M}$-Category.

To define a functor between $\mathcal{M}$-Categories, we want a functor that preserves the qualities of an $\mathcal{M}$-Category.

Definition 3.3. *An $\mathcal{M}$-Functor is a functor between $\mathcal{M}$-Categories, $F : (\mathbf{C}, \mathcal{M}) \longrightarrow (\mathbf{D}, \mathcal{N})$ such that*

MF1 $\forall m \in \mathcal{M}. F(m) \in \mathcal{N}$

MF2 *F preserves $\mathcal{M}$-pullbacks*

Lemma 3.2. *For any $\mathcal{M}$-Functors,*

1. *The composition of $\mathcal{M}$-Functors is an $\mathcal{M}$-Functor*
2. *The identity functor is an $\mathcal{M}$-Functor*

Proof.

1. Assume $(\mathbf{C}, \mathcal{M}) \xrightarrow{\mathbf{F}} (\mathbf{D}, \mathcal{N}) \xrightarrow{\mathbf{G}} (\mathbf{E}, \mathcal{L})$. The composite of functors is a functor. Also $F(m) \in \mathcal{N}$, and hence $G(F(m)) \in \mathcal{L}$, so MF1 is satisfied. Given any $\mathcal{M}$-pullback, P, F(P) is an N-pullback, and so G(F(P)) is a L-pullback, so MF2 is satisfied.

2. $1_C(m) = m \in \mathcal{M}$, so MF1 is covered. If P is a pullback, $1_C(P)$ is the same pullback, so MF2 is covered.

□

The above lemma, along with general results of associativity of functorial composition, shows that there is a category whose objects are $\mathcal{M}$-Categories and arrows are $\mathcal{M}$-Functors. Call this category **MCat**. Next, we develop **MCat** as a 2-category.

Definition 3.4. *An $\mathcal{M}$-Cartesian natural transformation between $\mathcal{M}$-Functors is a natural transformation, $\alpha : F \Rightarrow G : (\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{N})$ such that for every $m \in \mathcal{M}$, naturality about m produces a pullback, i.e. the following square is a pullback.*

$$\begin{tikzcd}
FA \arrow[r, "Fm"] \arrow[d, "\alpha_A"'] & FB \arrow[d, "\alpha_B"] \\
GA \arrow[r, "Gm"'] & GB
\end{tikzcd}$$

Lemma 3.3. *$\mathcal{M}$-Cartesian natural transformations are closed under vertical and horizontal composition.*

Proof. Given $\alpha : F \Rightarrow G : (\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{N}), \beta : \mathbf{G} \Rightarrow \mathbf{H} : (\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{N})$, $\mathcal{M}$-Cartesian natural transformations, we must show that $\alpha\beta : F \Rightarrow H : (\mathbf{C}, \mathcal{M}) \to (\mathbf{D}, \mathcal{N})$ is an $\mathcal{M}$-cartesian natural transformation for vertical composition, i.e. the outer square is a pullback in the following diagram.

$$\begin{tikzcd}
FA \arrow[r, "Fm"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(1)"] & FB \arrow[d, "\alpha_B"] \\
GA \arrow[r, "Gm" description] \arrow[d, "\beta_A"'] \arrow[dr, phantom, "(2)"] & GB \arrow[d, "\beta_B"] \\
HA \arrow[r, "Hm"'] & HB
\end{tikzcd}$$

By assumption, (1) and (2) are pullbacks, and pullbacks are closed to composition, hence the outer square is indeed a pullback.

For horizontal composition, we must show the horizontal composite of $\mathcal{M}$-Cartesian natural transformations is an $\mathcal{M}$-Cartesian natural transformation. Suppose

$$\begin{tikzcd}
(\mathbf{W}, \mathcal{M}) \arrow[r, bend left, "R_1"] \arrow[r, bend right, "S_1"'] \arrow[r, phantom, "\Downarrow\lambda"] & (\mathbf{X}, \mathcal{N}) \arrow[r, bend left, "R_2"] \arrow[r, bend right, "S_2"'] \arrow[r, phantom, "\Downarrow\xi"] & (\mathbf{Y}, \mathcal{L})
\end{tikzcd}$$

The burden is that $\lambda.\xi = (R_1\xi)(\lambda S_2)$ be shown $\mathcal{M}$-Cartesian, and so the outer square in the following diagram must be a pullback.

$$\begin{tikzcd}
R_2(R_1(w_1)) \arrow[r, "R_2(R_1(m))"] \arrow[d, "\xi_{R_1(w_1))}"'] \arrow[dr, phantom, "(1)"] & R_2(R_1(w_2)) \arrow[d, "\xi_{R_1(w_2)}"] \\
S_2(R_1(w_1)) \arrow[r, "S_2(R_1(m))"] \arrow[d, "S_2(\lambda_{w_1})"'] \arrow[dr, phantom, "(2)"] & S_2(R_1(w_2)) \arrow[d, "S_2(\lambda w_2)"] \\
S_2(S_1(w_1)) \arrow[r, "G_2(G_1(m))"'] & S_2(S_1(w_2))
\end{tikzcd}$$

Since, $R_1(m) \in N$, $R_2(R_1(m)), S_2(R_1(m)) \in L$ by the assumption that R_i, S_i are $\mathcal{M}$-Functors. Thus, since, ξ is assumed to be an $\mathcal{M}$-Cartesian natural transformation, (1) is a pullback. Next, note that

$$\begin{tikzcd}
R_1(w_1) \arrow[r, "R_1(m)"] \arrow[d, "\lambda_{w_1}"'] & R_1(w_2) \arrow[d, "\lambda_{w_2}"] \\
S_1(w_1) \arrow[r, "S_1(m)"'] & S_1(w_2)
\end{tikzcd}$$

Is a pullback by the assumption λ is an $\mathcal{M}$-Cartesian natural transformation. Since S_2 is an $\mathcal{M}$-Functor, it preserves $\mathcal{M}$-Pullbacks; therefore, (2) is a pullback. Since the composite of pullbacks is a pullback, the outer square is a pullback. □

Proposition 3.1. *$\mathcal{M}$-Categories, $\mathcal{M}$-Functors, and $\mathcal{M}$-Cartesian Natural Transformations are a 2-category.*

Proof. By lemma 3.2 and lemma 3.3. □

4 2-Functor MTotal

This section develops a 2-functor from $\mathbf{rCat_s}$ of split restriction categories to **MCat**. This construction demonstrates that for given any split restriction category, there is a natural way in which it transforms to an $\mathcal{M}$-Category.

Definition 4.1. *Let $\mathbf{X}$ be a split restriction category, then define $MTotal(X) = (\mathbf{Total(X)}, \mathbf{M_X})$*

Lemma 4.1. *Let $\mathbf{X}$ be a split restriction category, then $MTotal(X)$ is an $\mathcal{M}$-Category.*

Proof. By Corollary 2.1, $\mathbf{Total(X)}$ is a (split restriction) category, by lemma 3.1, M_X is a stable system of monics for this category. □

To extend MTotal to a functor, MTotal must be defined on maps, and to a 2-functor, defined on 2-cells. The definition given for MTotal will yield an $\mathcal{M}$-Functor for each restriction functor and an $\mathcal{M}$-Cartesian natural transformation for each restriction natural transformation.

Proposition 4.1. *Let $\mathbf{X}, \mathbf{Y}$ be split restriction categories. Define MTotal on a restriction functor $F : \mathbf{X} \longrightarrow \mathbf{Y}$ to just be F restricted to $\mathbf{Total}(\mathbf{X})$, i.e. $MTotal(F) : MTotal(\mathbf{X}) \longrightarrow \mathbf{MTotal(Y)} = \mathbf{F}_{\|\mathbf{Total(X)}} : \mathbf{MTotal(X)} \longrightarrow \mathbf{MTotal(Y)}$. Since restriction natural transformations are total at each component, it is appropriate to define $MTotal(\alpha) = \alpha$, for any restriction transformation α. Then MTotal, so defined, is a 2-functor.*

Proof. It is clear that the definition of $MTotal(F) = F_{\|\mathbf{Total(X)}}$ gives a functor. To claim that $MTotal(F)$ is an $\mathcal{M}$-Functor, it must preserve M_X elements,

MF1 By lemma 2.4.3, restriction functors preserve the splitting of idempotents, and so $\forall m \in M_X . F(m) \in M_Y$.

MF2 Functors preserve composition, domain, and codomain, so MTotal it will preserve $\mathcal{M}$-Pullbacks. Consider the construction of pullbacks, P_{M_X} by

Abbreviating $MTotal(F)$ by F since all the maps in the following are total, and by considering $F(f), F(m_e)$ with the note that restriction functors preserve idempotent splitting, we can construct a pullback in M_Y as

$$\begin{tikzcd}
FA' \arrow[r, "F(r_e)F(f)F(m)"] \arrow[d, "Fm"'] & FB' \arrow[d, "Fm_e"] \\
FA \arrow[r, "Ff"'] \arrow[u, bend left=60, "Fr"] \arrow[loop below, "F\overline{ef}"] & FB \arrow[u, bend right=60, "Fr_e"'] \arrow[loop below, "F\overline{e}"]
\end{tikzcd}$$

by construction P_{M_X}. But note the original pullback of, f, m_e, is constructed

$$\begin{tikzcd}
A' \arrow[r, "r_e f m"] \arrow[d, "m"'] & B' \arrow[d, "m_e"] \\
A \arrow[r, "f"'] \arrow[u, bend left=60, "r"] \arrow[loop below, "\overline{ef}"] & B \arrow[u, bend right=60, "r_e"'] \arrow[loop below, "\overline{e}"]
\end{tikzcd}$$

by construction P_{M_X}. Thus $MTotal(F)$ preserves $\mathcal{M}$-Pullbacks, as the constructed pullback is simply F of all the components of the original.

Therefore, MTotal is an functor. To prove MTotal is a 2-functor, we must show that 2-cells are sent to 2-cells; i.e. restriction natural transformations are sent to $\mathcal{M}$-Cartesian natural transformations.

Let $\alpha : F \Longrightarrow G : X \Longrightarrow Y$ be a restriction natural transformation. We need to show that

$$\begin{tikzcd}
FA \arrow[r, "Fm"] \arrow[d, "\alpha_A"'] & FB \arrow[d, "\alpha_B"] \\
GA \arrow[r, "Gm"'] & GB
\end{tikzcd}$$

is a pullback, since by assumption α is a natural transformation.

Note, that Gm is the monic part of the splitting of $\overline{G(mr)}$. Next, note that, the second part of constructing pullbacks via P_{M_X} is to split $\overline{G(mr)\alpha_B}$ Thus note,

$$\begin{aligned}
F(m)F(r) &= F(mr) & \\
&= F(\overline{mr}) & \text{lemma 2.1.4} \\
&= \overline{F(mr)} & \text{Restriction functor} \\
&= \overline{\alpha_B}\,\overline{F(mr)} & \alpha_B \text{ is total} \\
&= \overline{\alpha_B \overline{F(mr)}} & \text{R3} \\
&= \overline{\alpha_B F(mr)} & \text{lemma 2.1.4} \\
&= \overline{G(mr)\alpha_B} & \text{Naturality}
\end{aligned}$$

Next, across the top of the diagram of P_{M_X}, is expected to be $G(r)\alpha_B F(m)$, but note

$$\begin{aligned}
G(r)\alpha_B F(m) &= G(r)G(m)\alpha_A & \text{Naturality} \\
&= G(rm)\alpha_A & \\
&= \alpha_A & \text{Per splitting, G(rm) = 1}
\end{aligned}$$

Now, to redraw the original diagram as

$$\begin{tikzcd}
FA \arrow[r, "G(r)\alpha_B F(m)"] \arrow[d, "Fm"'] & GA \arrow[d, "Gm"] \\
FB \arrow[r, "\alpha_B"'] \arrow[loop below, "\overline{G(mr)\alpha_B}"] & GB \arrow[u, bend right=60, "Gr"'] \arrow[loop below, "\overline{G(mr)}"]
\end{tikzcd}$$

Which is exactly the construction of the pullback, via P_{M_X}, of α_B, and $G(m)$.
Therefore, $MTotal$ is a 2-functor. □

5 2-Functor Par

This section will give a construction that for any $\mathcal{M}$-Category, yields a split restriction category, called Par. Par will be extended to a 2-functor from **MCat** to $\mathbf{rCat_s}$.

Definition 5.1. *Let $(\mathbf{C}, \mathcal{M})$ be an $\mathcal{M}$-Category. Define $Par(\mathbf{C}, \mathcal{M})$ to be the category where*

Obj: *The objects of* $\mathbf{C}$

Arr: $A \overset{(m,f)}{\longrightarrow} B$ *are classes of spans* (m, f),

$$\begin{tikzcd}
 & A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\
A & & B
\end{tikzcd}$$

where $m \in \mathcal{M}$. The classes of spans are formed under the equivalence relation $\sim$ which is the relation, $(m, f) \sim (m', f')$ if there is an isomorphism, ϕ, such that both triangles in the following diagram commute.

$$\begin{tikzcd}
 & A' \arrow[r, dotted, "\alpha"] \arrow[dl, bend right, "m"'] \arrow[drr, "f"] & A'' \arrow[dr, bend left, "f'"] \arrow[dll, "m'"] & \\
A & & & B
\end{tikzcd}$$

Id: $A \overset{(1_A,1_A)}{\longrightarrow} A$

Comp: *By pullback; i.e. given* $A \overset{(m,f)}{\longrightarrow} B, B \overset{(m',f')}{\longrightarrow} C$*, the pullback*

$$\begin{tikzcd}
 & & A'' \arrow[dl, "m''"'] \arrow[dr, "f''"] & & \\
 & A' \arrow[dl, "m"'] \arrow[dr, "f"] & & B' \arrow[dl, "m'"'] \arrow[dr, "f'"] & \\
A & & B & & C
\end{tikzcd}$$

gives a composite $(mm'', f'f'') : A \longrightarrow C$. To note, this composition requires the above $\sim$, to get the associative law to hold.

Lemma 5.1. *$Par(\mathbf{C}, \mathcal{M})$ is a restriction category with the restriction structure given by $\overline{(m, f)} = (m, m)$.*

Proof. **R1** , consider $(m, f)(m, m)$, noting $m : A' \to A$ is monic iff

$$\begin{tikzcd}
A' \arrow[r, "1_{A'}"] \arrow[d, "1_{A'}"'] & A' \arrow[d, "m"] \\
A' \arrow[r, "m"'] & A
\end{tikzcd}$$

is a pullback, so the composite is

$$\begin{tikzcd}
 & & A' \arrow[dl, "1_{A'}"'] \arrow[dr, "1_{A'}"] & & \\
 & A' \arrow[dl, "m"'] \arrow[dr, "m"] & & A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\
A & & A & & B
\end{tikzcd}$$

Thus, $(m, f)(m, m) = (m1_{A'}, f1_{A'}) = (m, f)$, giving R1.

R2 , consider $\overline{(m, f)} = (m, m)$, and $\overline{(m', g)} = (m', m')$

$$\begin{tikzcd}
& & A' \arrow[dl, "1_{A'}"'] \arrow[dr, "1_{A'}"] & & \\
& A' \arrow[dl, "m"'] \arrow[dr, "m"] & & A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\
A & & A & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& & A' \arrow[dl, "1_{A'}"'] \arrow[dr, "1_{A'}"] & & \\
& A' \arrow[dl, "m"'] \arrow[dr, "m"] & & A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\
A & & A & & B
\end{tikzcd}$$

The above diagrams must be equal, which means we need α satisfying $\sim$. So take $\alpha = 1_{A''}$. Then we need, $m'x' = mx$, but this is given by the commutativity of the pullback.

R3 Take $A \overset{(m,f)}{\longrightarrow} B, A \overset{(m',f')}{\longrightarrow} C$, then we have,

$$\begin{tikzcd}
& A \arrow[dl, bend right, "m"'] \arrow[dr, bend left, "f"] & & A'' \arrow[dlll, "m'"] \arrow[dr, bend left, "f'"] & \\
A & & B & & C
\end{tikzcd}$$

First, consider $(m', f')(m, m)$,

$$\begin{tikzcd}
& & A'' \arrow[dl, "m''"'] \arrow[dr, "f''"] & & \\
& A' \arrow[dl, "m"'] \arrow[dr, "m"] & & B' \arrow[dl, "m'"'] \arrow[dr, "f'"] & \\
A & & A & & B
\end{tikzcd}$$

Thus, $\overline{(m'f')}(m, m) = \overline{(mm'', ff'')} = (mm'', mm'')$
Next, consider $\overline{(m', f')}\,\overline{(m, f)} = (m', m')(m, m)$,

$$\begin{tikzcd}
& & A'' \arrow[dl, "m''"'] \arrow[dr, "f''"] & & \\
& A' \arrow[dl, "m"'] \arrow[dr, "m"] & & B' \arrow[dl, "m"'] \arrow[dr, "m"] & \\
A & & A & & A
\end{tikzcd}$$

Next, note that $m'f'' = mm''$ by the commutativity of the pullback; hence, $\overline{(m', f')}\,\overline{(m, f)} = (mm'', m'f'') = (mm'', mm'')$. Therefore, [R3] holds.

R4 Take $A \overset{(m,f)}{\longrightarrow} B, B \overset{(m',g)}{\longrightarrow} C$ then the composite gives $(m', g)(m, f) = (mm'', gf')$, as

$$\begin{tikzcd}
 & & A'' \arrow[dl, "m''"'] \arrow[dr, "f'"] & & \\
 & A' \arrow[dl, "m"'] \arrow[dr, "f"] & & B' \arrow[dl, "m'"'] \arrow[dr, "g"] & \\
A & & B & & C
\end{tikzcd}$$

Next consider “$\overline{g}f$,” i.e. $(m', m')(m, f)$,

$$\begin{tikzcd}
 & & A'' \arrow[dl, "m''"'] \arrow[dr, "f'"] & & \\
 & A' \arrow[dl, "m"'] \arrow[dr, "f"] & & B' \arrow[dl, "m'"'] \arrow[dr, "m'"] & \\
A & & B & & B
\end{tikzcd}$$

Thus using the commutativity of the pullback to give $m'f' = fm''$, conclude $(m', m')(m, f) = (mm'', m'f') = (mm'', fm'')$
Next consider “$f\overline{gf}$, i.e. $(m, f)(mm'', mm'')$,

$$\begin{tikzcd}
 & & A'' \arrow[dl, "1_{A''}"'] \arrow[dr, "m''"] & & \\
 & A'' \arrow[dl, "mm''"'] \arrow[dr, "mm''"] & & A' \arrow[dl, "m"'] \arrow[dr, "f"] & \\
A & & A & & B
\end{tikzcd}$$

Thus, $(m, f)(mm'', mm'') = (mm'', fm'')$, and so [R4] holds.
Therefore, $Par(\mathbf{C}, \mathcal{M})$ under $\overline{(m, f)} = (m, m)$ is a restriction category.

□

Lemma 5.2. *The restriction structure on $Par(\mathbf{C}, \mathcal{M})$ is split.*

Proof. Consider that $(m, m) = (1, m)(m, 1)$, and $(1, 1) = (m, 1)(1, m)$ since,

$$\begin{tikzcd}
 & & A \arrow[dl, "1_A"'] \arrow[dr, "1_A"] & & \\
 & A \arrow[dl, "m"'] \arrow[dr, "1_A"] & & A \arrow[dl, "1_A"'] \arrow[dr, "m"] & \\
A & & A & & A
\end{tikzcd}
=
\begin{tikzcd}
 & A \arrow[dl, "m"'] \arrow[dr, "m"] & \\
A & & A
\end{tikzcd}$$

And

$$\begin{tikzcd}[column sep=small]
& & A \arrow[dl, "1_A"'] \arrow[dr, "1_A"] & & \\
& A \arrow[dl, "1_A"'] \arrow[dr, "m"] & & A \arrow[dl, "m"'] \arrow[dr, "1_A"] & \\
A & & A & & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=small]
& A \arrow[dl, "1_A"'] \arrow[dr, "1_A"] & \\
A & & A
\end{tikzcd}$$

□

Lemma 5.3. *Par extends to a restriction functor.*

Proof. Let, $(\mathbf{C},\mathcal{M})$, $(\mathbf{D},\mathcal{N})$ be $\mathcal{M}$-categories, and $F : (\mathbf{C},\mathcal{M}) \longrightarrow (\mathbf{D},\mathcal{N})$ be an $\mathcal{M}$-Functor. Define $Par(\mathbf{C},\mathcal{M}) \overset{\mathbf{Par(F)}}{\longrightarrow} \mathbf{Par}(\mathbf{D},\mathcal{N})$, by $Par(F)(A) = FA$ and $Par(F)(m,f) = F(m,f) = (Fm, Ff)$.
First, $F(1_A, 1_A) = (F1_A, F1_A) = (1_{FA}, 1_{FA})$. Next, Par(F) preserves composition. Let $(m', f) : A \longrightarrow B, (m, g) : B \longrightarrow C$. First, consider

$$\begin{tikzcd}[column sep=small]
& & A'' \arrow[dl, "m''"'] \arrow[dr, "f'"] & & \\
& A' \arrow[dl, "m'"'] \arrow[dr, "f"] & & B' \arrow[dl, "m"'] \arrow[dr, "g"] & \\
A & & B & & C
\end{tikzcd}$$

So that

$$\begin{aligned}
F((m,g)(m',f)) &= F(m'm'', gf') & & \\
&= (F(m'm''), F(gf')) & & \text{Definition of Par(F)} \\
&= (F(m')F(m''), F(g)F(f')) & & \text{F is an } \mathcal{M}\text{-functor}
\end{aligned}$$

On the other hand, $F(m,g)F(m',f) = (Fm, Fg)(Fm', Ff)$, and using the fact that $\mathcal{M}$-Functors preserve $\mathcal{M}$-pullbacks then,

$$\begin{tikzcd}[column sep=small]
& & FA'' \arrow[dl, "Fm''"'] \arrow[dr, "Ff'"] & & \\
& FA' \arrow[dl, "Fm'"'] \arrow[dr, "Ff"] & & FB' \arrow[dl, "Fm"'] \arrow[dr, "Fg"] & \\
FA & & FB & & FC
\end{tikzcd}$$

gives that $(Fm, Fg)(Fm', Ff) = (F(m')F(m''), F(g)F(f'))$. Thus $Par(F)$ is a functor.
To show Par(F) is a restriction functor, consider

$$
\begin{aligned}
\overline{Par(F)(m,f)} &= \overline{F(m,f)} \\
&= \overline{(Fm,Ff)} \\
&= (Fm,Fm) \\
&= F(m,m) \\
&= F\overline{(m,f)}
\end{aligned}
$$

Thus Par(F) is a restriction functor. $\square$

Proposition 5.1. *Par extends to a 2-functor.*

Proof. Given, $\alpha : F \Longrightarrow G : (\mathbf{C},\mathcal{M}) \longrightarrow (\mathbf{D},\mathcal{N})$ be an $\mathcal{M}$-Cartesian natural transformation, and define $Par(F)A \overset{Par(\alpha)_A}{\longrightarrow} Par(G)A =$

$$\begin{tikzcd}
 & FA \arrow[dl, "1_{FA}"'] \arrow[dr, "\alpha_A"] & \\
FA & & GA
\end{tikzcd}$$

Immediately, $Par(\alpha)$ is total at each component, since $\overline{(1_{FA},\alpha_A)} = (1_{FA},1_{FA})$. Thus, we need naturality, i.e. the following diagram must commute.

$$\begin{tikzcd}
FA \arrow[r, "{(Fm,Ff)}"] \arrow[d, "{(1_{FA},\alpha_A)}"'] & FB \arrow[d, "{(1_{FB},\alpha_B)}"] \\
GA \arrow[r, "{(Gm,Gf)}"'] & GB
\end{tikzcd}$$

Thus, it must be shown that $(1_{FB},\alpha_B)(Fm,Ff) = (Gm,GF)(1_{FA},\alpha_A)$

Note, that the fact that α is an $\mathcal{M}$-Cartesian natural transformation, gives that the following is a pullback.

$$\begin{tikzcd}
FA' \arrow[r, "Fm"] \arrow[d, "\alpha_{A'}"'] & FA \arrow[d, "\alpha_A"] \\
GA' \arrow[r, "Gm"'] & GA
\end{tikzcd}$$

Now, consider the composites under question in $Par(\mathbf{D},\mathcal{N})$.

$$\begin{tikzcd}
 & & FA' \arrow[dl, "1_{FA'}"'] \arrow[dr, "Ff"] & & \\
 & FA' \arrow[dl, "Fm"'] \arrow[dr, "Ff"] & & B' \arrow[dl, "1_{FB}"'] \arrow[dr, "\alpha_B"] & \\
FA & & FB & & GB
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & FA' \arrow[dl, "Fm"'] \arrow[dr, "\alpha_{A'}"] & & \\
 & FA \arrow[dl, "1_{FA}"'] \arrow[dr, "\alpha_A"] & & GA' \arrow[dl, "Gm"'] \arrow[dr, "Gf"] & \\
FA & & GA & & GB
\end{tikzcd}$$

The left gives, $(Fm,\alpha_B F(f))$ and the right gives $(Fm,G(f)\alpha_{A'})$. So to show naturality, it suffices to show that $\alpha_B F(f) = G(f)\alpha_{A'}$. But drawing the equality gives the naturality of α.

$$\begin{tikzcd}
FA' \arrow[r, "Ff"] \arrow[d, "\alpha_{A'}"'] & FB \arrow[d, "\alpha_B"] \\
GA' \arrow[r, "Gf"'] & GB
\end{tikzcd}$$

Thus $Par(\alpha)$ is a restriction natural transformation. Therefore, with lemma 5.1 and lemma 5.3, Par is a 2-Functor $Par : \mathbf{MCat} \longrightarrow \mathbf{rCat_s}$. $\square$

6 Equivalence of $\mathbf{rCat_s}$ and MCat

Theorem 6.1. *There is a 2-equivalence of 2-categories between* **MCat** *and* $\mathbf{rCat_s}$ *given by* $MTotal$ *and* Par.

Proof. By way of the following two propositions exerting the existence of 2-natural isomorphisms. One asserts that $1 \cong_2 Par \circ MTotal$, the other asserts that $MTotal \circ Par \cong_2 1$ $\square$

Proposition 6.1. $MTotal \circ Par \cong_2 1$

Proof. For any total map,$(m, f) : A \longrightarrow B$, in $Par(\mathbf{C}, \mathcal{M})$, $\overline{(m, f)} = (m, m) = (1_A, 1_A)$. Thus $m = 1_A$. Then the span is trivial, i.e.

$$\begin{tikzcd}
 & A \arrow[dl, "1_A"'] \arrow[dr, "f"] & \\
A & & B
\end{tikzcd}$$

.

Next, consider $M_{Par(C,M)}$. Since by lemma 5.2, $\overline{(m, f)} = (m, m) = (1, m)(m, 1)$; the restriction monic is just the $\mathcal{M}$-map m. Thus $M_{Par(C,M)}$ is just M. Thus, one can take the identity 2-natural isomorphism, from $MTotal \circ Par \Longrightarrow 1$, and conclude that $MTotal \circ Par \cong_2 1$ $\square$

Proposition 6.2. $1 \cong_2 Par \circ MTotal$.

Proof. To give an explicit construction of the unit of the adjuntion, define $\varphi : 1 \Rightarrow Par \circ MTotal$ to be the following family of maps.

For each split restriction category, $\mathbf{X}$, define

$$\varphi_X : \mathbf{X} \longrightarrow Par(MTotal(\mathbf{X}))$$

$$A \in \mathbf{X} \mapsto A \in Par(MTotal(\mathbf{X})) \qquad \text{id. on objects}$$

$$A \xrightarrow{f} B \in \mathbf{X} \mapsto \begin{tikzcd}
 & A' \arrow[dl, "m"'] \arrow[dr, "fm"] & \\
A & & B
\end{tikzcd}$$

Since $fm \in \mathbf{Total}(\mathbf{X})$, by assumption, fm must be total. This is true since,

$$\begin{aligned}
\overline{fm} &= \overline{\overline{f}m} && \text{lemma 2.1.3} \\
&= \overline{mrm} && \text{assumption } \overline{f} = mr \\
&= \overline{m} && \text{per splitting, rm = 1} \\
&= 1 && \text{m is total}
\end{aligned}$$

Next, it must be shown that φ_X is a functor. Consider $\varphi_X(1_A)$,

$$\begin{tikzcd}
& A \arrow[dl, "1_A"'] \arrow[dr, "1_A 1_A"] & \\
A & & A
\end{tikzcd}$$

does give $(1_A, 1_A)$, so φ_X preserves identities. Next consider two maps $A \xrightarrow{f} B, B \xrightarrow{g} C$, s.t. $\overline{f} = mr$ and $\overline{g} = ns$. Then

$$\varphi_X(g)\varphi_X(f) = \begin{tikzcd}
& & A'' \arrow[dl, "m'"'] \arrow[dr, "f'"] & & \\
& A' \arrow[dl, "m"'] \arrow[dr, "fm"] & & B' \arrow[dl, "n"'] \arrow[dr, "gn"] & \\
A & & B & & C
\end{tikzcd}$$

Now, we can inquire as to what m', and f' are. We must form the pullback of

$$\begin{tikzcd}
& B' \arrow[d, "n"] \\
A' \arrow[r, "fm"'] & B
\end{tikzcd}$$

We use the construction of pullbacks, P_{M_X} in partial map categories, to get the following data as usual.

$$\begin{tikzcd}
A'' \arrow[r, "s(fm)m'"] \arrow[d, "m'"'] & B' \arrow[d, "n"] \\
A' \arrow[r, "fm"'] \arrow[u, bend left, "s'"] \arrow[loop below, "\overline{gfm}"] & B \arrow[u, bend right, "s"'] \arrow[loop below, "\overline{g}"]
\end{tikzcd}$$

Thus, $f' = sfmm'$ and m' is the monic part of the splitting for $\overline{gfm}$. The proof of lemma 3.1.2, makes it clear that mm' is the splitting for $mm's'r$. But

$$\begin{aligned}
mm's'r &= m\overline{(gf)mr} && \text{by pullback construction of m'} \\
&= \overline{gf}mr && \text{R4} \\
&= \overline{gf}\,\overline{f} && \text{assumption } \overline{f} = mr \\
&= \overline{gf} && \text{1.1.ii}
\end{aligned}$$

Thus, mm' is the restriction monic for $\overline{gf}$. Thus we can construct

$$\varphi_X(gf) = \begin{tikzcd} & A' \arrow[dl, "mm'"'] \arrow[dr, "gfmm'"] & \\ A & & C \end{tikzcd}$$

Therefore, to prove that $\varphi_X(g)\varphi_X(f) = \varphi_X(gf)$, it must be the case that $(mm', gfmm') = (mm', gnsfmm')$. Thus it suffices to show that $gfmm' = gnsfmm'$, but this is clear since by R1, $g = gns$.

Also, φ is a restriction functor at each component, since let $\overline{h} = mr = \overline{mr}$ be a restriction idempotent. Then, by lemma 2.2, m splits $\overline{mr} = \overline{r}$. Hence

$$\begin{aligned}
\varphi_X(\overline{h}) &= \varphi_X(mr) \\
&= (m, mrm) \\
&= (m, m) \\
&= \overline{(m, hm)} \\
&= \overline{\varphi_X(h)}
\end{aligned}$$

Next, we must show φ_X is invertible at each component X. Since, φ_X is the identity on objects, it suffices to show φ_X is full and faithful. For fullness, let $(m, f) : A \longrightarrow B \in Par(MTotal(X))$. Then m is a restriction monic. X is a split restriction category, so by lemma 2.2, there is a unique r, s.t. $mr = \overline{r} = \overline{mr}, rm = 1$. So consider

$$\begin{aligned}
\overline{fr} &= \overline{\overline{f}r} && \text{lemma 2.1.3} \\
&= \overline{r} && \text{f is total} \\
&= mr
\end{aligned}$$

Thus m is the restriction monic for $\overline{fr}$.

$$\varphi_X(fr) = \begin{tikzcd} & A' \arrow[dl, "m"'] \arrow[dr, "frm=f"] & \\ A & & B \end{tikzcd}$$

Therefore, φ is full. To show faithfullness, suppose $\varphi_X(g) = (m, f)$. Then it is true that m splits $\overline{g}$, so that $\overline{g} = mr$. Further, $f = gm$. This means that $g = gmr = fr$, and this holds for any g, so that if $\varphi_X(h) = \varphi_X(g) = (m, f)$, then it is immediate that $h = fr = g$. Therefore, φ is faithful.

Therefore we have that φ is an isomorphism at each component, thus φ is an isomorphism. Thus it remains to show that φ is a 2-natural transformation. First, consider that φ is a natural transformation.

Let $\mathbf{X}, \mathbf{Y}$ be split restriction categories, and $F : \mathbf{X} \longrightarrow \mathbf{Y}$ be a restriction functor. Then consider the naturality square.

$$\begin{tikzcd}
\mathbf{X} \arrow[r, "\varphi_X"] \arrow[d, "F"'] & Par(MTotal(\mathbf{X})) \arrow[d, "Par(MTotal(F))=Par(F)"] \\
\mathbf{Y} \arrow[r, "\varphi_Y"'] & Par(MTotal(\mathbf{Y}))
\end{tikzcd}$$

Then on objects $\varphi_Y(Par(F)(A)) = \varphi_Y(FA) = FA = Par(F)(A) = Par(F)(\varphi_X(A))$. Let $A \xrightarrow{h} B \in X$. Then $Par(F)(\varphi_X(h)) = Par(F)(m, fm) = (Fm, F(hm))$, where $\overline{h} = mr$, and $\varphi_Y(Par(F)(h)) = \varphi_Y(Fh) = (n, F(h)n)$, where $\overline{F(h)} = ns$. Thus

$$\begin{aligned}
\overline{F(h)} &= F(\overline{h}) && \text{Restriction functor} \\
&= F(mr) \\
&= F(m)F(r)
\end{aligned}$$

But F is a restriction functor, so it preserves restriction monics by lemma 2.4.3, which are unique by lemma 2.2, thus $F(m) = n$. Thus $\varphi_Y(Fh) = (n, F(h)n) = (Fm, F(h)F(m))$, and so φ is a natural transformation.

To establish that φ is indeed a 2-natural transformation, let G be another restriction functor $G : \mathbf{X} \longrightarrow \mathbf{Y}$ and let $\alpha : F \Longrightarrow G : \mathbf{X} \longrightarrow \mathbf{Y}$ be a restriction natural transformation. The following equality must be proved.

$$\begin{tikzcd}
\mathbf{X} \arrow[r, bend left, "\varphi_Y F"{name=U}] \arrow[r, bend right, "\varphi_Y G"'{name=D}] & Par(MTotal(\mathbf{Y})) \arrow[Rightarrow, from=U, to=D, "\varphi_Y \alpha"']
\end{tikzcd} \qquad =$$

$$\begin{tikzcd}
\mathbf{X} \arrow[r, bend left, "Par(MTotal(F))\varphi_X"{name=U}] \arrow[r, bend right, "Par(MTotal(G))\varphi_X"'{name=D}] & Par(MTotal(\mathbf{Y})) \arrow[Rightarrow, from=U, to=D, "Par(MTotal(\alpha))\varphi_X"']
\end{tikzcd}$$

The typing in the above for $\varphi_Y \alpha$ and $Par(MTotal(\alpha))\varphi_X$ is actually the same and this follows from the naturality of φ. Thus it remains to show that for any component $A \in X$, $(\varphi_Y \alpha)_A = (Par(MTotal(\alpha))\varphi_X)_A$.

First, consider $(\varphi_Y \alpha)_A = \varphi_Y(\alpha_A) = (1_{FA}, \alpha_A)$, since α_A is total, and φ_Y is a restriction functor it preserves total maps by lemma 2.4.2.

Next, consider $(Par(MTotal(\alpha))\varphi_X)_A = (Par(MTotal(\alpha)))_{\varphi_X(A)} = Par(\alpha)_{\varphi_X(A)}$, since by prop. 4.1, $MTotal(\gamma) = \gamma$ for any restriction transformation γ. Next, $Par(\alpha)_{\varphi_X(A)} =$

$(Par(\alpha))_A$, since φ_X is the identity on objects. By the definition of Par on any $\mathcal{M}$-Cartesian natural transformation (in proposition 5.1), conclude that

$$(Par(\alpha))_A = \begin{tikzcd} & FA \arrow[dl, "1_{FA}"'] \arrow[dr, "\alpha_A"] & \\ FA & & GA \end{tikzcd} = (1_{FA}, \alpha_A)$$

Therefore, φ is a 2-natural transformation, giving the desired 2-natural isomorphism.

□

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